THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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LANCASTER, PA., AND ANN ARBOR, MICH.

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MATHEMATICAL MONTHLY

THE NOVEMBER MEETING OF THE IOWA SECTION.

The eleventh regular meeting of the Iowa Section was held at the West Des Moines High School, Des Moines, Iowa, on November 3, 1922, the Section, as is usual at the fall meeting, uniting with the Iowa Association of Mathematics Teachers. The meeting consisted of one session with Professor R. B. McClenon, president of the Iowa Association, in the chair.

There were about one hundred and fifty in attendance, including the following twenty-one members of the Association:

O. W. Albert, R. P. Baker, I. S. Condit, Marian E. Daniells, R. M. Deming, F. C. Earhart, C. W. Emmons, Iva Ernsberger, Fay Farnum, C. Gouwens, G. E. King, R. B. McClenon, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, C. W. Wester.

The next meeting will be held at Cornell College, Mount Vernon, Iowa, April 27–28, 1923, in conjunction with the annual meeting of the Iowa Academy of Science.

The following two papers were read:

(1) "The appreciation of four dimensions" by Professor R. P. Baker, University of Iowa;

(2) "How can we improve the teaching of mathematics in Iowa" by Miss Wheeler, teacher of mathematics, Grinnell High School; and Professor E. R. Smith, Iowa State College.

A general discussion on the improvement of mathematics teaching followed the presentation of these papers, abstracts of which follow below, the numbers corresponding to the numbers in the list of titles:

1. In this paper Professor Baker discussed Poincaré's suggestion that by making independent our two methods of judging depth—shading and binocular convergence—an appreciation of four dimensions might be reached. A set of stereoscopic pictures was shown illustrating retinal rivalry where the compromise between the judgments of two eyes is an alternation in time; and another set where, by putting the usual right and left eye pictures in opposite places, the shading and binocular convergence work against one another. In this case the compromise is by majority rule of associations. The standard method of models in three dimensions of projections of four-dimensional solids was discussed and illustrated by models of the tesseract and the square on the triangle.

2. To improve the teaching of mathematics Miss Wheeler placed emphasis on the following three points: (*a*) the teacher should feel that the subject taught

is worth while, (*b*) the mathematics department should be supplied with equipment as liberally as is any other department, and (*c*) the individual student should be kept in mind and the course adapted as far as possible to his needs.

Professor Smith asserted that at the present time the Colleges and Universities of the State are supplying annually only about fifty per cent. of those required to meet the demand in the high schools and in the industries. He recognized four groups of students studying mathematics in the high schools: (*a*) those taking tradesmen's courses, (*b*) those taking such courses as will prepare them to read current literature, (*c*) those who will become technicians, and (*d*) those who will be mathematics specialists. In order to improve the instruction given in the high school, especially for the latter two groups, he would like to see a larger number of ideal teachers whose preparation consisted of not less than four hours per week for four years in College, and one year of professional work in teaching.

J. F. REILLY, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE MISSOURI SECTION.[1]

The sixth regular annual meeting of the Missouri Section was held at the Junior College of Kansas City on Saturday morning, November 18, 1922, in connection with the annual meeting of the Missouri State Teachers' Association. Professor W. A. Luby, vice-chairman of the section, presided. Mr. Alfred Davis acted as secretary in the absence of Professor P. R. Rider.

There were twelve in attendance, including the following six members of the Association:

L. V. Cutting, Alfred Davis, S. Lefschetz, W. A. Luby, A. D. Pierson, R. A. Wells.

The present officers were elected for another year. They are Professor E. R. HEDRICK, University of Missouri, chairman; Professor W. A. LUBY, Junior College of Kansas City, vice-chairman; Professor P. R. RIDER, Washington University, secretary-treasurer.

The following papers were presented:

(1) "A study of the data determining the sun-spot maximum of 1829" by Professor W. A. LUBY, Junior College of Kansas City;

(2) "Problems concerning the teaching of secondary mathematics" by Mr. ALFRED DAVIS, Soldan High School, St. Louis;

(3) "Mathematics in Europe" by Professor S. LEFSCHETZ, University of Kansas (by invitation).

Abstracts of papers follow below, the numbers corresponding to the numbers in the lists of titles:

1. The observations on which the date of the sun-spot maximum of 1829 is based are those of Schwabe, Stark and Arago. Professor Luby's paper presented

[1] The secretary of the section is indebted to Mr. Alfred Davis for the material for this report

an abstract of the observations and pointed out the smaller weight given to the more reliable data of Schwabe.

2. Mr. Davis's paper stated that recent activities among some of the prominent educators of the country, with reference to mathematics in our high schools, have emphasized some problems and have created other problems that need solution. Some of these have been partially solved by such agencies as the National Committee on Mathematical Requirements; by special schools, such as the Horace Mann School, New York; and by individual teachers. Much however remains to be done. And their complete solution, let us hope that this may not be long delayed, will help to restore mathematics to its proper place in the class room, as well as outside. Mathematics is developing with great rapidity along many lines and it is becoming increasingly important as an aid to progress in many branches of science. In view of this and of the educational value of the subject we need to define more clearly, and without any trace of apology, the aims and purposes of the subject. Much time is wasted and vantage lost because pupils are not classified according to ability to learn. The dull are overwhelmed and the bright are retarded. Not only do the junior high school courses need to be more clearly outlined, but, and no less important, the senior courses need re-organizing. The traditional courses have served their day. Pupils are not now merely prepared for college; the majority of those who graduate from high school do not go to college. All plane and solid geometry beyond the junior high school might be condensed into one year. Similarly all the algebra beyond that given in junior high school might be taught in one year. This leaves the senior year for some advanced courses. The latter may consist of trigonometry, of history of mathematics, or advanced work which would include the elements of analytics and the calculus. By such means the pupil might be given a working knowledge of mathematics and so have the field for further study and application of the subject opened to him. The standing of the subject among people generally could be greatly improved by giving the brighter of our high school pupils this advanced work. Mathematics would thus seem to have a real purpose in the work-a-day world. The climax of our problems is the securing of the inspired and the inspiring teacher. Let us remember that every good cause needs championing by its friends. If these problems are solved it will be because we solve them. Indifference must vanish.

3. Professor Lefschetz recited some of the impressions gathered during his recent trip and spoke of the outlook, which is anything but brilliant, for science in general, and for mathematics in particular. (See *1922*, 144.)

P. R. Rider, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The twelfth regular meeting of the Maryland-Virginia-District of Columbia Section was held at the Bureau of Standards, Washington, D. C., on Saturday, December 9, 1922. The Chairman of the Section, Professor Frank Morley, presided at both morning and afternoon sessions. At the close of the morning session, those in attendance at the meeting had dinner in the dining-hall of the Bureau of Standards as guests of the Washington members of the Section. The hour following dinner was spent in an inspection of some of the work being carried on at the Bureau of Standards.

The next meeting of the section will be held on May 12, 1923, at Baltimore, Maryland.

There were fifty-eight in attendance, including the following thirty-seven members of the Association:

O. S. Adams, J. J. Arnaud, R. N. Ashmun, H. G. Avers, Sarah Beall, G. A. Bingley, W. H. Bixby, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, F. W. Darling, C. H. Davis, A. Dillingham, H. English, J. B. Eppes, W. M. Hamilton, W. E. Heal, W. D. Lambert, A. E. Landry, E. A. LeLacheur, Florence P. Lewis, Nannie J. McKnight, F. Morley, C. A. Mourhess, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, H. M. Robert, Jr., H. A. Robinson, R. E. Root, C. A. Shook, E. W. Woolard.

The following papers were read:

(1) "A problem in the theory of numbers" by Dr. J. R. MUSSELMAN;

(2) (*a*) "A remarkable formula for prime numbers," (*b*) "A method of distinguishing between prime and composite numbers of large size" by Dr. PAUL R. HEYL, Bureau of Standards (Introduced by Mr. W. D. LAMBERT);

(3) "Remarks on the proposed plan of reorganization of secondary school mathematics" by Professor A. E. LANDRY;

(4) "Dynamic symmetry" by Professor H. M. ROBERT, Jr.;

(5) "Remarks on a problem in geometry" by Professor FRANK MORLEY;

(6) "A property of a system of partial differential equations" by Dr. C. A. NELSON.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Dr. Musselman discussed the problem of finding even "perfect" numbers, which reduces to finding values of $2^n - 1$ which are primes. He also mentioned what is known concerning the expression of a number as a sum of cubes.

2. Dr. Heyl uses the expression $(2^{n-1} - 1)/n$, which is integral for all odd *prime* values of n, and non-integral for all even values of n. Computation shows that it is also non-integral for all composite odd values of n up to 1,000 with the exception of 341 and 645. A practical method of handling the very large numbers to which this formula gives rise may be based on the device of expressing large

powers of 2 as powers of powers; and the making use of the principle that if the product of several numbers be divisible by n with a certain remainder, this same remainder will be obtained if for any of these numbers there be substituted its remainder when divided by n. The remainder resulting from this formula will in about 75 per cent. of the cases examined (1,000 in number) contain a factor of the original number, which may then be found by the usual process for highest common factor. Time tests have shown that a number of the order 10^8 may thus be reduced in one hour as against 20 hours required for the trial of prime divisors.

3. Professor Landry's paper was an exposition and criticism of the plan of reorganization of secondary school mathematics drawn up by the National Committee on Mathematical Requirements, and printed in part by the United States Bureau of Education in 1921 as Bulletin No. 32. Attention was devoted chiefly to Chapters III and IV, on the junior and senior high school periods respectively. The proposed junior high school course was on the whole approved except as to (1) the extent to which unification is possible or desirable, and (2) a profitable lessening in the amount of *time* allotted to algebraic drill. The suggestion that calculus be introduced as an elective in the senior high school course was examined and approved, but with the accompanying expression of serious doubt of the likelihood of its early adoption.

4. Professor Robert discussed the theory of Mr. Jay Hambidge regarding the proportions of Greek vases as explained in his book *Dynamic Symmetry*, and in *Geometry of Greek Vases* by Dr. L. D. Caskey, Curator of Classical Antiquities of the Museum of Fine Arts, Boston. Mr. Hambidge finds that the Greeks used geometry rather than arithmetic in design. Rectangles having the ratio of longer to shorter side equal to $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, are called root-two, root-three, and root-five rectangles, respectively. These rectangles are combined with the square to form other rectangles. A rectangle derived from the root-five rectangle, having the ratio $\frac{1}{2}(\sqrt{5} + 1)$ is called the whirling square rectangle and is much used. Mr. Hambidge finds that if the over-all proportions of a vase, that is, the ratio existing between its height and greatest width, is expressible in terms of one of these three root-rectangles, then the heights and widths of all its parts can be expressed in terms of that rectangle and of no other; *e.g.*, root-two or root-three never appear in connection with root-five. In Dr. Caskey's book dimensions to the nearest half-millimeter of most of the vases in the Museum are given. The outline of each vase is enclosed in a rectangle belonging to one of the systems of dynamic symmetry, the margin of error allowed averaging less than one millimeter. The interrelation of details is shown by subdivision of the containing rectangles and by intersections of diagonals.

Professor Robert gave illustrations of the analysis of vases and pointed out misunderstanding on the part of critics and failure to recognize that all constructions must be geometrical. In particular, he discussed the suggestion in a published criticism (this MONTHLY, *1922*, 164) that .9393 might be mistaken for .927. He showed the drawing for the kylix in question and the simplicity of the

geometrical construction of the diameter of the foot as $2 - 3\sqrt{2}/4 = .9393$ in relation to the over-all rectangle.

5. Professor Morley made some remarks on recent work on the three-bar curve, especially on the memoir on the analytical treatment of the three-bar curve by F. V. Morley in *Proceedings of the London Mathematical Society*, series 2, volume 21, pp. 140–160.

6. Dr. Nelson pointed out the relation between the solutions of a completely integrable system of linear partial differential equations of the second order and of the adjoints, in the sense of Riemann, of these equations.

G. R. CLEMENTS, *Secretary-Treasurer.*

THE ORGANIZATION OF COLLEGE COURSES IN MATHEMATICS FOR FRESHMEN.[1]

By J. W. YOUNG, Dartmouth College.

The movement toward so-called unified courses in mathematics for freshmen reached a certain rather definite stage of growth with the publication in 1914 of Professor Slichter's *Elementary Mathematical Analysis*. It is true that before this date other texts embodying a breaking down of the traditional barriers between various subjects had appeared, such as the well-known texts of Smith and Granville, Woods and Bailey, Brenke, Ziwet and Hopkins. But these were either intended for special classes of freshmen, or for sophomores, or they involved the correlation merely of two specific subjects, such as algebra and trigonometry or algebra and analytic geometry.

Professor Slichter's text was, as far as I know, the first that attempted to give a unified course for the entire freshman year based on the minimum college entrance requirements. It is rather interesting to note, in passing, that this first attempt was written specifically for students preparing to study engineering rather than for the general student. This text was followed by the texts of Young and Morgan (1917), Karpinski, Benedict and Calhoun (1918), Webber and Plant (1919), Gale and Watkeys (1920) and Griffin (1921). I note that the next speaker is to discuss the historical aspects of this movement. I do not wish to trespass on his domain; this much, however, seems to be necessary as an explanation of what follows.

It seemed to me that it might be of interest to try to get some idea as to the extent to which the new type of course has been adopted. I accordingly appealed to the publishers of the textbooks listed in the preceding paragraph with the purpose of securing a list of colleges and universities using the various texts in the various years since their publication. These publishing firms were all very cordial in their replies and furnished me with such data as they had—they pointed out, however, that their data were very unreliable for my purposes. I accord-

[1] Read at the meeting of the Association, University of Rochester, Sept. 6, 1922.

ingly sent a questionnaire to certain colleges and universities selected as follows: I included first all institutions listed in the World Almanac for 1922 as having an endowment of at least $1,000,000; I then added all the larger state Universities and Colleges, and then selected other institutions having at least 1,000 students. This gave a list of 165 institutions. The questionnaire was sent out Aug. 4, when many of those addressed were away from their institutions. It is, therefore, it seems to me, a fair showing that replies were received from 98 institutions;[1] of these apparently 59 have at one time or another within the last ten years given some form of unified courses to freshmen. If we may assume that the 98 institutions which replied form a typical selection from the institutions of the country, it is clear that the new type of course has been at least tried in a very substantial proportion of our institutions. The lists furnished me by the publishers indicate that seven of the eight textbooks listed (no list was secured from one of them) have been used in about 150 different institutions in this country and Canada. It would seem that the time has come when a critical estimate of the results secured by unified courses could be made with profit. The combined experience of all these institutions would be extremely valuable could it be secured and evaluated.

We must be content today, however, with a less ambitious program—one that is commensurate with our limited data and with the limited time at our disposal.

Of the 59 institutions which have given unified courses, 41 gave detailed information as to the years in which certain texts were used. From these data it is possible to derive an answer to the question whether the movement toward a unified course for freshmen is at present on the increase or the wane. Of these 41 institutions, 14 were giving the new type of course in 1917–18, 20 were giving such a course in 1918–19, 27 in 1919–20, 23 in 1920–21, and 23 in 1921–22. These figures may indicate a slight decrease in enthusiasm during the past two years; but the data are very incomplete and too much reliance must not be placed on them. The figures do show conclusively that a strong body of institutions still believe in the new type of course—and that those who are interested actively in furthering the cause have no need to feel discouraged.

Those of us, however, who believe in the new type of course must by no means feel that the battle is won—or that our work is done. The opposition is strong, the criticisms severe and often just, the problem of constructing the ideal freshman course is far from solved—and the solution presents many very serious difficulties. It is still to my mind an open question whether these difficulties can be overcome.

The replies to my questionnaire have been very helpful in my attempt to determine and evaluate these difficulties. A brief summary of the arguments against unified courses will indicate some of these difficulties.

A number of arguments relate to administrative problems. Transfer of credits from one institution to another is difficult; the unified course is not well

[1] The figures here given differ slightly from those presented at Rochester, since I have included replies received since that time.

adapted to institutions operating on the term or quarter plan; at many institutions students can not be counted on to take mathematics for a whole year; the course is not well adapted to the varied preparation with which freshmen enter college, etc. With such difficulties I do not propose to deal today; not because they are not real or important, but because their discussion would take us too far afield. Perhaps some of the other speakers will touch upon them.

A number of other arguments relate to the needs of special classes of students, such as prospective engineers, scientists, agriculturists, etc. These also I shall pass over without comment, as I think it desirable to limit my own discussion to a fairly definite problem.

Before proceeding further with the enumeration of difficulties, let me attempt to define this problem with reasonable precision:

I have in mind a student who enters college with the mathematical preparation implied by a year of plane geometry and a year or a year and a half of elementary algebra. I realize that students especially in our eastern colleges often have somewhat more preparation; above all I realize that in our western institutions large numbers enter with less. But I conceive the preparation indicated to be the natural starting point for our discussion. I will suppose furthermore that this student expects to take one full year of mathematics, three hours per week, during his freshman year. I assume, furthermore, that he does not know what his life work is to be, nor in what department of college he expects to specialize. What is the best course in mathematics which we can plan for this student? That is the problem I wish to approach and to which I propose to limit my own part in this discussion. I venture to believe that the problem as thus formulated is the central problem from the solution of which other modified problems will have to result.

I return to the difficulties in the way of a solution. One of the most insistent claims made by the adverse critics of a unified course is to the effect that it does not produce clear-cut concepts in the mind of the student, that he is confused by the multiplicity of ideas and methods presented to him. This is a serious defect of the course, if it is a fact—and we must remember that I am giving you the testimony of men who have tried out the unified course for one or more years.

Another group of critics claims that the new type of course offers a poor foundation for later more advanced work in mathematics, that students taking it are weak in manipulation, that the course lacks thoroughness, that it is sketchy, superficial, provides only a smattering of various topics, that at the end of the course the student does not feel that he has mastered anything. This also is a serious charge—for our typical student *may* want to take more mathematics, he may want to go into a profession where he needs mathematical proficiency—and in any case none of us advocating the use of the new type of course can rest easy under the charge of superficiality and lack of thoroughness.

Another group of criticisms states that the textbooks now on the market present too much material for a year's course, that the program is too ambitious.

And finally we find serious argument to the effect that the goal set by the advocates of a unified course is in the nature of the case unattainable, that the

unifying principle fails to unify, that the desired correlation has not yet been and probably can not be made, that, as one correspondent puts it, the attempt to secure "the larger unity is at the expense of more elementary and fundamental unities without which the larger is almost unattainable."

Are we here facing a fundamental obstacle? Is our goal contrary to the laws of the learning process and of mental growth? Are we attempting the impossible?

Very many of my correspondents state that they are theoretically in favor of a unified course but that in practice it does not seem to work as well as the older traditional courses, that the results secured by the new type of course are not as good as those secured by the old. Many of these critics seem to have reached the conclusion that it is better policy to teach the separate subjects separately and then at the end attempt a general unification, that courses in the separate subjects are satisfactory if only the teacher does his part properly—that it is the teacher's duty to call attention to the various interrelations and interdependencies of the different subjects, and that this is the psychologically natural way to secure an appreciation of whatever underlying unity exists.

This leads us, as it seems to me, directly to the crux of the argument. The critics I have been quoting are arguing for the most part in favor of the traditional arrangement of the traditional material—trigonometry, algebra, analytic geometry—and against a new, unified arrangement of this traditional material. With respect to this traditional content of the freshman course they may be right, their arguments are strong, based as they are on experience, although personally in spite of manifest difficulties I have faith in the possibility of a so-called unified arrangement that will give better results than the separate-subject arrangement.

But the problem that faces us today concerning the needs of the typical general student I have described has passed entirely beyond this stage. At the present time *it is the traditional selection of material itself that is challenged.* The problem of unification is no longer the primary problem—it is a secondary problem forced on us by a new standard for the selection of content.

Most teachers, who have given serious thought to the needs of my general student, are I believe convinced that the traditional courses in trigonometry, college algebra, and analytic geometry can no longer be justified as best meeting his needs.

Can we formulate these needs? Recognizing that he probably will not take more than one year of mathematics in college, but keeping in mind the fact that he may do so and that hence we should provide a good foundation for possible subsequent courses, what would we like him to secure in the way of mathematical information and training during his freshman year?

I shall attempt to formulate these needs, as I see them. Before doing so, however, I want to say that these needs when enumerated are far beyond what can be accomplished in a one-year course. The topics which one would naturally select to satisfy these needs make a list entirely too long and extensive. It seems to me desirable, nevertheless, to list them, in order that we may have before us in all its difficulty the problem of selection that still remains. Furthermore, the

time at my disposal makes it necessary for me to be very brief. I can only indicate the general line of my thought and must count on my hearers to supply the details of the argument.

With these preliminary remarks, I now proceed to formulate the needs of my general student as follows:

1. *I should want this student to gain such mathematical information and training as will be of most use to him in later college courses in the physical and social sciences and in his later life as a citizen, parent, and educated man, without reference to his vocation or profession.*

To satisfy this demand as to the physical sciences our course must include the elements of trigonometry, linear and quadratic functions and equations, graphs, proportion and variation, familiarity with formulas and their use, and should certainly include if possible the fundamental ideas of the calculus and their applications. As to preparation for the social sciences (economics, etc.) we should want to include the elements of statistical methods, the elements of the mathematics of finance and investment, as well as some of the topics previously listed. As to his needs as citizen, parent, etc., we should again include a number of the topics just mentioned, notably statistics and the mathematics of investment, which are fundamental in many questions of public affairs, and would also want him to be familiar with the elements of probability, which in connection with the laws of compound interest are necessary for any adequate understanding of problems of insurance, to say nothing of games of chance.

2. *I should want him to gain a clear working knowledge of the fundamental general concepts in terms of which the quantitative thinking of the world is done.*

This aim to a large extent duplicates the preceding but it is broader in that it is not only utilitarian but also interpretative. Some of the concepts which I have in mind are number, ratio, measurement, congruence, similarity and proportionality, functional dependence in general, rates, limits, and so, again, the ideas of the calculus. It is under this aim that the value of training in functional thinking makes itself most strongly felt.

3. *I should want him to secure from his study of mathematics certain norms, standards and ideals of logical rigor, precision of thought, logical structure of a science or body of knowledge.*

This aim appears to me of very great importance—and here we doubtless have one of the strong arguments in favor of the separate-subject treatment, at any rate until such time as we can organize the new type of course in such a way as to exhibit a clear logical structure. I should like to go even further, however. I should like my general student to have some conception of the meaning of a logical demonstration, an understanding what is meant by a deductive system, an appreciation of the significance of assumptions, axioms, postulates, of what is meant by a formal deductive proof. To satisfy this demand we should apparently have to include in our course something concerning the foundations of mathematics—either of algebra or of geometry or of both. This leads me, however, to my fourth aim.

4. *I should want my typical student to gain some appreciation of the rôle that mathematics has played and continues to play in the development of civilization, in its material, scientific, intellectual, philosophical and spiritual aspects.*

All of the topics thus far listed can be made to serve this end. But it seems to me that we must not ignore the fact that mathematics has something fundamental to offer to the cultural development of the individual in those topics, of rather recent development, which have close contact with philosophy and possibly with religion. I refer to the various developments relating to the concept of infinity and the theory of classes, non-euclidean and n-dimensional geometry, etc. The proposal to include such material in an elementary course must appear very fanciful—and I do not myself see how it can be done in a one-year course. But interest in these topics is great and widespread among educated people—and it seems to me that one of the important higher functions of education should be to minister to such interests. Is it not precisely by furnishing the means of satisfaction to such latent interests that the general intellectual level can be raised? It seems to me altogether proper, therefore, that in a discussion of this sort we should at least recognize the existence of this need and appreciate its importance.

If you agree with my formulation of needs for the general student, I think you will have to agree that the traditional content of trigonometry, algebra and analytic geometry, whether taught in separate courses or in a unified course, fails to offer the best material to satisfy them. In the first place too much time is spent in these traditional courses on drill in manipulation. As has often been said in recent years, manipulative technique must be regarded as a means to an end, not an end in itself. In so far as it contributes to an understanding of principles it is not only desirable but essential. But beyond that it is of little value to our general student.

In the second place, much of the traditional content is not the most suitable for the satisfaction of our needs. The properties of the conic sections and the methods of analytic geometry in general can not compare in value with the elements of the calculus, most of the work in college algebra is of little use compared with the algebra of finance, etc.

It should be clear, then, that our problem is not primarily one of unification of the traditional material, *it is primarily a problem in the selection of new material.* This material having been selected, some form of unification becomes necessary in order to secure a simple logical structure rather than a mass of isolated topics. The secondary, but equally important problem, relates then to *the organization of the material selected.*

The plea of lack of time will hide my inability to solve either of these problems. Any one who has attempted a solution will realize how difficult they are. But all of us interested in the improvement of mathematical education in this country must contribute what we can toward their solution. As one of my correspondents has pointed out: "On the evolution of a satisfactory textbook depends the continued growth of the movement."

I will not attempt to characterize the ideal textbook of the future. By comparing the earlier ones of 1914 and 1917 with the later ones of 1920 and 1921 some

idea of present tendencies can be gained. It seems safe to say that at present there is a well-defined tendency toward the diminution of most of the traditional material in analytic geometry and toward the inclusion of a considerable amount of calculus. But beyond this it is not safe to indulge in prophecy.

I desire rather, in the remaining time at my disposal, to analyze a bit further some of the difficulties enumerated by our critics—and to offer one or two constructive suggestions.

First of all, I must remove the impression of general dissatisfaction with unified courses, if I have created it by an exclusive insistence on the difficulties. My correspondents include quite as many enthusiastic supporters of the movement as adverse critics. It should be noted that the testimony of those who have used the unified courses is on many points conflicting. While some say that the new type of course is too difficult, others say it is easier than the traditional; while some have found it to provide a poor foundation for subsequent work in mathematics, others have found it to give an excellent foundation; while some claim that it is not well adapted to the varying preparation of students, others find that it is well adapted thereto, etc. And there are many who appear to be firmly convinced that the new type of course is more interesting to the student; that it offers him a far richer and more generally valuable training; that the utility and power of mathematics can be made more apparent to him, and that hence his appreciation of mathematics is increased; that more emphasis is placed on an understanding of principles and less on memorizing formulas, and that it, therefore, is more effective in stimulating thought; and, finally, that the earlier introduction of certain fundamental concepts, such as those of the calculus, extends their period of growth and hence their effectiveness. This last point appears to me of very great importance, since it seems to be in line with a psychological principle which experience has seemed to establish.

Why then this feeling on the part of many, who are theoretically in favor of the new type of course, that the results secured are not as satisfactory as they should be? I wish these gentlemen had given me more detailed specifications as to these unsatisfactory results. The claim that students are less proficient in technique we may pass over, since we ought not to expect them to be as proficient. The claim, however, that the student does not gain as clear-cut concepts under the new as under the older course requires consideration. I wonder if the claim is valid; I wonder if those who make this claim have really used any valid test to find out the facts in the matter. I recently had an illuminating experience that is related to this question. I was teaching a traditional course in analytic geometry and we were on the topic of the point dividing a segment from one point to another in a given ratio. I did my best to explain the meaning of the concept, including the significance of the order of the two given points and of the algebraic sign involved, assigned the lesson and a few days later gave a written quiz on the subject. The members of the class were quite good at using the formula and solving the traditional types of problems. But I passed around slips of paper on which were drawn two parallel straight lines each with a scale

marked on it. There were marked also on each of the lines two points A and B. In addition, on the first of the two lines were marked two points C and D, one of them lying between A and B, the other not between A and B. Under these two lines were written the following questions:

In what ratio does the point C divide the segment from A to B?

In what ratio does the point C divide the segment from B to A?

In what ratio does the point D divide the segment from A to B?

In what ratio does the point D divide the segment from B to A?

Then with regard to the second line, which in addition to the scale divisions had only two points A and B marked, I called for the following:

Mark on the line the point which divides the segment from A to B in the ratio 1 to 3 and label it P. Mark the point which divides the segment from B to A in the ratio 1 to 3 and label it Q. Mark the point which divides the segment from A to B in the ratio -1 to 2 and label it S.

Not more than two or three men in a class of something over twenty could answer these simple questions correctly. These students could do the traditional things well—but they were clearly doing them mechanically. They certainly had no clear idea of the concept involved.

My first constructive suggestion is that we need to develop a new technique of teaching for the new type of course, a technique that is directed toward the new aims. As an essential part of this technique must be developed a new type of test, designed to measure the achievement of students with respect to the fundamental aims of the course. Until such new tests, that are adequate to their purpose, are developed our movement will labor under a severe handicap. For it is manifestly unfair to judge the results secured under the new course by means of tests developed for the traditional course. And is not that what many of us have been doing? Is not here one of the reasons why some of those who have experimented with the new material have found the results discouraging?

I would offer one other suggestion. We need a textbook for the second year's work designed for those who expect to have professional use for mathematics—mathematicians, scientists, engineers and the like; a textbook to follow the first year's work. This second-year course will take up those topics omitted from the first year's work but needed by the prospective engineer let us say—notably analytic geometry, some advanced algebra, and the remainder of the calculus. It will strongly emphasize drill in technique because the student taking this course will need to be technically expert. Such a course would appear to be entirely feasible, if a proper text is provided. To give it from existing texts is awkward and wasteful. Will not some of our authors address themselves to this problem. It seems to me important.

In closing I would merely express my own conviction as to the continued vitality of the movement we have been discussing and my belief that a solution of the various difficulties will be found by a process of successive approximations. Moreover, I believe the time has come when the primary emphasis must be placed on aims, such as those I have attempted to formulate, and on the selection of

material to meet these aims, rather than on unification as such. Unification may not be possible at this stage beyond a certain point; in any case it is a possible means to an end, not an end in itself.

SOME CURIOUS FALLACIES IN THE STUDY OF PROBABILITIES.

By ROBERT E. MORITZ, University of Washington.

Part I.

Quale è'l geometra che tutto s'affige
Per misurar lo cerchio, e non ritruova,
Pensando qual principio ond' egli indige.
Dante, *Paradiso*, canto 33.

It is a strange anomaly that that branch of mathematics, known as the theory of probability, which rests upon the fewest, simplest, and least controvertible fundamental principles, which demands practically no mathematical prerequisites for its pursuit, which throughout occupies itself with innumerable interesting and important problems that even the layman can understand, should be, at the same time, that branch of mathematics which has presented the greatest number of pitfalls to its most illustrious devotees. Indeed it seems that the simplicity, obviousness, and certainty of its basic principles, the easily understood character of its subject matter, and the ease with which it ensnares into error the most skilful dialecticians, are the three outstanding characteristics of this science.

Cardan, as will be shown presently, may be said to have inaugurated the study with a mistaken solution; Pascal, another pioneer of the subject, committed a fallacy in his problem of points involving three players;[1] Leibnitz fell into error in thinking that a throw of twelve with two dice is as probable as a throw of eleven.[2] D'Alembert stumbled time and again when dealing with probabilities. James Bernoulli, in his *Ars Conjectandi*, recorded two erroneous solutions of his nineteenth problem which occurred to him before he obtained its true solution.[3]

Some of the problems that today we expect every schoolboy to solve have been the occasion of serious, and sometimes even acrimonious, contention on the part of mathematicians of the first rank. Take the simple question as to the probability of throwing heads at least once in the course of two tosses of a coin. D'Alembert[4] reasoned that the required probability is 2/3, since there are but three conceivable cases two of which include heads, namely, head on the first toss, or, if head fails to turn up on the first toss then the second toss must show either head or tail, making three possible cases HH, TH, TT. The same sort of reasoning would, of course, make the probability of every future event equal to one-half since there are but two conceivable cases, the occurrence or the non-occurrence of the event.

[1] I. Todhunter, *History of the Theory of Probability*, Cambridge and London, 1865, p. 15.
[2] Leibnitz, *Opera Omnia* (Dutens), vol. 6, pt. 1, p. 217.
[3] I. Todhunter, *l.c.*, p. 69.
[4] I. Todhunter, *l.c.*, p. 258.

D'Alembert [1] considered it obvious that, when on tossing a coin heads have turned up three times in succession, it is more likely that the next toss will result in tail than in head. Bequelin[2] was of the same opinion and concluded hat, when heads have turned up n times in succession, the probability that the next toss will result in head is only $1/(n + 1)$ as against a probability of $n/(n + 1)$ that the result will be tail.

But little less obvious than the preceding is the following fallacy. If three coins are tossed at once, what is the probability that either three heads or three tails are turned up? Now of three coins at least two must show heads or tails, the probability that the third coin will show like the other two is one-half, hence the probability that all three coins will show alike is 1/2. The fallacy was exposed by Francis Galton.[3]

In the preceding fallacies the errors in logic are easily discerned but there are other problems in which apparently sound logic seems to lead to contradictory results.

Suppose that a poor man and a rich man engage in a game of chance on the following conditions: Each stakes one dollar after which they toss a coin to see who takes the pot. If the rich man wins, the play is ended, for we suppose that the poor man has nothing further to lose, but if the poor man wins they play again the next day, each man doubling his stake. If the rich man wins the second day the play ends, for the poor man will have lost his original stake together with his winnings from the first play and has therefore nothing further to lose. On the other hand, if the poor man wins they continue to play again doubling their stakes, and so on indefinitely.

Common sense tells us that the rich man is sure to win, for no matter how many times in succession the poor man may have won, if he loses the next play he will have lost his original stake together with all of his previous winnings, and, since he has nothing further to lose the play stops automatically, so that he has no chance to recoup. In fact the poor man's losing seems to be a necessary condition of the problem since by hypothesis the play is to continue until the rich man wins.

This problem, known as the martingale, was, according to Cantor,[4] first introduced into mathematical literature by Cardan, who discussed it in his *Practica Arithmeticae* published in 1539. Cardan is said to have shown that the condition of the play imposes a great disadvantage on the rich man. Cantor gives no indication of Cardan's method of proof.

Whitworth seems to hold the opposite view. While he does not discuss the martingale in detail, he announces a principle which clearly covers the case. He says, "If any condition is introduced which requires the play to stop when a certain position is reached, and if that position is more favorable to one player

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[1] I. Todhunter, *l.c.*, p. 263.
[2] E. Czuber, Grunert's *Archiv*, vol. 67, p. 8.
[3] *Nature*, vol. 49, p. 365.
[4] M. Cantor, *Geschichte der Mathematik*, vol. 2, Leipzig, 1900, p. 502.

than to the other, the latter is at a disadvantage."[1] The martingale introduces such a condition in that it requires the play to stop when the rich man wins.

Bachelier,[2] one of the most recent writers on probabilities, accepts neither of the foregoing conclusions. Assuming the play to stop when the poor man loses or when he has won n games in succession, he finds the poor man's gain to be $p^n \cdot 2^n - 1$, where p is his probability of winning one game. As n increases indefinitely, this expression approaches ∞, 0, or -1, according as p is greater than, equal to, or less than 1/2. In our case $p = 1/2$, so that the poor man's gain is 0, that is, the poor man has neither an advantage nor a disadvantage over the rich man.

Bachelier's formula is unassailable, yet it leads to a curious paradox if we consider a case in which p is slightly greater than 1/2. Let us assume that the coin used by the players has a tendency when tossed to show heads somewhat more frequently than tails, let us say an average of 51 heads to 49 tails, and that the poor man wins whenever head shows up. Then $p = 51/100$, and Bachelier's formula gives ∞ as the poor man's expectation. This conclusion amounts to saying that no finite sum, however great, would compensate the rich man for the risk he assumes in accepting the conditions of the play provided the coin used in tossing has the slightest tendency to favor the poor man. Common sense assures us that the rich man might well accept the risk on even terms.

A striking illustration of the confusion of thought which has possessed eminent mathematicians when dealing with probabilities is found in Montmort's solution of one of the five problems proposed to him by Nicolas Bernoulli in a letter dated September, 1713, and which Montmort published in the second edition of his *Essai d'analyse sur les jeux de hazard* (Paris, 1714). The third of these problems was as follows:

A deposits a crown. *B* throws a common die and if an even number turns up, he takes a crown, if an odd number turns up he deposits a crown. Then *A* throws the die and if an even number turns up he takes a crown, but deposits nothing if an odd number turns up. Then *B* throws the die again, taking a crown if he throws an even number, depositing a crown if he throws an odd number. Then *A* throws the die again under same conditions as before, and so on indefinitely so long as there remains any sum on deposit. Required the advantage of *A* or *B*.

In his reply to Bernoulli, Montmort states that he had not tried the first two problems, that the last two presented no difficulty, but that it had taken him a long time to solve the third problem, that he had finally come to the conclusion that there would be neither advantage nor disadvantage to *B*, as had also Waldegrave, an English mathematician, who had worked with him on the problem.

The modern reader can only wonder what it was that Montmort and Walde-

[1] W. A. Whitworth, *Choice and Chance*, 5th edition, Cambridge, England, 1901, p. 220.

[2] L. Bachelier, *Calcul des Probabilités*, vol. 1, Paris, 1912, p. 38.

grave worked over, since the problem states explicitly that B has at every trial an even chance of winning or losing.[1]

Bernoulli's fifth problem, which Montmort said presented no difficulty, is the problem which in a slightly modified form has since become celebrated under the name "Petersburg Problem." As first stated the problem admitted of several interpretations, the modified form in which it has become generally known was given it by Daniel Bernoulli, the nephew of Nicolas, who first discussed the problem in the transactions of the Petersburg academy,[2] whence the name of the problem.

In its original form the problem was in substance as follows: B throws a common die. If six turns up on the nth trial, A is to give B a sum of money which is some known function of n. Required B's expectation.

If we denote the sum by $f(n)$, and assume that the play ends when six turns up for the first time, B's expectation is $\Sigma[f(n)\cdot 5^{n-1}/6^n]$. If $f(n)$ is such that $\lim_{n\to\infty}[f(n+1)/f(n)] > 6/5$ or if $f(n+1)/f(n)$ is constant and equal to $6/5$, say $f(n) = (6/5)^n$, B's expectation becomes infinite.

Daniel Bernoulli's modification of the problem makes it read thus: A coin is tossed until head turns up. If this happens on the first trial Peter is to pay Paul 1 crown, if on the second trial 2 crowns, if on the third 4 crowns, and so on, the sum which Peter is to pay Paul when head finally does turn up being double the sum he should have paid had head turned up on the immediately preceding trial. Required Paul's expectation.

This, then, is the Petersburg problem. By Paul's expectation is meant the sum which Paul ought to pay Peter at the outset in order that the play may be fair to both Peter and Paul. The terms of the play are considered fair if, provided the play is repeated indefinitely, neither Peter nor Paul has an advantage.

If an event whose probability is p carries with it a gain P it is easily seen that the expectation of this contingent gain is pP. It follows that the expectation of a series of contingent gains is equal to the sum of the expectations of the separate contingent gains.

In the Petersburg problem the probability that head will turn up for the first time on the nth trial is $1/2^n$, if head does then turn up Paul is to receive 2^{n-1} crowns, hence so far as the nth trial is concerned Paul's expectation is $2^{n-1}/2^n = 1/2$. Now n may have any integral value, hence Paul's total expectation is $\sum_{n=1}^{\infty}(2^{n-1}/2^n) = 1/2 + 1/2 + \cdots = \infty$.

But this conclusion contradicts common sense which assures us that Paul would not pay any considerable sum for the advantage which Peter offers him.

It is of interest to observe that if the foregoing reasoning is valid Paul's expectation will remain infinite even if the conditions of the play are modified in a number of ways, each modification resulting in a distinct disadvantage to Paul.

[1] I. Todhunter, *l.c.*, p. 134.

[2] "Specimen theoriae novae de mensura sortis," *Comment. Acad. Petrop.*, vol. 5.

1. In the first place the expectation remains infinite whether the initial prospective gain is one crown or any smaller sum, say 10^{-10} crowns.

2. In the second place the expectation remains infinite even though it were stipulated that Paul should receive nothing unless head turned up for the first time at any arbitrarily designated trial, say the one-millionth trial, and if head does not show then he were to receive nothing unless head did not turn up until some other arbitrarily designated trial, say the two-millionth trial, and so on provided only that if head does turn up for the first time on one of the designated trials Paul is to receive the stake attached to that particular trial by the conditions of the problem.

3. The expectation will remain infinite if the stake attached to the nth trial is $2^{n-1}/(kn)$ instead of 2^{n-1} times the initial stake, where k is any constant, however large.

4. Paul's expectation will remain infinite even if the conditions of the play combined all the disadvantages enumerated under 1, 2, and 3.

The proof for this assertion is simple. Paul's expectations of the favorable trials are

$$\text{for the one-millionth trial } \frac{1}{1000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{1000000-1}}{2^{1000000}},$$

$$\text{for the two-millionth trial } \frac{1}{2000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{2000000-1}}{2^{2000000}},$$

$$\text{for the three-millionth trial } \frac{1}{3000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{3000000-1}}{2^{3000000}},$$

$$\cdots\cdots\cdots\cdots,$$

and the sum of these expectations is

$$\frac{1}{1000000}\cdot\frac{10^{-10}}{k}\cdot\frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]=\infty.$$

In this generalized form of the Petersburg problem Paul would receive nothing unless tail turns up 999,999 times in succession, and his gain, if head turns up on the next trial, is

$$\frac{10^{-16}\cdot 2^{999,999}}{k}\text{ crowns},$$

a number small at will, say less than 1 crown, since k can be chosen as large as we wish. Yet the theory asserts that Paul's expectation is infinite, that is to say, no matter how large a sum Paul should offer Peter for a single game, Paul would come out winner if the game were repeated on the same terms a sufficient number of times.

Absurd as this conclusion seems, it is an incontrovertible consequence of the fundamental theorem that if A's probabilities of receiving the separate sums $P_1, P_2, P_2, \cdots$, are $p_1, p_2, p_3, \cdots$, respectively, his total expectation is the sum of the separate expectations

$$p_1P_1 + p_2P_2 + p_3P_3 + \cdots.$$

(*To be concluded in the next issue.*)

THE USE OF AN EXISTENCE THEOREM IN DEVELOPING THE PROPERTIES OF THE SINE AND COSINE.

By H. T. DAVIS, University of Wisconsin.

Introduction. In an interesting section of Professor Osgood's *Funktionentheorie* (Second edition, volume 1, Leipzig and Berlin, 1912, pp. 571–582) many of the properties of the sine and cosine are developed through their definition by means of a differential equation. One method there employed is that of obtaining explicit expansions for the functions by means of substituting series in the equation.

The purpose of the following paper is to show how this same development can be accomplished without the use of series by means of the existence theorem for linear differential equations. The power and elegance of this kind of analysis makes it seem worth while to record a simple example of its use.

The theorem in question is the following:[1]

"A linear homogeneous differential equation, in which the coefficient of the highest derivative is a constant, possesses one and only one integral which, for a preassigned value of the independent variable in whose neighborhood the coefficients of the differential equation remain analytic, assumes a preassigned value and whose first $n-1$ derivatives assume preassigned values at the same point. The circle of convergence of the ordinary power series development, representing the integral in the neighborhood of such a value, is never smaller than the largest circle within which the power series developments of the coefficients are all convergent."

The power of the method which is here developed attaches to the fact that the theorem just quoted assures us of the uniqueness as well as the existence of the two functions U and V which we shall choose to satisfy the following conditions:

U and V are both solutions of the equation

$$\frac{d^2y}{dx^2} + n^2y = 0, \tag{1}$$

$$\begin{aligned} U(O) &= 1, & V(O) &= 0, \\ U'(O) &= 0, & V'(O) &= n. \end{aligned} \tag{2}$$

We shall now develop some of the properties of these functions, ultimately to be identified with the sine and cosine, on the basis of this definition.

THEOREM I. $U^2 + V^2 = 1$.

By hypothesis $U'' + n^2U = 0$, $V'' + n^2V = 0$. Multiply the first by V and the second by U and subtract. Then

$$VU'' - UV'' = (VU' - UV')' = 0.$$

Consequently $VU' - UV' = c$, and by making use of conditions (2) we have $UV' - VU' = n$.

[1] L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 1, Leipzig, 1895, p. 25.

We next show that $V' = nU$. Since nU is a solution of the equation, we shall require V' also to be a solution, a fact verified by differentiating the equation and comparing the result with that of substituting V' for y. That V' and nU are actually the same solution follows at once from the uniqueness part of the existence theorem, because

$$\begin{aligned} V'(0) &= n, & nU(0) &= n, \\ V''(0) &= 0, & nU'(0) &= 0. \end{aligned}$$

It follows similarly that $U' = -nV$ and these results substituted above establish the theorem.

The Oscillation Properties. We shall next prove the following theorem:

THEOREM II. *When n is a real number the functions U and V vanish an infinite number of times along the real axis and their zeros alternate with one another.*

Consider the interval $0 \leqq x < \infty$. We have, if $Z = U'/U$,

$$Z' = \frac{UU'' - U'^2}{U^2} = \frac{-n^2U^2 - U'^2}{U^2} = -n^2 - Z^2.$$

Hence, by integration,

$$U' = -U\int_0^x (n^2 + Z^2)dx,$$

and from the differential equation $U'' = -n^2U$.

If we now suppose that U is positive for all values of x, we are led to a contradiction because U' and U'' would then both be negative which is evidently a sufficient condition that U should vanish at some finite point of the axis of x. From this contradiction we see that U must vanish at least once in the interval.

Moreover, when U vanishes it follows from Theorem I that $V = 1$, and the argument may be repeated for V. Since this may be continued indefinitely, it is seen that U and V must vanish an infinite number of times along the positive, and by similar reasoning, along the negative axis of x.

The second part of the theorem is apparent from the identity:

$$\left(\frac{U}{V}\right)' = \frac{UV' - VU'}{V^2} = \frac{n}{V^2}.$$

Suppose that a and b are two successive zeros of U and that V vanishes at no point in the interval. Then U/V is a continuous function which vanishes at two points and whose derivative must, therefore, by Rolle's theorem, vanish at some point in the interval. But this is clearly a contradiction so that V must vanish in the interval. Since the same argument holds for V, the theorem is seen to be established.

The Addition Formulas. As an example of the application of this method to proving the addition formulas we give the following:

THEOREM III. $U(x + y) = U(x)U(y) - V(x)V(y)$.

[1] If we set $z = x + y$, then by hypothesis the left-hand member satisfies the equation

$$\frac{d^2U}{dz^2} + U = 0,$$

and the additional conditions $U(0) = 1$, $U'(0) = 0$.

Consequently letting $U(x)U(y) - V(x)V(y) = W(x, y)$, it is sufficient to show that W is a function of z, that it satisfies the equation in z, and that $W(0, 0) = 1$, $(dW/dz)_{00} = 0$.

A necessary and sufficient condition that W shall be a function of z is that the Jacobian[2]

$$J = \frac{D(z, W)}{D(x, y)} \equiv 0.$$

But $J = U(x)U'(y) - V(x)V'(y) - U'(x)U(y) + V'(x)V(y) \equiv 0$, since $V' = U$ and $U' = -V$.

Also, by means of the theory of transformations,[3] it follows that

$$\frac{d^2W}{dz^2} = \frac{\partial^2 W}{\partial x^2} = U''(x)U(y) - V''(x)V(y),$$

$$= -U(x)U(y) + V(x)V(y).$$

Finally, since

$$\frac{dW}{dz} = \frac{\partial W}{\partial x} = U'(x)U(y) - V'(x)V(y),$$

we have $W(0, 0) = 1$, and $(dW/dz)_{00} = 0$.

By virtue of the uniqueness part of the existence theorem, these three conclusions are sufficient to establish the identity.

De Moivre's Theorem. Since the existence theorem applies to complex as well as real functions, it is possible to give a simple proof of De Moivre's theorem.

THEOREM IV. $[U(x) + iV(x)]^n = U(nx) + iV(nx)$.

It is at once clear that the right-hand member of the equation is a solution of the differential equation (1) since it is a linear combination of two particular solutions, and that for $x = 0$ it takes the value 1 and its derivative with respect to x, the value ni.

Consequently it will be sufficient to show that $[U(x) + iV(x)]^n$ is a solution of the differential equation and that this solution and its first derivative at the point $x = 0$ assume the values 1 and ni, respectively.

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[1] In preceding paragraphs U and V stand for $U(nx)$ and $V(nx)$, U' for $dU(nx)/dx$, etc. In the proofs of Theorems III and IV U and V when used alone stand for $U(x)$ and $V(x)$. n may be considered as taken equal to 1, but it should be noted that the n which is introduced in Theorem IV is not the n which is taken equal to 1. Thus we have there $V' = U$ and $U' = -V$, and at the end we have $V'(0) = 1$, although we also have an n which is not supposed to be equal to 1—EDITORS.

[2] Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, p. 52.

[3] Goursat-Hedrick, *loc. cit.*, p. 65.

If $Y = [U(x) + iV(x)]^n$, then it follows readily that

$$
\begin{aligned}
Y'' + n^2 Y &= (U + iV)^{n-2}[n(n-1)(U' + iV')^2 + n(U + iV)(U'' + iV'') \\
&\qquad + n^2(U + iV)^2], \\
&= (U + iV)^{n-2}\{n^2[V^2 - U^2 + U^2 - V^2 + 2i(UV - VU)] \\
&\qquad + n[V^2 - U^2 - V^2 + U^2 + 2i(UV - VU)]\} \\
&= 0.
\end{aligned}
$$

Also

$$Y(O) = U(O) + iV(O) = 1,$$

and

$$Y'(O) = n[U(O) + iV(O)]^{n-1}[U'(O) + iV'(O)] = ni,$$

which establishes the theorem.

The Geometrical Interpretation. In order to complete the discussion by showing that the functions U and V are the ordinary cosine and sine, we may interpret them geometrically. We have already shown that $U^2 + V^2 = 1$. Hence if U and V are taken as rectangular coördinates depending upon a parameter, their locus will be a circle of unit radius with center at the origin. It merely remains to give a geometrical interpretation of the parameter nx.

From the relations $V' = nU$ and $U' = -nV$ it follows that $U'^2 + V'^2 = n^2$. Hence $ds/dx = n$ and upon integration, $s + \text{const.} = nx$. This means that the parameter nx is a length of arc measured along the unit circle and, therefore, is also a measure of the angle at the center.

It is interesting to observe that we might have argued as follows: By hypothesis $U'' + n^2U = 0$, $V'' + n^2V = 0$. If we multiply the first equation by V' and the second by U' and subtract, recalling the fact that $UV' - VU' = n$, we have

$$V'U'' - U'V'' = n^2(VU' - UV') = -n^3,$$

or since $U'^2 + V'^2 = n^2$,

$$\frac{(U'^2 + V'^2)^{\frac{3}{2}}}{U'V'' - V'U''} = 1.$$

Hence the radius of curvature of the curve whose parametric representation is $U = U(nx)$, $V = V(nx)$, is 1, which means that the curve itself is a unit circle.

THE EFFECT OF CHANGE OF SCALE ON CURVATURE.

By JAMES K. WHITTEMORE, Yale University.

An important problem of applied mathematics is the determination of an empirical formula to represent the relation between two measured quantities. This problem is fully discussed in Joseph Lipka's *Graphical and Mechanical Computation*,[1] chapter VI. The simplest and most important of the empirical laws are the two-constant laws such as (1) $y = ax + b$, the straight line law;

[1] New York, 1918.

(2) $y = ax^b$, the power law; (3) $y = ae^{bx}$, the exponential law. In each of these equations x and y are corresponding values of the two measured quantities, and a and b are constants. If the form of the law is known, the problem is the determination of the constants. If the form of the law is not known, the first and more difficult problem is the determination of the law best adapted to represent the relation between the measured x, y. The first step in the determination of the form of the law is to plot on ruled paper the points whose coördinates are corresponding numbers x, y. If the plotted points appear to lie nearly in a straight line the law (1) may be tried. It sometimes happens that, even though the plotted points lie approximately in a straight line, the law (1) gives unsatisfactory results. It is then necessary to try a different law. In the choice of a two-constant law other than (1) a most important question is whether the curve joining the plotted points is concave up or down. If the points are approximately in a straight line it is difficult if not impossible to answer this question. "Change of scales from those first used is sometimes advisable in order to bring out the curvature, if any." [1] It is the purpose of this paper to show how different changes of scale affect the numerical value of the curvature, hereafter called simply the curvature, of the graph of a set of points of given coördinates. It is hoped that the results obtained, which are certainly not obvious, may have a practical value in the determination of empirical laws, as well as some general interest.

Consider first a simple example. If we plot the points x, y, which satisfy the equation, $x^2 + y^2 = 1$, with equal scales on the two axes the graph is the unit circle. Suppose the points with the same coördinates to be again plotted with the x scale doubled, the y scale remaining unchanged. Evidently we shall obtain the same curve as in the second plot if we plot with the original equal scales the points of coördinates, $x_1 = 2x$, $y_1 = y$, that is the locus of the equation, $x_1^2/4 + y_1^2 = 1$. The curvature at a point x_1, y_1 of this ellipse is

$$K_1 = \frac{16}{(x_1^2 + 16y_1^2)^{3/2}}\,.$$

For $x_1 = 2$, $y_1 = 0$, $K_1 = 2$; for $x_1 = 0$, $y_1 = 1$, $K_1 = 1/4$. Clearly doubling the x scale has had quite different effects on the curvature of the circle, unity for every point, at the two points, 1, 0 and 0, 1. Let us next suppose plotted the coördinates of the points of the same circle with the x scale increased in the ratio $m : 1$, m being a positive number greater than one, the y scale unchanged as before. The graph is the same as that obtained by plotting with the original equal scales the locus of the equation, $x_1^2/m^2 + y_1^2 = 1$. If y is not zero,

$$K_1^2 = \frac{m^2}{y^6(m^2 + y'^2)^3}\,,$$

where y and $y' = dy/dx$ refer to the circle, $x^2 + y^2 = 1$, and do not change

[1] *Sheffield Scientific School Sophomore Mathematics*, pamphlet by P. F. Smith, 1921, p. 15.

with m. To find for what points of the circle the curvature increases as m increases from unity we compute $d(K_1^2)/d(m^2)$ for $m = 1$, obtaining

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{y'^2 - 2}{1 + y'^2}, \qquad m = 1.$$

It follows that as the x scale is infinitesimally increased the curvature at a point of the circle increases or decreases as y'^2 is greater or less than 2, that is as the acute angle between the tangent to the circle and the x axis is greater or less than $54° 44'$. We proceed to show that the result is the same for any curve.

For any curve the square of the curvature is

$$K^2 = \frac{y''^2}{(1 + y'^2)^3},$$

where $y' = dy/dx$ and $y'' = d^2y/dx^2$. If the coördinates x, y of this curve are plotted with the x scale increased in the ratio $m : 1$ and the y scale unchanged we shall have the same locus as that obtained by plotting with the original equal scales the points of coördinates, $x_1 = mx$, $y_1 = y$. The equation of the new curve is obtained from that of the given curve by replacing x and y in the latter by x_1/m and y_1 respectively. The curvature at x_1, y_1, the point corresponding to x, y, is given by

$$K_1^2 = \frac{y_1''^2}{(1 + y_1'^2)^3} = \frac{m^2 y''^2}{(m^2 + y'^2)^3},$$

since $y_1' = dy_1/dx_1 = y'/m$ and $y_1'' = d^2y_1/dx_1^2 = y''/m^2$. Taking the logarithmic derivative of K_1^2 with respect to m^2, then writing $m = 1$, we find as in the case of the circle

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{y'^2 - 2}{1 + y'^2}, \qquad m = 1.$$

The result is the same for all curves: as the x scale is increased, the y scale remaining unchanged, the acute angle between the tangent to the curve and the x axis decreases, since $y_1'^2 = y'^2/m^2$, and the curvature increases or decreases as y'^2 is greater or less than 2. It is evident that if we consider a definite point of a given curve for which $y' > \sqrt{2}$ and gradually increase the x scale the curvature at the corresponding point will gradually increase until $y_1' = \sqrt{2}$; further increase of the x scale will decrease the curvature. For maximum curvature at a point corresponding to a given point we must choose $m^2 = y'^2/2$, and find that the maximum curvature is given by

$$K_1^2 = \frac{4}{27}\frac{(1 + y'^2)^3}{y'^4}K^2.$$

It is easy to show directly that this maximum K_1^2 is greater than K^2 when y' is greater than 2 and that the ratio of maximum increase K_1^2/K^2 increases with y'^2.

If the equation of a curve is not given but the coördinates of a number of points are known, as in the case where an empirical law connecting corresponding pairs of given numbers is to be found, the effect of change of scale on curvature is the same as if an equation were given. If a number of pairs of coördinates are given and plotted, if it is desired to increase the curvature of the graph at a point where the slope y' is greater than $\sqrt{2}$, and if for other reasons an increase of the x scale is not undesirable, then for maximum curvature the x scale should be increased approximately in the ratio $y' : \sqrt{2}$. If, for example, $y' = 3$, since $3/\sqrt{2} = 2.1$ the x scale should be doubled and $K_1 = 1.4\,K$ approximately.

A discussion similar to the preceding shows that if the y scale is increased in the ratio $n : 1$, the x scale remaining unchanged, the slope y' increases numerically and the curvature increases or decreases as y'^2 is less than or greater than $1/2$. Then if y'^2 is between $1/2$ and 2 an increase of either scale will decrease the curvature while a decrease of either scale will increase the curvature.

We consider finally the effect on curvature of simultaneous changes in both scales. Suppose a curve plotted originally from a given equation or from pairs of given coördinates x, y, with equal scales on the two axes; suppose then the same coördinates plotted with the x scale increased in the ratio $m : 1$ and the y scale increased in the ratio $n : 1$. The second plot may be obtained by plotting with the original equal scales the points whose coördinates are $x_1 = mx$, $y_1 = ny$. For the new curve

$$K_1^2 = \frac{y_1''^2}{(1 + y_1'^2)^3} = \frac{m^2n^2y''^2}{(m^2 + n^2y'^2)^3},$$

since $y_1' = ny'/m$ and $y_1'' = ny''/m^2$. We have

$$\frac{d(K_1^2)}{K_1^2} = \frac{(n^2y'^2 - 2m^2)d(m^2) + (m^2 - 2n^2y'^2)d(n^2)}{m^2n^2(m^2 + n^2y'^2)}.$$

In this equation we set $m = n = 1$ to obtain the change in curvature for infinitesimal changes in scale:

$$\frac{d(K_1^2)}{K_1^2} = \frac{(y'^2 - 2)d(m^2) + (1 - 2y'^2)d(n^2)}{1 + y'^2}, \qquad m = n = 1,$$

an equation which contains the results given before for changes in the x scale alone and in the y scale alone. It is clear that the exact effect on curvature of simultaneous change of both scales is not determined till the relative rate of change of scales is given. We consider two cases: (1) The two scales shall be changed in the same ratio, $m = n$. Since $dm = dn$ we have, for $m = 1$,

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{1}{K_1}\frac{dK_1}{dm} = -1.$$

As the scales increase the curvature decreases as is indeed evident; the rate of change of curvature with respect to scale is exactly equal to the curvature.

(2) The two scales shall be changed in inverse ratio, $m = 1/n$. Suppose for example the x scale to increase, $m > 1$, while the y scale decreases, $n < 1$. In this case the slope, $y_1' = y'/m^2$, decreases numerically; for infinitesimal changes of scale, $m = n = 1$, we have $d(m^2) = -d(n^2)$ and

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{3(y'^2 - 1)}{y'^2 + 1},$$

so that an increasing x scale and a y scale decreasing in inverse ratio will increase curvature as long as $y'^2 > 1$. Suppose that the curvature at a definite point of a given curve for which $y' > 1$ is to be increased by simultaneous inverse changes of scale. For the maximum increase $m^2 = y'$ and

$$K_1^2 = \frac{1}{8}\frac{(1 + y'^2)^3}{y'^3}K^2.$$

We may see directly that this maximum K_1^2 is greater than K^2 when y' is greater than 1, that the ratio of maximum increase K_1^2/K^2 increases with y', and that for values of y' greater than $\sqrt{2}$, as might have been anticipated, this maximum curvature obtained by simultaneous inverse change of scale is greater than the maximum previously found by change of the x scale alone. If again we suppose, for example, $y' = 3$ we obtain the maximum increase of curvature by choosing $m = 1/n = \sqrt{3} = 1.7$ and $K_1 = 2.2\ K$ approximately.

In conclusion we remark that in making a plot of given coördinates for the determination of an empirical law other things besides curvature must be considered. We quote from Lipka:[1] "The scales with which these values are plotted are generally chosen so that the length of the axis represents the total range of the corresponding variable, and so that the line or curve is about equally inclined to the two axes. There is no advantage in choosing the scale units on the two axes equal. Care should be taken not to choose the units either too small or too large; for in the former case the precision of the data will not be utilized, and in the latter case the deviations from a representative line or curve are likely to be magnified. The drawing of a good plot is evidently a matter of judgment." From the results obtained in this paper we know that "if the line or curve is about equally inclined to both axes," and if the plot is not actually a straight line, the curvature will be increased by a decrease in either or both scales, but what change of scale is most advantageous will in the end depend on the precision of the data, the form of the plot, and the size of the paper as well as on a desired magnification of the curvature.

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[1] *Loc. cit.*, pp. 122–123.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

Among the unphilosophical there is generally no very clear notion of what is involved in a mathematical theory; and many of the attacks on this science are attacks, not on mathematics itself, but on the hypotheses on which it happens to operate. Hence arises the distinction, unmeaning in any exact sense, between what is mathematically true and what is true in practice. It is therefore not unlikely that the rather fanciful doctrine of Golden Section, as the most perfect proportion, is by many persons regarded as something proved by mathematics (namely, by the study of the equation $x^2 + x - 1 = 0$), and their unwillingness to apply the doctrine in a wholesale fashion would no doubt be excused on the ground that it is one of those things that are mathematically true only. Readers of the MONTHLY, though innocent of these errors, will be interested in Professor Bennett's article, which goes further and exposes on general ground the impossible character of the theory in the sweeping form in which it is often stated. For notes on the history and literature of Golden Section, compare this MONTHLY, *1918*, 232–235.

The two formulas given by Dr. Weisner, which have a very simple application to a number of elementary summation problems, may well be considered in relation to the operators D and Δ, where $Df(n) = (d/dn)f(n)$ and $\Delta f(n) = f(n+1) - f(n)$. We know that the inverse operators D^{-1} and Δ^{-1} give rise to arbitrary complementary terms, the constants of indefinite integration and indefinite summation. It is also well-known that the operators D, Δ, D^{-1}, Δ^{-1} are commutative, provided the complementary terms are not forgotten. When we pass from the indefinite integral (supposed to exist) to the definite integral with upper limit n, and in like manner to the definite sum, the complementary terms will become fixed, and the results of $D^{-1}\Delta^{-1}$ and $\Delta^{-1}D^{-1}$ will now be found to differ by a linear function of n. The determination of this linear function depends on the lower limits of summation and integration. Dr. Weisner's integration formula (2) may be thought of as giving the results for the lower limits 1 and c; for the left member is of the form $\int_c^n \varphi(n)dn + An + B$. The differentiation formula (1) has the same relation to the change of order of D and Δ^{-1}.

I. THE "MOST PLEASING RECTANGLE."

By ALBERT A. BENNETT, University of Texas.

As has been frequently emphasized, the Golden Section is even richer in interesting associations than was imagined by those who first bestowed upon it this grandiloquent title. We have indeed the regular pentagon, the regular decagon, the regular five-pointed star or pentagram, and related figures depending for their usual construction directly or indirectly upon the Golden Section. Yet

the relation of the segments thus obtained figures numerically in other familiar connections. The proportion, $1 : x = x : (1 - x)$ gives $x^2 + x - 1 = 0$, whence $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, which enters into the study of the so-called Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, where each term after the second is the sum of the two preceding terms. It might seem at first strange that the sum of n terms of a series which is so essentially integral should be represented by an expression involving $\sqrt{5}$, but the following formula is readily verified inductively and constitutes merely the solution of a simple problem in differences:

$$S_n = [(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^{n-1} - (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1}]/\sqrt{5}.$$

The form is more suggestive of the Golden Section, if we alternate signs and write $0, -1, 1, -2, 3, -5, 8, -13, 21, \cdots$, where now each term after the second is the difference obtained by subtracting the preceding element from the one preceding that. In this form the solution for the sum is

$$[(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1} - (-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1}/\sqrt{5}.$$

From the properties of the Golden Section a direct proof that $\sqrt{5}$ is irrational follows easily, since the determination of a common measure for 1 and $-\frac{1}{2} + \frac{1}{2}\sqrt{5}$ can be shown to be impossible owing to the recurrent appearance of this ratio at each step in the classical process.

Despite the wealth of geometrical and arithmetical material that is available in connection with the Golden Section, one sometimes sees in mathematical texts a further statement to the effect that the Golden Section provides segments which when used for the sides of a rectangle result in a "rectangle of the most pleasing shape." One can object to this statement in mathematical texts on three grounds: (1) it is not coördinate with the other propositions, not being mathematical in its content; (2) it could not be proved even it if were true; (3) if not meaningless it is false.

The assertion as to the "most pleasing rectangle" is at its face value incapable of mathematical demonstration, and if intended as a scientific fact can rely only upon psychological grounds. This is not a fatal objection to making a reference to the claims of some psychologists but is a reason for criticizing the abrupt statement sometimes made. It is surely obvious that no mere experimentation can serve to select an irrational number exactly. No matter how many subjects be tested or how many reactions recorded, the determination of an irrational number with absolute precision is out of the question. Two methods of testing occur to mind: (1) A *finite* collection of shapes might be compared, (2) the persons whose esthetic judgments are being invoked might be requested to draw a figure or adjust a mechanism and so select the most pleasing rectangle out of a possible *infinite* number. In either case, at best all might agree on a figure differing but imperceptibly from that given by the Golden Section; and at worst, a distribution of choices might occur with a maximum or mode, near this point. But the statement will be meaningless in the abstract unless it is applicable to all of the numerous concrete instances that come to mind. To be specific, one

may ask what is the most pleasing shape of a sheet of note paper, of a newspaper, of a photograph, of a picture frame, of a postage stamp, etc. The whole question reduces to the character of our esthetic preferences in a matter of this sort. Is there a certain abstract pleasing rectangle which we prefer to see embodied in these various examples, or do practical commercial considerations familiarize us with certain shapes which thereby acquire a certain sanction? The question seems to be one which can be answered readily. One has merely to show that most pleasing proportions in one application are not those in another, and that in some of the most familiar cases at least commercial advantages are sufficient to explain the usual and for these purposes most pleasing proportions.

The proportions of the usual postage stamp when measured from perforation center to center, are $\frac{7}{8}$ in. $\times$ 1 in. These dimensions are selected for the convenience in handling sheets of stamps consistent with ease of handling individually. The factors of the problem are the original size of the sheets to be used, the even decimal number of stamps to a row and to a column desired to facilitate sales, the average size of a clerk's thumb. Long familiarity has accustomed us to finding esthetic satisfaction in this size and shape. Any other size attracts attention by its unfamiliarity, and unless it is an even multiple of the present size looks awkward to the average American. But these are not the proportions of the Golden Section. The American dollar bill and paper money in general show a certain size and shape with which we are all familiar. The dimensions are approximately $3\frac{1}{8}$ in. $\times$ $7\frac{3}{8}$ in. We are accustomed to the handling and folding of bills of this size. When in foreign travel we come upon assorted sizes and shapes, there is nearly always a feeling that the American bill is of more pleasing proportions. In this case again the dimensions are not those of the Golden Section but lie to the other extreme. If a picture that is to be framed should itself show certain proportions, and if the frame is to be of the usual sort to the extent of constituting a uniform border to the picture, it is clear that the exterior dimensions of the frame cannot be in the same ratio as those of the picture. The most pleasing dimensions for a picture frame if such there could be ought to be those presumably fitted to the picture of most pleasing dimensions and therefore the most pleasing rectangle for the exterior of a frame would more nearly approximate a square.

Thus far most of these remarks have been directed to the proportions of a sheet of paper. The most familiar example of such a rectangle is undoubtedly that of a page of a book or magazine. And it is here that simple commercial considerations are most obvious. Changes in shape and in size are both disconcerting, but changes in size are commercially necessary while marked variations in ratio can be and therefore usually are avoided. Small pages are obtained by folding larger sheets, whence the terms, folio, quarto, octavo, duodecimo, etc. The question amounts to one as to how a sheet of paper shall be originally planned so that after these successive foldings the pages shall show sensibly similar shapes, the solution being complicated by the demand for even integral multiples of an inch or at least of a quarter inch. Instead of making one dimen-

sion exactly $\sqrt{2}$ times as long as the other, we can make use of the fact that for most purposes not requiring precision, 2/3 and 3/4 are not sensibly distinct, and are such that their product is 1/2. Thus a sheet, 18×24 when folded gives two sheets, 12×18 of sensibly similar shape, and after another folding, we obtain four sheets of 9×12 which are exactly similar to the original sheet.

By referring to architectural details the folly of speaking of a most pleasing rectangle apart from its use becomes even more evident. The most pleasing rectangle for the cross section of the capital to a column is obviously a square, while the most pleasing shape of a rectangular pilaster has proportions between 1 : 8 and 1 : 15 probably. The most pleasing shape of the tread on a stairway differs again from the most pleasing proportions for a brick or for a fence board, or for a door. The abbreviated regimental colors look awkward to most civilians accustomed to the proportions of the national flag.

There is still another element in the problem. As soon as commercial demands become secondary to the vagaries of fashion, the assumption that there is a most pleasing rectangle even for a given specific purpose becomes doubtful. Exclusiveness and novelty often outweigh the advantages of traditional familiarity. Fancy bricks come in a wide variety of sizes and shapes, ladies' stationery shows continual variations. If there be indeed a most pleasing rectangle why is it that visiting cards whose proportions need be dictated by no considerations other than esthetic show such periodic fluctuations?

Mathematics presents interest enough on its own account, it is called upon in ever increasing measure in the elucidation of natural laws and the furthering of arts. Why must mathematicians persist in seeking extraneous mystical significance in numbers?

II. Note on the Summation of Series.

By Louis Weisner, Columbia University.

The object of this note is to present a method for determining the sum of the first n terms of a series whose nth term is $f'(n)$ or $\int_c^n f(n)dn$, provided that the sum of the first n terms of the series whose nth term is $f(n)$ is known.

If $\varphi(n) = \sum_1^n f(n)$, then $\varphi(n) - \varphi(n-1) = f(n)$.

For most series which occur in practice this difference equation is satisfied for all values of n, with the possible exception of isolated values. If this is the case, and $\varphi(n)$ and $f(n)$ are differentiable functions, we have

$$\varphi'(n) - \varphi'(n-1) = f'(n).$$

If $\varphi'(x) \neq \infty$ when $x = 0, 1, 2, \cdots n$, we may let $n = 1, 2, \cdots n$ in this equation. Adding the n equations thus obtained, we find that

$$\varphi'(n) - \varphi'(0) = \sum_1^n f'(n). \tag{1}$$

For example, from $\frac{1}{4}n^2(n+1)^2 = \sum_1^n n^3$ we get $\frac{n}{2}(n+1)(2n+1) = \sum_1^n 3n^2$.

Again, from $\log \Gamma(n+1) = \log 1 + \log 2 + \cdots + \log n$, we obtain

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(1)}{\Gamma(1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\,.$$

Integrating our difference equation between limits c and n, where c is an arbitrary constant for which the integrals exist, we find that

$$\int_c^n \varphi(n)dn - \int_c^n \varphi(n-1)dn = \int_c^n f(n)dn,$$

whence

$$\int_c^n \varphi(n)dn - \int_{c-1}^{n-1} \varphi(n)dn = \int_c^n f(n)dn,$$

or

$$\int_{n-1}^n \varphi(n)dn - \int_{c-1}^c \varphi(n)dn = \int_c^n f(n)dn.$$

Letting $n = 1, 2, \cdots n$, assuming the integrals to have finite values, and adding the n equations thus obtained, we find that

$$\int_0^n \varphi(n)dn - n\int_{c-1}^c \varphi(n)dn = \sum_1^n \int_c^n f(n)dn. \tag{2}$$

For example from $\frac{1}{4}n^2(n+1)^2 = \sum_1^n n^3$ we get $\dfrac{6n^5 + 15n^4 + 10n^3 - n}{120} = \sum_1^n \dfrac{n^4}{4}\,.$

The reader will have little difficulty in finding other applications.

RECENT PUBLICATIONS.

REVIEWS.

Analytische Behandeling van de Rationale Kromme van den Vierden Graad in een Vierdimensionale Ruimte. By J. Fr. de Vries. The Hague, Martinus Nijhoff, 1922. 8vo. xi + 158 pages. Price 4 guilders.

This book gives an analytical discussion of the rational quartic curve in 4-space, looked upon by the author as the analogon of the twisted cubic in 3-space. The classical analytical treatment of the latter curve[1] is taken as a guide in the study of the C_4^4. The following analysis of the contents of the work gives an indication of its scope:

Chapter I: Some parametric representations of the rational curve of fourth order in 4-space. Conical spaces containing the curve; Chapter II: Simply, doubly and triply tangent spaces, osculating spaces, tangent and osculating planes, tangent lines. Polar space of a point with respect to the curve; Chapter III: The space of the osculating planes, the locus of the tangent lines, the space of the bisecants, polar space. The Plücker numbers; Chapter IV: Quadratic

[1] See, for example, O. Staude, *Analytische Geometrie der kubischen Kegelschnitte*, Leipzig, 1913; or P. W. Wood, *The Twisted Cubic with Some Account of the Metrical Properties of the Cubical Hyperbola*, Cambridge, England, 1913.

spaces containing the curve; Chapter V: Quadratic and cubic involutions on the curve; Chapter VI: Quadratic conical spaces through C_4^4; Chapter VII: Generation of the curve by means of projective systems of spaces and rays; Chapter VIII: The curves C_4^4 through six given points; Chapter IX: Properties of the rational C_4^3 obtained from those of the rational C_4^4 by central projection.

ARNOLD DRESDEN.

The Calculus. By E. W. DAVIS and W. C. BRENKE. Revised edition, New York, The Macmillan Company, 1922. 345 + 65 pages. Price $2.75.

In writing an elementary text on the calculus an author may try to build up a logical treatment of some of the elementary facts as free as possible from intuition and sacrifice manipulative skill to logical rigor, or he may trust to intuition in the more delicate matters and try to develop an intelligent manipulative grasp of the subject.

The first edition of this book followed strongly this second alternative and while it had certain undoubted merits was not altogether a satisfactory text.[1]

The revision by Professors Hedrick and Brenke which has just appeared seems, to the writer, to have cured all the faults of the earlier text and to have given us about as good a book for the average calculus class as one could wish. The authors show an accurate and detailed knowledge of the beginner's difficulties and clearly indicate the right methods of meeting them. The book is not too long and the topics omitted show good judgment—asymptotes and multiple points of curves are not treated at all and the theory of infinite series is given little space. For the first hundred pages the only functions treated are algebraic (mostly polynomials). Reduction formulas do not occupy their usual imposing position and mechanical and formal problem solving receives no encouragement. The treatment of the notion of area as intuitional (which it is not) together with the ambiguity of the integral sign (a now unavoidably defective symbolism) is responsible for the obliteration of the distinction between $D_x^{-1}f(x)$ and $\int_a^b f(x)dx$, which is to be found in practically all the elementary texts. It is responsible for the mental fog which the treatment of the inverse operation as a direct limit usually produces.

Since it is unfair to an author to say nothing but nice things about his book, we must follow the usual custom of casting a few stones of an innocuous sort as certain infelicities of expression. For instance, one would like to know why it it *unfortunate*[2] that positive series whose terms approach zero do *not* converge. Also how can *all*[3] absolutely convergent series be tested by the ratio test or, indeed, by any test? The beginner might imagine they could be successfully tested in this way. It is *not* true, as stated in the foot-note on page 198, that the two definitions of integral are identical. The word *practical* will hardly bear the strain put upon it on page 86 when it is asserted that $x^2 + 5$ and $x^2 + C$ are practically the same thing.

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[1] See review in this MONTHLY, *1912*, 202.
[2] p. 270. The beginner might not see the humor unless it is pointed out to him.
[3] p. 263.

The book ends with an excellent formulary of algebra, trigonometry, and analytic geometry with tables and graphs. This last feature though necessary is an eloquent testimonial to the state of preparation of many of our neophytes in the calculus. Let us hope that the day is not far distant when no one will be allowed to enter college without suitable visual training (at least six reels of Charlie Chaplin) and a psychological test certifying that he can name all the brands of automobiles made in the city of Detroit.

M. B. PORTER.

Weather Prediction by Numerical Process. By L. F. RICHARDSON. Cambridge, University Press, 1922. 4to. 12 + 236 pages. Price 30 shillings.

This book is intended primarily for scientific meteorologists, and yet inasmuch as it lays down a suggested program involving preferably international coördination, and certainly much governmental expenditure in Great Britain, it may make a secondary appeal to such legislators, if any there be, as can understand the scientific advantages proposed and weigh these against the financial drawbacks.

The book appears to have three principal purposes. One is to establish the claims of a proposed arrangement of observation stations and a suitable division of labor among them, aiming toward a consistent, inclusive and reasonably refined compilation of scientific data on the weather, for Great Britain at least and preferably for the whole earth. The second purpose is to exhibit and justify a set of partial differential equations in vectorial form which are to account for all the principal meteorological variables. This aim is carried out in a scholarly manner involving the careful weighing of much data and with a breadth of vision of which only an experienced scientist could be capable. The third purpose, one might say, is to explain a set of computing forms by which the partial differential equations may be integrated, on the basis of known meteorological constants and of the periodically observed data secured at the regularly distributed observation stations assumed to exist.

It is not entirely fair to the author of this valuable work to examine only its mathematical features and to ignore the real substance of the discussion. And yet the mathematical form contributes an inherent and important part to the value of the entire undertaking. I will quote some extracts on this point from the preface.

"The extensive researches of V. Bjerknes and his school are pervaded by the idea of using the differential equations for all that they are worth. I read his volumes on *Statics and Kinematics* soon after beginning the present study, and they have exercised a considerable influence throughout it; especially, for example, in the adoption of conventional strata, ... But whereas Prof. Bjerknes mostly employs graphs, I have thought it better to proceed by way of numerical tables. The reason for this is that a previous comparison of the two methods, in dealing with differential equations, had convinced me that the arithmetical procedure is the more exact and the more powerful in coping with otherwise awkward equations. Graphical methods are sometimes elegant when the problem involves irregularly curved boundaries. But the atmospheric boundary, at the earth, nearly coincides with one of the coordinate surfaces, so that the graphs would have no advantages over arithmetic in that respect. . . . The question then arises: how

chapter, involving an abandonment of the serial numeration of sections or paragraphs, and the adoption of a method of expression which indicates, at a glance, not only the subdivision number, but also the division and chapter. For example, a subdivision will be marked 3/10/2, to designate the second subdivision, of the tenth division, of chapter three. This is good but not nearly as satisfying as the recently introduced decimal fractional notation. The other minor item worth mentioning is an index of terms with equivalents in Ido, one of the competing "international languages."

The author assumes that the theoretical study and practical forecasting of air conditions is worth the cost and labor involved in the scientific location, equipment and conduct of observation stations, working not independently, but as efficient unit sources of periodic information, to be interpreted in accordance with established physical theory at a central office. This book appears to be the first important expression of this rather generally accepted conviction. Much intelligent scientific effort has been spent in the study of the dynamics of the air. Every year the subject of the prediction of atmospheric conditions becomes increasingly important, while the results capable of simple mathematical treatment continue to be disappointingly meager. Books of this sort, ambitious, original and scholarly, serve to inform and train the novice, and to stimulate fellow scientists.

ALBERT A. BENNETT.

The Fourth Dimension and the Bible. By W. A. GRANVILLE. Boston, Richard G. Badger, The Gorham Press, 1922. 8vo. 9 + 119 pages. Price $1.50.

This work is meant seriously and can be taken in no other sense, however whimsically one might be inclined to approach it at first. One's sympathy with the attempt will depend largely upon one's scientific and religious attitude, and the author admits that he is "on a 'no man's land' exposed to fire from the mathematical trenches on the one side and the theological trenches on the other." For this reason a statement of the content and attitude of the book with quotations in the author's words will doubtless constitute the most satisfactory review.

The preface seeks to establish the reasonableness of the whole inquiry while emphasizing its originality as "practically a virgin field for theological research." The interested reader will do well however to glance at C. F. Bragdon's *Fourth Dimensional Vistas,* New York, A. A. Knopf, 1916. The author first explains why "philosophy and the physical sciences when called to the defence of Christianity have so often proven to be broken reeds," concluding that "it is evident that we cannot reasonably expect that much constructive light will be thrown on Christianity now or in the near future by either philosophy or the physical sciences." On the other hand mathematics "is the only exact science that God has revealed to man and the truths which it contains are the only truths that can be absolutely established through pure reason. Because pure mathematics reveals absolute truth it is part of God himself, for God is the essence of all truth." "That mathematics will ultimately prove to be a valuable aid in solving many of

the perplexing problems connected with our Christian religion is the firm belief of many of our sound thinkers" (with citation from C. J. Keyser).

Approximately half of the book is an exposition of the notion of a fourth dimension and of higher spaces. This portion is in no respect original or scientifically ambitious, but it is thoroughly readable and actually elementary. It is as a whole accurate, calm and logical, with an appeal indeed to one's imagination but none to one's preferences or prejudices. In the remainder of the book, the mathematician is silent, and the discussion can be most generously characterized perhaps, with respect to its philosophical features, as mystical. There comes to mind, as possibly an unfair parallel, the story of Cardan who as a devout and conscientious astrologer is said to have cast the horoscope of Christ, and incidentally to have suffered the consequences of having shocked the medieval church. There seems to be a curious progress from spontaneous analogy to surmise and from surmise to certainty. But the following excerpts may be allowed to speak for themselves:

"There are no restrictions of time, or restrictions of the space of our experience, in dreamland." . . . "Perhaps dreamland is located in four-dimensional space." . . . "If our mental vision is four-dimensional it points to the possibility that our mental or spiritual self is four- (or higher-) dimensional." . . . "In this higher space our grammatical tenses would then have no meaning." . . . "The soul of man, being higher-dimensional, can leave the human body, which is three-dimensional, without doing any violence to the body or leaving any trace on it of the point of emergence." . . . "No image of the soul, which is higher-dimensional, can be secured by material (three-dimensional) means; nor can the human body imprison the soul." . . . "The raising of the dead involves the re-entrance of the souls into their former corporeal habitations."

As to the laws of nature, the author indulges in such remarks as the following:

"This suggests that their (referring to certain snails) external form is the expression of an internal difference due to a right or left twist of their atoms by a four-dimensional force.". . . "A congenital blind and deaf individual . . . could not perceive even any *effect* of those subtle marvelous vibrations that produce light and color and music." "Astronomers have observed the sudden appearance in the heavens of stars and comets and also the apparent vanishing of such heavenly bodies, all in a manner suggesting that they were either entering or leaving our universe. For these and other reasons there has been for some time a suspicion in the minds of many scientists that the total mass of our material universe is not constant. . . . It follows that the principle of the conservation of energy also fails. Numerous instances are recorded in the Bible where new matter or new energy was apparently added to our material universe by supernatural means." "This difficulty (of accepting the miracles) vanishes if instead we look upon the miracles of the Bible as the perfectly logical results of the working out of laws connected with higher spaces."

The author has a chapter on "Spaces as Heavens and Hells." One may not agree with the assertions and surmises made, but in any case, explicit statements are not wanting. Lazarus seems not to have been so fortunate. "Where was the immortal part of Lazarus during the four days in which his body was in the grave, what did he do, what were his experiences? A message from Mars would be of trivial interest to dwellers on earth compared with the importance of the message that Lazarus should have brought back from the regions beyond the grave. It would have answered the great burning question of past, present, and future ages, the question beside which all other questions vanish into insignificance. . . . The reason why Lazarus did not answer this question, did not

relate his experiences between his death and his return from the grave, was because he could not. . . . The impressions he received were therefore received in a space totally foreign to us, impressions which could not be described to us because we have no words, no language which will convey such a description to human (three-dimensional) beings. . . . On being again clothed with his material body his spirit at once became subject to the restrictions and limitations imposed on beings living in our space of three dimensions." But the case is even worse than this. After showing how a three-dimensional material universe might be conceivably transferred bodily into our three-dimensional space and therefore be represented by a material creation, "out of nothing" so far as our space was concerned, and by a similar translation be "annihilated" with respect to our sense perceptions, the writer has the boldness to remark with regard to the limitations of the Holy Scriptures, "The Bible does not and could not tell us how something can become nothing any more than it can tell us how nothing can become something because it was written for human (three-dimensional) beings wholly incapable of comprehending such (higher-dimensional) mysteries."

The author turns readily from mathematics to miracles and again from religious verities to real variables, and does so in a way that is likely to carry conviction to many readers. An orthodox religious attitude, a light and pleasing literary style, and the crisp chapter divisions will retain many as readers who may originally intend merely to glance at this book.

ALBERT A. BENNETT.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 44, April, 1922: "A primary classification of projective transformations in function space" by L. L. Dines, 87–101; "A general theory of limits" by E. H. Moore and H. L. Smith, 102–121; "Substitution groups whose cycles of the same order contain a given number of letters" by G. A. Miller, 122–128; "Boundary values and expansion problems" by R. D. Carmichael, 129–152; "On a theorem in general analysis" by E. W. Chittenden, 153–162.

ANNALS OF MATHEMATICS, volume 23, December, 1921: "Transformations of trajectories on a surface" by J. Lipka, 101–111; "On the structure of finite continuous groups with one two-parameter invariant sub-group" by S. D. Zeldin, 112–117; "On the simplification of the structure of finite continuous groups with more than one two-parameter invariant sub-group" by S. D. Zeldin, 118–121; "The automorphic transformation of a bilinear form" by J. H. M. Wedderburn, 122–134; "A direct determination of the minimum area between a curve and its caustic" by O. Dunkel, 135–140; "The Poisson integral and an analytic function on its circle of convergence" by A. Arwin, 141–143; "Systems of circuits on two-dimensional manifolds" by H. R. Brahana, 144–168; "Two generalizations of the Stieltjes integral" by P. J. Daniell, 169–182.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, October, 1922: "Cremona transformations and applications to algebra, geometry, and modular functions" by

A. B. Coble, 329–364; Reviews by S. Lefschetz of F. Severi, *Vorlesungen über algebraische Geometrie* (Leipzig, 1921), 365–366; by P. Field of A. E. H. Love, *Theoretical Mechanics* (3d ed., Cambridge, 1921), 366; by F. L. Hitchcock of A. Naess, *Zur Theorie der Triaden* (Kristiania, 1921), 366–367; by J. B Shaw of C. Isenkrahe, *Untersuchungen über das Endliche und das Unendliche* (Bonn, 1920), 367; by R. B. McClenon of T. L. Heath, *The Copernicus of Antiquity* (*Aristarchus of Samos*) (London and New York, 1920), 368; by J. B. Shaw of P. Appell, *Eléments de la Théorie des Vecteurs et de la Géometrie Analytique* (Paris, 1921), 368; Notes, 369–392; New publications, 373–376.

Bulletin des Sciences Mathématiques, volume 57, October, 1922: Review of É. Picard, *Traité d'Analyse*, volume 1, 3d ed. (Paris, 1922), 353–354; "Sur quelques transformations d'équations aux dérivées partielles"(to be continued) by E. Goursat, 370–384—November: Review by H. Andoyer of Bureau des Longitudes, *Annuaire pour l'an 1922*, 385–389; "Sur quelques transformations d'équations aux dérivées partielles" (conclusion) by E. Goursat, 390–403; "Deux leçons sur certaines équations fonctionnelles et la géométrie non-euclidienne" (to be continued) by E. Picard, 404–416.

L'Intermédiare des Mathématiciens, second series, volume 1, May–June, July–August, 1922: Fifty questions are proposed or re-proposed and thirty-two are answered. The death of H. Brocard, "un des collaborateurs les plus fidèles et les plus érudits du journal," is announced (cover).

Mathematical Gazette, volume 11, December, 1922: "Approximate integration" by A. Buxton, 181–187; "The complete angle and geometrical generality" by D. K. Picken, 188–193; "Isochordic circles related to the triangle" by W. W. Taylor, 194–199; "Some incidental writings by De Morgan," 200–203; Mathematical notes, 204–208; Review by E. H. Neville of Weatherburn's *Elementary Vector Analysis*, 209; Review by H. T. H. Piaggio of Griffin's *Introduction to Mathematical Analysis*, of Passano's *Calculus and Graphs* and of Phillips' *Differential Equations*, 210–211. ["In America a wider range is covered and a broad general outline obtained. The price that has to be paid for this is the sacrifice of much of the mental training obtained by working problems."]

Mathematics Teacher, volume 15, November, 1922: "The case for general mathematics" by W. D. Reeve, 381–391; "Errors in computation and the rounded number" by H. Rice, 392–404; "The constitution of algebraic abilities" by E. L. Thorndike, 405–415; "Romance in science" by Bessie I. Miller, 416–422; "Mathematics of the calculating machine" by L. L. Locke, 423–428; "'Steradians' and spherical excess" by G. W. Evans, 429–434; and ten pages of discussion, news, notes and research—December: "Non-euclidean geometry" by W. H. Bussey, 445–459; "Study of mathematics under individual system" by Mary M. Reese, 460–466; "Problems concerning the teaching of secondary mathematics" by A. Davis, 467–477; "Future development of mathematical education" by C. N. Moore, 478–483; "Function concept in H. S. mathematics" by J. M. Kinney, 484–495; and twelve pages of discussion, news and reviews.

Messenger of Mathematics, volume 52, May, June, 1922. "Factorization of N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = n$]" by A. Cunningham, 1–32.

Nature, volume 110, September 16, 1922: Notices of A. Dakin, *Practical Mathematics* (London, 1921) and of J. Haag, *Cours complet de mathématiques spéciales*, volume 2, *Géométrie* (Paris, 1921), 375; "The theory of numbers" by G. H. Hardy (Presidential address delivered to Section A of the British Association at Hull, September 8, 1922), 381–385—September 23: Notice of F. F. P. Bisacre, *Applied Calculus* (London, 1921), 411; "Summary of the theory of relativity" by H. J. H. Piaggio, 432–434—October 14: "Bergson and Einstein" [review of H. Bergson, *Durée et Simultanéité* (Paris, 1922)] by H. W. Carr, 503–505—October 21: "Mersenne's numbers" by G. H. Hardy, 542 [correction of the statement made in the presidential address (see above), to the effect that for $n = 137$ it was not yet known whether $2^n - 1$ is prime or composite; the note refers to A. Gérardin's article in *Comptes Rendues du Congrès des Sociétés Savantes*, 1920, 53–55, and to this Monthly, *1921*, 380. In the address $n = 139$ should accordingly be substituted for $n = 137$, throughout.]—October 28: Notice of volume 20 (1922), *Proceedings of the London Mathematical Society*, 570; "Relativity and physical reality" by A. A. Robb, 572.

Philosophical Review, volume 31, September, 1922: "$7 + 5 = 12$" by G. W. Cunningham, 495–504—November: "$7 + 5 = 12$" by B. Bosanquet, 593–598 [reply to the discussion in the preceding number].

Proceedings of the National Academy of Sciences of the U. S. A., volume 8, October, 1922: "New properties of all real functions" by H. Blumberg, 283–288; "Generalized limits in general analysis" by C. N. Moore, 288–293.

Revista Matematica Hispano-Americana, volume 4, September–October, 1922: "Prime numbers in arithmetic progression" by E. Landau, 113–128; Appreciation of Camille Jordan by C. de la Vallée Poussin, 129–131; Review of C. P. Steinmetz' *L'Industrie Electrique* by J. A. Perez del Pulgar, 134–135, and fourteen pages of short notes, problems and solutions.

Revue de Mathématiques Spéciales, volume 33, November, 1922: "On the integrals $\int \frac{dx}{(ax^2+bx+c)^n}, \int \frac{P(x)dx}{(ax^2+bx+c)^n}$" (to be continued) by R. Dontot, 313–315; Solutions of problems in algebra, analytic geometry and physics, 316–324, 330–332; Examination questions, 324–329, 333–336; New problems, 336.

Science, volume 56, November 10, 1922: "The order of scientific merit" by J. McK. Cattell, 541–547—November 17: Review by H. Blumberg of E. W. Hobson, *Theory of Functions of a Real Variable* (2nd ed., vol. 1, Cambridge, 1921), 574–575; Review by H. L. Rietz of O. Veblen, *Analysis Situs* (New York, 1922), 575.

Tôhoku Mathematical Journal, volume 21, October, 1922: "A general view of the theory of summability" by S. Takenake, 193–221; "Notes sur la géométrie du tétrahèdre" by V. Thébault, 222–233; "Notes on differential geometry in non-euclidean space" by T. K. Kubota, 234–243; "New demonstration of Euler-Maclaurin sum formula" by C. Jordan, 244–246; "Note on the theory of approximation of irrational numbers by rational numbers" by K. Kurosu, 247–260; "Beweise einiger Sätze über Eiflächen" by T. Kubota, 261–265; "A mathematical game" by S. Fukazawa, (in Japanese) 265–270; "Pentaspherical geometries in non-euclidean space, III" by T. Takasu, 271–309; "Plane algebraic curves invariant under a quadratic Cremona transformation" by A. Emch, 310–326; "Projective generalization of some theorems on algebraic curves and surfaces" by T. Takasu, 327–348; "On the convergencies of series of functions" by T. Matsumoto, 349–355; "Some integral equations in the theory of diffusion of mixed gases" by T. Hayashi, 355–359.

Transactions of the American Mathematical Society, volume 23, January, 1922 [published November, 1922]: "Relatively uniform convergence and the classification of functions" by E. W. Chittenden, 1–15; "Periodic functions with a multiplication theorem" by J. F. Ritt, 16–25; "Note on Dirichlet and factorial series" by T. Fort, 26–29; "Functions of infinitely many variables in Hilbert space" by W. L. Hart, 30–50; "Prime and composite polynomials" by J. F. Ritt, 51–66; "Some two-dimensional loci connected with cross-ratios" by J. L. Walsh, 67–88; "On transformations with invariant points" by J. W. Alexander, 89–95; "Invariant points in function space" by G. D. Birkhoff and O. D. Kellogg, 96–115.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

The Mathematics Club of Brown University, Providence, R. I.
[*1922*, 77.]

The program for 1922–23 is announced in printed form, as follows:

November 3, 1922: "Huge numbers" by Philip Welch '23; "The Moscow mathematical papyrus and formulas for the volume of a truncated square pyramid" by Amelia Harris '24; "Sophie Kovalevski, a mathematician of Russia" by Dorothy Bundy '24.

December 15: "Mathematics of genetics" by Doris Anthony '24; "Ramanujan, a mathematician of India" by Everard Ketcham '24; "A Diophantine problem" by John Miner, Jr. '25.

January 12, 1923: "Foundations of our geometric notions" by James Pierpont, professor of mathematics, Yale University.

February 16: "Approximate methods for trisecting an angle" by Nellie Stokes '23; "Robert Recorde, a mathematician of England" by Rose Whelan '25; "Mathematical paradoxes" by George Smith '23.

March 23: "Egyptian mathematics" by Arnold Chace, chancellor of Brown University.

April 27: "Leibnitz, a mathematician of Germany" by Sarah Jacobson '23; "Methods of work of mathematicians" by Richard Whipple '25; "The game of Nim" by Evelyn Wiggin, Gr.

May: Picnic.

The officers are: Chairman, Professor Richardson; committee on program, Professor Burgess, Professor Gilman, Mildred Carlen, Sp., Amelia Harris '24, Philip Welch '23, Henry Bodwell '24; committee on arrangements, Mr. MacPherson, Mary Appel '23, Charlotte Perry '25, Clarence Bennett '23, John Miner, Jr. '25.

THE MATHEMATICS CLUB OF COOPER UNION, NEW YORK CITY.
[*1922*, 24–25.]

For the year 1921–22, the following officers were elected: President, Thomas Peterson '23; secretary, Hector Audino '25; faculty adviser and honorary treasurer, Professor H. W. Reddick. Meetings were held on alternate Tuesdays, with an average attendance of twenty-five. A polyphase duplex slide-rule was awarded by the Club to Fred Buhrendorf '25 at commencement as a prize for the highest average in first-year mathematics.

The following papers were presented:

November 1, 1921: "Einstein's theory of relativity" by Professor Reddick.
November 15: "Magic squares" by Harry Serper '24.
November 29: "The geometry of numbers" by R. D. Irby '24.
December 13: "Survey of ancient mathematics" by Reginald Overton '23.
January 10, 1922: "Perpetual calendars" (with presentation of wooden models to the Club) by Harry Powell '25 and Albert Johns '25.
January 24: "Geometric representation of indeterminate forms" by William Hoffman '24.
February 7: "Non-Euclidean geometries" by David Samson '24.
February 21: "Amusing numbers" by D. A. Lunden, instructor.
March 7: "The trisection of an angle" by Fred Buhrendorf '25.
March 21: "The fundamental theorem of algebra" by George Agins '22.
April 4: "The slide-rule" by Thomas Peterson '23; "The planimeter" by Battista Sola '23.

At the last meeting the following officers were elected for the year 1922–23: President, Peter Kosting '25; vice-president, Barnett Emmerich '23; secretary, Arthur Cook '25.

(Report by Hector Audino.)

PI MU EPSILON OF SYRACUSE UNIVERSITY, SYRACUSE, N. Y.
[*1922*, 80.]

The Pi Mu Epsilon Fraternity has now four chapters. These are located at Syracuse University, Ohio State University, University of Pennsylvania, and University of Missouri, the latter having been recently admitted with fifteen charter members.

At the Syracuse Chapter in 1921–22 there were elected to membership one faculty member, seven graduate students, and nineteen undergraduates. In addition to the formal meetings, a picnic and a Christmas party were given. The officers for the year 1921–22 were: Director, Professor W. G. Bullard; vice-director, Professor May Harwood; secretary, Elizabeth Lyons '22; treasurer, Carl Bye '22; librarian, Eunice Davidson '22. The officers for the year 1922–23 are: Director, Professor Mary Harwood; vice-director, Otto Gelormini, Gr.; secretary, Olive Jackway '23; treasurer, Otis Hendershot '23; librarian, Helen Franklin '23.

Papers were read as follows:

November 7, 1921: "History and plans of Pi Mu Epsilon" by Professor E. D. Roe, Jr.
November 28: "The volume of a tetrahedron in terms of its sides" by Helen Houck '22.
April 24, 1922: "Practical problems in maxima and minima" by Eunice Davidson '22; "Factors in the development of interest in America" by Elizabeth Lyons '22; "Calculus and chemical kinetics" by Otto Gelormini, Gr.
May 18: "The monomolecular reaction constant" by Howard Post, Gr.

(Report by Olive Jackway.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2999. Proposed by M. B. PORTER, University of Texas.

Given n positive numbers: $a_1, a_2, a_3, \cdots, a_n$, then

$$\Sigma a_i \Sigma a_j^{-1} > n^2 \qquad (i, j = 1, 2, 3 \cdots n)$$

unless $a_i = a_j$, for which case equality occurs; show by passing to limits that, if $\varphi(x) > 0$ and is continuous, $\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)}$ takes on its minimum for $\varphi(x) =$ constant over the interval a to b.

3000. Proposed by J. ROSENBAUM, Milford, Conn.

With use of compasses alone, to construct a circle whose area shall be n times the area of a given circle, where n is any positive integer.

3001. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

In the plane of a given circle a second circle with a given radius is drawn so that the radical axis of the two circles passes through a given fixed point. Find the locus of the center of the second circle.

3002. Proposed by C. N. MILLS, Heidelberg College, Tiffin, O.

The diagonals of any quadrilateral are in length a and b respectively and are inclined at an angle A. Show that the greatest rectangle which can be drawn with its four sides passing through the four corners of the quadrilateral is $\frac{1}{2}ab(1 + \sin A)$.

3003. Proposed by R. M. MATHEWS, Wesleyan University.

The angle PAM rotates around A and meets a line rotating around B in P and M. When M moves along the perpendicular bisector of AB the locus of P is an equilateral hyperbola of which the mid-point of AB is the center. Generalize.

SOLUTIONS

2908 [1921, 326]. Proposed by L. E. DICKSON, University of Chicago.

If f is a homogeneous polynomial in n variables and H is its Hessian determinant, prove that the Hessian of f^2 is cHf^n, where c is a constant.

SOLUTION BY CONSTANCE R. BALLANTINE, Chicago, Ill.

This is a special case of the general theorem that the Hessian of f^m is $cHf^{n(m-1)}$, which may be proved as follows:

$$H(f^m) = \left| \frac{\partial^2 f^m}{\partial x_i \partial x_j} \right| \quad (i, j = 1, 2, \cdots, n)$$
$$= m^n \left| f^{m-1} \frac{\partial^2 f}{\partial x_i \partial x_j} + (m-1) f^{m-2} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right| \quad (i, j = 1, 2, \cdots, n).$$

This determinant may be broken up into the sum of 2^n determinants, one of which is $H(f)$ with every element multiplied by f^{m-1}, and the others each the same as this determinant with the elements in the jth column replaced by $(m-1)f^{m-2}(\partial f/\partial x_i)(\partial f/\partial x_j)$, for one or more values of j. Of these last, all containing more than one such column will be zero; for suppose the jth and kth columns so replaced, then we may factor out $\partial f/\partial x_j$ and $\partial f/\partial x_k$ and leave these columns the same. Thus,

$$H(f^m) = m^n\left\{Hf^{n(m-1)} + (m-1)f^{n(m-1)-1}\sum_{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}C_{ij}\right\},$$

where C_{ij} is the cofactor of $\partial^2 f/\partial x_i\partial x_j$ in H, and where the indices of summation run from 1 to n.

By Euler's identity, if p is the order of f in the variables,

$$\sum_j x_j\frac{\partial f}{\partial x_j} = pf, \qquad \text{and} \qquad \sum_k x_k\frac{\partial^2 f}{\partial x_k\partial x_i} = (p-1)\frac{\partial f}{\partial x_i},$$

$\partial f/\partial x_i$ being a homogeneous polynomial of order $p-1$ in the variables. Thus

$$\sum_i\frac{\partial f}{\partial x_i}C_{ij} = \frac{1}{p-1}\sum_{ik}x_k\frac{\partial^2 f}{\partial x_i\partial x_k}C_{ij} = \frac{1}{p-1}x_jH,$$

since

$$\sum_i\frac{\partial^2 f}{\partial x_i\partial x_k}C_{ij} = \begin{cases}0 \text{ when } k \neq j,\\ H \text{ when } k = j.\end{cases}$$

Then

$$\sum_{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}C_{ij} = \frac{1}{p-1}H\sum_j x_j\frac{\partial f}{\partial x_j} = \frac{p}{p-1}Hf,$$

and

$$H(f^m) = \frac{m^n(mp-1)}{p-1}Hf^{n(m-1)}.$$

In particular, if $m = 2$,

$$H(f^2) = \frac{2^n(2p-1)}{p-1}Hf^n.$$

Also solved by ELSIE J. MCFARLAND, J. R. MUSSELMAN, F. D. MURNAGHAN, and H. L. OLSON.

2910 [1921, 326]. Proposed by DANIEL KRETH, Wellman, Iowa.

The segments formed on the base of a triangle by the perpendicular from the opposite vertex are m and n. The product of the other two sides is p. Compute the two unknown sides and give a simple construction for the triangle.

PARTIAL SOLUTION BY M. L. YÜ, Nanking, China.

COMPUTATION. In the triangle ABC, let AD be the perpendicular on the side BC, $DB = m$, $DC = n$, the unknown sides $AB = x$, and $AC = y$.

Then we have $x^2 - y^2 = m^2 - n^2$ and $xy = p$.

From these two equations,

$$2y^2 = Y_1 = n^2 - m^2 + \sqrt{(m^2-n^2)^2 + 4p^2},$$

or

$$Y_2 = n^2 - m^2 - \sqrt{(m^2-n^2)^2 + 4p^2}.$$

Y_2 renders y imaginary. $y = \sqrt{Y_1/2}$ gives $x = p\sqrt{2/Y_1}$.

CONSTRUCTION.[1] Draw the straight line RQS, making $RQ = m$ and $QS = n$, and the perpendicular

$$PQ = \sqrt{y^2 - n^2} = \left[\sqrt{(m^2-n^2)^2 + 4p^2} - (m^2+n^2)\right]^{1/2}.$$

Join P and R and P and S and then PRS is the triangle required.

DISCUSSION. In order that there be a real triangle, $PQ^2 = y^2 - n^2$ must be positive, that is, $\sqrt{(m^2-n^2)^2 + 4p^2} - (m^2+n^2) > 0$, which means that $mn < p$.

[1] This is not what is generally meant by a construction but is merely a formula for computation. The geometric construction of PQ from m, n and p^2 is possible but not simple—EDITORS.

Also solved by T. M. BLAKSLEE, J. B. REYNOLDS, A. R. NAUER, and F. L. WILMER.

2912 [1921, 326]. Proposed by T. W. JACKSON, Jamestown College, N. D.

Given c, the chord of a circle, determine r the radius, so that $3c$ is equal to the major arc of the circle.

SOLUTION BY A. H. LAUDER, Laramie, Wyo.

Let 2φ be the angle subtended by the minor arc, and let $x = \sin\varphi = c/2r$. Then $\pi - \varphi = 3x$ and after eliminating φ we have

$$x - \sin 3x = 0, \qquad r = \frac{c}{2x}\cdot$$

The first equation has only one positive root different from zero. Solving for this root and inserting its value in the second equation, it is found that

$$x = .759622, \qquad r = .658222c.$$

Also solved by T. M. BLAKSLEE, A. M. HARDING, J. B. REYNOLDS, and F. L. WILMER.

2913 [1921, 326]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given

$$S(z) \equiv nz\left[1 + \sum_1^\infty (-1)^p \frac{(n^2-1)(n^2-3^2)\cdots\{n^2-(2p-1)^2\}}{(2p+1)!} z^{2p}\right]$$

$$C(z) \equiv 1 + \sum_1^\infty (-1)^p \frac{n^2(n^2-2^2)(n^2-4^2)\cdots\{n^2-(2p-2)^2\}}{(2p)!} z^{2p}$$

show that, for any value of n, $|\cos x| < 1$, $|\sin x| < 1$:

$$\sin(nx) \equiv S(\sin x) \equiv \sin\left(\frac{n\pi}{2}\right) C(\cos x) - \cos\left(\frac{n\pi}{2}\right) S(\cos x);$$

$$\cos(nx) = C(\sin x) \equiv \cos\left(\frac{n\pi}{2}\right) C(\cos x) + \sin\left(\frac{n\pi}{2}\right) S(\cos x).$$

SOLUTION BY THE PROPOSER.

I. Let $z = \sin x$ and $y = \sin nx$, and let $dy/dz = y'$, $d^2y/dz^2 = y''$. Then $y' = n\cos nx/\cos x$, $y'' = (-n^2\cos x\sin nx + n\cos nx\sin x)/\cos^3 x$, so that $(1-z^2)y'' - zy' + n^2y = 0$.

It is readily seen that this differential equation also arises if we let

II. $z = \sin x$, $y = \cos nx$,

III. $z = \cos x$, $y = \sin nx$,

IV. $z = \cos x$, $y = \cos nx$.

Now let $y = a_0 + a_1z + a_2z^2 + \cdots + a_kz^k + \cdots$, so that $y' = a_1 + 2a_2z + 3a_3z^2 + \cdots + ka_kz^{k-1} + \cdots$, $y'' = 2a_2 + 3.2a_3z + 4.3a_4z^2 + \cdots + k(k-1)a_kz^{k-2} + \cdots$.

If we substitute these values for y, y', and y'' in the differential equation, we shall have as the coefficient of z^k, $(k+2)(k+1)a_{k+2} - (k^2 - n^2)a_k$, and this coefficient must vanish identically. Hence

$$y = a_0C(z) + \frac{a_1}{n}S(z).$$

This is the law of the series that satisfies the differential equation $(1-z^2)y'' - zy' + n^2y = 0$.

Sets of corresponding values sufficient to complete the definition of the series for each of the particular cases I, II, III, IV are exhibited in the following table; from them and the law above, the required developments are found:

x	Case	z	$a_0(=y)$	$a_1(=y')$
0	I	0	0	n
0	II	0	1	0
$\pi/2$	III	0	$\sin\frac{n\pi}{2}$	$-n\cos\frac{n\pi}{2}$
$\pi/2$	IV	0	$\cos\frac{n\pi}{2}$	$n\sin\frac{n\pi}{2}$

It is interesting to develop other elementary functions in this way; for instance, to develop $\tan nx$ into a series of powers of $\tan x$, from the equation $(1+z^2)y''+2(z-ny)y'=0$.

2923 [1921, 392]. Proposed by C. N. SCHMALL, New York City.

The corner of a page of a book is turned down in such a manner that the triangle formed has a constant area. Show that the locus of the corner is an oval of the curve,

$$r^2 = a^2 \sin 2\theta.$$

SOLUTION BY W. G. HUBERT, College of the City of New York.

Take the corner in its initial position as origin, in its turned down position as $P(r, \theta)$, and let the ends of the crease be Q and R. By identity, the triangles OQR and PQR are equal, and QR is the perpendicular bisector of OP at E, whose coördinates are $r/2$, θ. The constant area K is then equal to the combined areas of triangles OEQ and OER, or

$$K = \tfrac{1}{2}OE(QE + ER)$$

$$= \frac{1}{2}\frac{r}{2}\left(\frac{r}{2}\tan\theta + \frac{r}{2}\cot\theta\right);$$

whence $r^2 = 4K\sin 2\theta$. The constant a is then equal to $2\sqrt{K}$.

Also solved by P. E. BASYE, EDNA E. KRAMER, R. M. MATHEWS, L. C. MATHEWSON, ADELE A. M. MATZKE, A. R. NAUER, E. J. OGLESBY, H. L. OLSON, ARTHUR PELLETIER, H. P. ROBERTSON, J. B. REYNOLDS, CONSTANCE RUMMONS, ANNA SCHIFF, the PROPOSER, and by MOE BUCHMAN, A. L. DUNA, HERMAN HENKIN, T. F. PETERSON, and BATTISTA SOLA of Cooper Union, New York.

2925 [1921, 393]. Proposed by F. V. MORLEY, New College, Oxford, Eng.

A regular polygon of $2n+1$ sides will have only $n-1$ diagonals of different lengths (*e.g.*, the regular heptagon has two distinct diagonals). Call the side of such a polygon a_1, and the $n-1$ diagonals in order of size $a_2 \cdots a_n$. Then if the circumscribed circle has radius unity, $\sum_{i=1}^{n} a_i^2 = 2n+1$; in words, the sum of the squares of the distinct lengths obtained by joining an odd number of regularly spaced points on a unit circle is equal to the number of such points.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let P be any vertex of the polygon, A_i the ith vertex after P, and $\angle POA_1 = \alpha$. Then $a_i^2 = PA_i^2 = OP^2 + OA_i^2 - 2OP\cdot OA_i \cos i\alpha = 2 - 2\cos i\alpha$. Hence $\sum_1^n a_i^2 = 2n - 2\sum_1^n \cos i\alpha$. Since $\alpha = 2\pi/(2n+1)$ and $2\sin(\alpha/2)\sum_1^n \cos i\alpha = \sin[(2n+1)\alpha/2] - \sin(\alpha/2)$, we have $\sum_1^n a_i^2 = 2n+1$.

Also solved by W. H. ECHOLS, R. M. MATHEWS, L. C. MATHEWSON, H. L. OLSON, J. B. REYNOLDS, F. L. WILMER, M. L. YÜ.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Mr. E. W. SCHREIBER, Proviso Township High School, Maywood, Ill., a member of the Association, is recording secretary of the Chicago Mathematics Club, which meets monthly to consider questions of mathematical or pedagogical interest.

The twenty-second meeting of the Central Association of Science and Mathematics Teachers was held at Hyde Park High School, Chicago, on December 1 and 2, 1922, under the presidency of Mr. ALFRED DAVIS, Soldan High School, St. Louis. Mr. E. L. THOMPSON, Township High School, Joliet, Ill., is vice-chairman of the Mathematics Section. The program of the Mathematics Section was as follows: "Organization of secondary mathematics" by W. W. HART, the discussion being opened by J. M. KINNEY; "Consistency in grading mathematics papers" by E. J. MOULTON; "The slide rule" by W. W. GORSLINE; "Preparation of teachers of mathematics for Junior High School" by J. R. OVERMAN; "Inspection of some old mathematics manuscripts from Armour Institute and other sources" by M. J. NEWELL. All the men mentioned are members of the Mathematical Association of America.

Mr. RALEIGH SCHORLING of the Lincoln School, New York, has been appointed principal and head of the mathematics department of the new model high school at the University of Michigan.

Announcement is made that the annual summer meeting of the Association for 1923 will be held at Vassar College, Poughkeepsie, N. Y., beginning Wednesday afternoon, September 5, and continuing through Wednesday and Thursday. The Thursday afternoon session will be held jointly with the American Mathematical Society whose meetings will continue through Friday.

The following reports of Summer Sessions to be held in 1923 have been received.

University of Chicago, June 18–August 31. In addition to the usual courses in Trigonometry, College algebra, Plane analytic geometry, and Calculus, the following advanced courses are offered. By Professor E. J. MOULTON: Advanced calculus. By Professor J. W. A. YOUNG: Selected topics in mathematics. By Professor A. F. CARPENTER: Solid analytic geometry. By Professor L. J. MORDELL: Theory of definite integrals; Analytic theory of numbers. By Professor A. C. LUNN: Theory of sound; The theory of relativity. By Professor L. E. DICKSON: Arithmetic and algebra of hypercomplex numbers; Differential equations from the standpoint of Lie. By Professor F. R. MOULTON: Analytic mechanics; Theory of functions of the complex variable. By Professor E. J. WILCZYNSKI: Theory of linear differential equations. By Dr. MAYME I. LOGSDON: Higher plane curves.

Columbia University, July 9–August 17. In addition to the usual courses in Logarithms and trigonometry, Solid geometry, College algebra, Analytic geom-

etry, and Calculus, the following courses are offered. By Professor DUNHAM JACKSON: Theory of functions of real variables; Calculus of variations. By Dr. K. W. LAMSON: Differential equations. By Dr. JESSE DOUGLAS: Differential geometry. By Professor W. B. FITE: Theory of infinite series. Each of these courses meets five times per week.

Cornell University, July 7–August 17. By Professor VIRGIL SNYDER: Course for teachers; Projective geometry. By Professor C. F. CRAIG: Analysis. The following reading and research courses are also offered. By Professor SNYDER: Algebraic curves and surfaces. By Professor F. R. SHARPE: Applied mathematics. By Professor W. B. CARVER and Professor F. W. OWENS: Foundations of geometry and problems in synthetic geometry. By Professor D. C. GILLESPIE and Professor W. A. HURWITZ: Advanced analysis. By Professor C. F. CRAIG: Functions of a complex variable. By Professor W. L. G. WILLIAMS: Theory of algebraic forms.

University of Illinois, June 18–August 11. By Professor R. D. CARMICHAEL: Elliptic functions. By Professor A. R. CRATHORNE: Mathematics of statistics. By Professor A. J. KEMPNER: Functions of a complex variable. By Professor G. E. WAHLIN: Theory of equations and determinants. By Dr. H. R. BRAHANA: Projective geometry. By Dr. GEORGE RUTLEDGE: Advanced calculus.

University of Michigan, June 25–August 17. Besides the usual elementary courses, the following more advanced courses are offered. By Professor W. B. FORD: Advanced calculus. By Professor L. C. KARPINSKI: History of mathematics. By Professor T. R. RUNNING: Graphical methods. By Professor PETER FIELD: Vector analysis. By Professor J. W. BRADSHAW: Geometry of pictorial representation. By Professor H. C. CARVER: Theory of probability. By Professor T. H. HILDEBRANDT: Functions of a complex variable. By Professor C. J. COE: Differential equations. By Professor R. B. ROBBINS: Finite differences.

University of Pennsylvania, July 2–August 11. In addition to courses in Solid geometry, Plane trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered. By Professor H. H. MITCHELL and Dr. J. D. ESHLEMAN: Advanced plane and spherical trigonometry. By Professor M. J. BABB: Review of college mathematics. By Professor G. G. CHAMBERS: Advanced calculus. By Professor H. H. MITCHELL: Theory of algebraic numbers. By Professor J. R. KLINE: Foundations of geometry.

University of Wisconsin, June 25–August 3. Besides the usual elementary courses, the following more advanced courses are offered. By Mr. O. H. RECHARD: Differential equations. By Professor L. W. DOWLING: Projective geometry, Higher geometry. By Professor W. WEAVER: Vector fields, Restricted theory of relativity. By Professor C. S. SCHLICHTER: Theory of probabilities. By Mr. E. B. MILLER: Theory of equations and determinants. By Professor ARNOLD DRESDEN: College geometry for teachers, Elliptic integrals, Algebra of matrices. By Mr. H. T. DAVIS: Theory of numbers.

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Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or September, 1918—fifty cents; September, 1915— seventy-five cents; May, 1915—one dollar (See MONTHLY, March, 1921, p. 152)

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December, 1922

The following are dates of Section meetings of the Association in 1921 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29, 1922

IOWA, Simpson College, Indianola, April 30, Des Moines, November 4

KANSAS, Topeka, January 22; Topeka, January 21, 1922

KENTUCKY, University of Kentucky, May 7; Georgetown College, 1922

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 13, 1922; Washington, December, 1922

MINNESOTA, College of St. Thomas, St. Paul, June 4

MISSOURI, Washington University and Soldan High School, St. Louis, November 25–26; Kansas City, November, 1922

OHIO, Columbus, March 25–26; Columbus, April 14–15, 1922

ROCKY MOUNTAIN, Denver, March 25–26; Greeley, Colo. April 14–15, 1922

TEXAS, Dallas, November 25; Houston, December 1–2 1922,

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THE DEVELOPMENT OF MATHEMATICS IN BOHEMIA.

By QUIDO VETTER, University of Prague.

Translated from the Italian[1] and edited with notes by

RAYMOND CLARE ARCHIBALD, Brown University.

In the following observations on the development of mathematics in the past, I must consider also astronomy since these two sciences, especially with us, have advanced together. And wishing to show foreigners the scientific movement in a nation little known up to the time of the Great War in the wide circle of an international public, I must sketch at least roughly the surroundings in which it developed. Bohemia, in Central Europe, is surrounded not only by mountains but also by Germans and Magyars, who were quite too frequently enemies of the Czechs. Hence life in this country was fostered as much through the effort of the Czech nation as by the influence of its neighbors.

In the Middle Ages the Czechs knew how to create a strong state which stood at the height of culture of the time. In 1348 the Emperor Karel IV, Czech in origin but educated in Paris (the seat of the famous university), on his return from Italy, where several fine universities flourished, founded the University of Prague which he tried to organize on the models of France and Italy; it was the first university in Central Europe. From that time also our scanty record of the study of the sciences in Bohemia begins. In the Faculty of Arts three professors were named for the Trivium, and four for the Quadrivium that is for the mathematical sciences.[2] Unfortunately their names are not preserved; it is probable that among them were some teachers from Italy. We know, however, the mathematical works which the professors followed in their academic courses. They are the *Algorismus* of Sacrobosco,[3] the six books of Euclid's Elements, the

[1] "Lo sviluppo della matematica nella Boemia" by Q. Vetter, *L'Europa Orientale*, Rome, vol. 2, February 15, 1922, pp. 99–104. This periodical is a monthly issued under the auspices of the Istituto per l'Europa Orientale, founded at Rome (Via Nazionale 89) in 1921, to develop and diffuse, by purely scientific methods, studies relative to Eastern Europe. Among those directing the Institute's work is the scholar Dr. Aurelio Palmieri, a remarkable linguist, ever cordially helpful to inquirers. An excellent library is being built up.

Doctor Vetter has also published "La storia della matematica presso i Cechi" in *Archivio di Storia della Scienza*, vol. 2, June, 1921, pp. 199–201. In the present article he refers to the work of J. Smolík (1863) telling of Bohemian mathematicians before 1750, and to Father Vydras's history (1778) as his chief sources of information. Another useful publication is F. J. Studnička, *Bericht über die mathematischen und naturwissenschaftlichen Publicationen der k. Böhm. Ges. der Wissenschaft während ihres Bestehens*, Prague, 1884. The footnotes I have added contain a number of further references to the literature of the subject; ten of these references were kindly furnished to me by Doctor Vetter. The two footnotes wholly by Doctor Vetter have his name attached to them.

For assistance in the translation I am much indebted to Miss Anne Hathaway of the New York Public Library.

[2] In the Middle Ages the Trivium was the lower division of the liberal arts, comprising grammar, rhetoric, and logic; the higher division was the Quadrivium comprising the mathematical sciences (arithmetic, geometry, astronomy, and music).

[3] Sacrobosco, Latinized form of Holywood; John of Holywood, an Englishman, who flourished in the thirteenth century.

Tractatus de Sphaera of Sacrobosco, the *Sphaera theoretica* of an unknown writer, the *Almagest* of Ptolomey, the *Computus Cyrometricalis,* written perhaps by a teacher called Erford or by Sacrobosco, the *Perspectiva communis* by John of Peckham.[1] They studied also the theory of the planets according to the book of Giovanni da Campano, or according to Gerardo da Cremona or Bonalo da Forli.

The first names of Czech professors known to us are the astronomer Master Gallus and the mathematician Jenek of Prague,[2] both of the second half of the fourteenth century; their names are about all we know of them. The first scientific manuscripts preserved are those of Křištian z Prachatic (1368–1439). He was an intimate friend of Huss[3] and professor in the Faculty of Arts, and later rectór, in the University of Prague. Among the authors of the time, Křištian has the reputation of being a prolific writer, but of his mathematical writings only an *Algorismus*[4] and a *Computus Cyrometricalis* are preserved.

These promising beginnings were interrupted by that movement which made sacred the historic mission of the Czech nation and glorified it, but at the same time arrested the development of the sciences in Bohemia, namely the Hussite movement. The profoundly religious and particularly ethical spirit of the older Czechs of the fourteenth and fifteenth centuries suffered severely through the deplorable lowering of morality in general and in the church in particular. With all the passion of their trusting hearts, with all the fervor and seriousness of their moral sense, they proclaimed the necessity of strict morality, and in their slavic individualism, the necessity of freedom of religious belief. All were drawn into the vortex of religious discussions, but as soon as the most intellectual people were carried into the strife then every other activity became absolutely impossible, or at least greatly curtailed. This psychological reason, it seems to me, hindered the development of science more than the long Hussite wars in which villages were destroyed, cities and monasteries burned. Křištian z Prachatice, whom I have mentioned, is better known for the part he played in the scenes of religious and political struggle than for his scientific activity. And in the conflagration of war the memories of his earlier work perished. That is why our knowledge of him is so meagre. The little nation was forced bitterly to expiate the courage it displayed in raising on high the torch of progress, a hundred years before others were capable of grasping it.

[1] John of Peckham, Archbishop of Canterbury from 1278 till his death in 1292, was equally celebrated as a theologian and mathematician. His *Perspectiva* was later printed, and has appeared in many editions, for example: The first edition, 1480; Leipzig, 1504; edition by G. Hartmann, Nürnberg, 1542; and Venice, 1593.

[2] Presumably this is "Zdeněk," the astrologer and physician to King Johann of Luxemburg, regarding whom some biographical notes are given by K. Teige on page 244 of *Časopis pro pěst. matem.*, vol. 22, 1893.

[3] John Huss (1369–1415), the Bohemian reformer and martyr.

[4] Compare the papers by F. J. Studnička: (a) "Ueber den Algorismus Křišt'ans von Prachatic," *Sitzungsberichte der kgl. böhm. Gesellschaft der Wissenschaften*, Math.-Naturwiss., 1892, pp. 100–104; (b) "Algorismus prosaycus magistri Christiani anno fere 1400 scriptus," *idem*, 1893, no. VI, 14 pp. Also *Bibliotheca Mathematica*, series 3, vol. 10, 1910, p. 58.

In the midst of this political turmoil the name of the astronomer Jan Šindel[1] (b. 1375) stands out. He is of interest to Italians as the friend of Giovanni Bianchi[2] and a man greatly esteemed by Enea Silvio, afterwards Pope Pius II. Even a man like Tyge Brahe[3] studied the works of Šindel.

I do not wish to try the patience of the reader with the names of all the mathematicians of minor importance who, however, prepared the field for scientific study of the stars; for the most part these were called from foreign countries to the court of Rudolph II.

At the end of the fifteenth century a monument was set up which showed the astronomical attainment of the Czechs at that time, namely the Clock of Prague whose maker was called, according to tradition, Master Hanus. Not only the movable statues of the clock diverted the public, but its construction won the admiration of the mathematical astronomers of the fifteenth and sixteenth centuries. The clock showed the motion, the rising and setting of the stars, the sun, moon, and planets, the phases of the moon, the height and depth of the sun and moon above and below the horizon, the calendar of feast days, the hours according to the ancient numeration, that is from one to twenty-four commencing with sunset, and finally the astrological rulers of every day and every hour.[4]

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[1] This name is usually spelled Szindel (other forms: Syndelius and Schinttel). Jan Szindel (in German, Johann Schindel) was born in Králové Hradec (Königgrätz), Bohemia, not earlier than 1370 nor later than 1375; it is not known that he was born in 1375. He died in Prag about 1450. After being director of the St. Niclas School in Prague, he was dozent in mathematics and astronomy in Vienna, 1407–1409. From 1410 on, he was "doctor et lector ordinarius Universitatis studii Pragensis." The following sources may be consulted for material treating of Szindel's life and work: L. J. Scherschnik, *De doctis Regiopradecensibus commentarius*, Prague, 1775; Truhlář, *Počátky humanismu v Čechách* (The beginning of humanism in Bohemia), 1892, pp. 24–27; K. Teige, "New information completing the history of mathematical science. 2. Jan Sindel" (in Bohemian), *Časopis pro pěst. matem.*, vol. 22, 1893, pp. 244–246; A. Czerny, "Aus dem Briefwechsel des grossen Astronomen Georg von Peurbach," *Archiv für österreichische Geschichte*, vol. 72, 1888, pp. 299–301; J. Teige, "Ein Beitrag zur Lebensgeschichte des Magister Joannes de Praga," *Zeitschrift für Mathematik und Physik*, hist.-literar. Abt., vol. 28, pp. 41-44; F. J. Studnička, *Sitzungsberichte der kgl. böhm. Gesellschaft der Wissenschaften*, Math.-Naturwiss. 1892, pp. 103–104; S. Günther, *Geschichte des math. Unterrichts im deutschen Mittelalter bis Zum Jahre 1525*, Berlin, 1887, p. 228; Schepfs, "Der Mathematiker Joh. Schindel (1444)," *Anzeiger für Kunde, deutschen Vorzeit*, 1879, new series, vol. 26, col. 262 (see also 1878, vol. 25, col. 1); and J. P. Kalina von Jätenstein, *Nachrichten über böhmische Schriftsteller und Gelehrte*, Prague, 1818.

[2] It seems certain that this name should be Giovanni Bianchini, that of the fifteenth century astronomer of Ferrara, and the correspondent of Regiomontanus. Compare P. Riccardi, *Biblioteca Matematica Italiana*, Modena, 1893, (I), cols. 133–134; (II), cols. 7, 94–95; R. Wolf, *Geschichte der Astronomie*, Munich, 1877; A. R. Grant, *History of Physical Astronomy*, London, 1852; S. Günther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter bis zum Jahre 1525*, Berlin, 1887, p. 225; and M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Leipzig, vol. 2, second edition, 1900, various references in the index.

[3] Tyge Ottesen Brahe (1546–1601) is popularly known as Tycho Brahe. For valuable bibliographical and historical notes concerning him see N. Nielsen, *Matematiken I Danmark 1528–1800* Copenhagen, 1912, pp. 27–37.

[4] This clock, one of the great marvels of Bohemia, is situated in the outer wall at the foot of the tower in the old Town Hall, on the market place, and was constructed in 1490 by Master, Hanus, an astronomer and mathematician who taught at the University. The old dial plate is protected by a Gothic canopy with ornamental conventional forms of leaves and animals, motions

In the sixteenth and at the beginning of the seventeenth century some manuals of arithmetic and mensuration were published. Their purpose was practical. The material was limited to the calculations of lines and figures, to the simple and complex rule of three and to similar problems, to arithmetic, and to geometric progressions.

Above the common level rise two astronomers, Kerásek Lvovický z Lvovicz (1514–1574) [1] and more especially Tadeáš Hájek z Hájek (1525–1600): [2] the first as author of several astronomical books, the other as professor of mathematics at the University and then as doctor of medicine to the Emperor Maximilian II. We know a group of his astronomical books, an *Oratio de laudibus geometriæ*, [3] and besides this, calendars, and minutiæ, and finally several books on medicine and chemistry. His greatest fame came through his fine description of the new star of the year 1572; [4] it is this star which is connected so closely with the scientific career of Tyge Brahe.[5] The great Danish astronomer considered the

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of the heavenly bodies, etc. Above the dial are two windows which open as an hour is struck, and figures of Jesus followed by the apostles appear. Under the dial is another recent one in an ancient border representing the signs of the zodiac; it contains scenes depicting seasonal events in Bohemia (fishing time, hunting season, vintage, etc.). In the time of Hanus this calendar was an integral part of the clock; it was removed to the shelter of the museum of the Rudolphinum in the nineteenth century, and replaced by the above-mentioned modern copy by the painter Josef Manes.

Master Hanus had charge of the Prague clock until his death; then a pupil held the office till his death in 1530. He was followed by a third keeper who served for forty years and left a work on 15 strips of parchment describing the origin and mechanism of the clock. After his death the clock stopped, no one being found able to care for it till 1787. It stopped again in 1824 and remained out of commission till 1866.

An excellent picture of the clock is given in F. M. Feldhaus, *Die Technik der Vorzeit*, Leipzig, 1914. Another good picture is given in the article "orloj" (from which most of the information given above is taken) in *Ottův Slovník Naučný* (Slovak Cyclopædia of Knowledge), Prague, vol. 18, 1902. See also L. P. M. Leger, *Prague* (Les villes d'art célèbres), Paris, 1907, pp. 93–94; F. H. H. Lützow, *The Story of Prague*, London, second edition, 1907, pp. 80, 168. There is also a fine picture in C. Wakefield, *Visit to the City of Prague*, London, 1921, p. 44.

[1] Compare "Der Astronom Cyprianus Leovitius (1514–1574) und seine Schriften" by Jos. Mayer, *Bibliotheca Mathematica*, series 3, vol. 4, pp. 134–159, 1903. Reference may also be given to: C. F. Dechales Milliet, *Cursus seu mundus mathematicus*, Leyden, 1690, vol. 1, p. 88; P. Bayle, *Dictionnaire historique et critique*, Rotterdam, 1696, third edition 1720, etc.; J. F. Weidler, *Historia astronomiae, sive de ortu et progressu astronomiae*, Wittenberg, 1741, p. 369; C. G. Jöcher, *Allgemeines Gelehrten-Lexicon*, Leipzig, vol. 2, 1750; J. S. Bailly, *Histoire de l'astronomie moderne*, new edition, Paris, 1785, vol. 2, p. 236; A. G. Kästner, *Geschichte der Mathematik*, Göttingen, vol. 2, 1797, pp. 344, 538; R. Wolf, *Geschichte der Astronomie*, Munich, 1877, p. 303.

[2] Reference may be given to: F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, vol. 4, Prague, 1782; J. S. Bailly, *Histoire de l'astronomie moderne*, new edition, Paris, 1785, vol. 1, pp. 375, 411. There is a brief paragraph concerning this scientist in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 1; this seems to have been the source of information for the paragraph on Hájek in D. E. Smith, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, New York, 1917, p. 136.

[3] Published at Prague in 1557.

[4] *De investigatione loci stellæ novæ in Zodiaco*, Vienna, 1573; *Dialexis de novæ et prius incognitæ stellæ inusitatæ magnitudinis apparitione et de ejusdem stellæ vero loco constituendo . . .*, Frankfurt, 1574.

[5] T. Brahe, *De nova et nvllivs. œvi memoria privs visa stella, . . . Anno . . . 1572. mense Novembrj primùm conspecta, contemplatio mathematica*, Copenhagen, 1573. See also *Tychonis Brahe Dani Opera Omnia* edidit I. L. E. Dreyer, vol. 1, Copenhagen, 1913.

observations of Hájek the best after his own. It is interesting that Hájek writes against astrological imposture. He believes, with his time, that comets and phenomena of the universe have an importance in the life of men, but, in his deeply religious nature, especially during his later years, he sees in them certain admonitions of God for the sins of men. He says expressly that it is not possible to predict the modes of divine punishments and that the omnipotent God can mitigate the punishment of a penitent world. And again we come to what I have already said. Hájek was greatly concerned with religious questions, he even maintained a polemic against an attack on the Bohemian Brothers though not of their religion. A hindrance to his astronomical activities was the lack of accurate instruments such as Tyge had. Not finding a patron like the beneficent Frederick II of Denmark, he had to make a living from his medical practice and gave only his spare time to science. Hájek was in active correspondence with the mathematical societies of his time. He continued his relations with the Italian scientists he had known while studying at Bologna; it was for this reason he visited Girolamo Cardano at Milan.[1] That he had an intimate friendship with Tyge Brahe, we see from their copious correspondence, a part of which remains still unpublished in the Bibliothek Nazional in Vienna. To him belongs the principal credit of calling Tyge Brahe to Prague. Hájek and Martin Bacháček, (1539–1612), professor of mathematics in the University, were the principal representatives of that "native" spirit of which I have already spoken.

The sojourns of Brahe, Kepler, and Bürgi[2] would have had a profound and beneficial influence on the development of the sciences in Bohemia, if there had not occurred a long prepared catastrophe. Scarcely had the Hapsburg dynasty got possession, in 1526, of the Bohemian crown, for which during two centuries and a half it had worked with a tenacity peculiar to itself, when it sought means completely to subjugate the hated Czech nation. The unfortunate battle of

[1] Cardano (1501(?)–1576) was a practising physician and professor of mathematics in Milan 1534–1559. It was in his *Artis magnae sive de regulis Algebrae Liber unus*, Milan, 1545, that he published the rule given him by Tartaglia under a pledge of secrecy six years earlier.

[2] Tyge Brahe (1546–1601) went to Prague in 1599 and died there. Johann Kepler (1571–1630) was in Prague from the time of Brahe's death to 1615. Joost Bürgi (1552–1632), chief clock-maker to Rudolph II, 1603–1622, is entitled to the honor of independent invention of logarithms and constructed the tables, *Arithmetische und Geometrische Progress-Tabulen*, published in 1620. Napier was working on logarithms at least as early as 1594; compare "Question of priority" by F. Cajori in *Napier Tercentenary Memorial Volume*, 1915. See also R. Wolf, *Johann Kepler und Jost (sic) Bürgi*, Zürich, 1872; and J. v. Hasner, *Tycho Brahe und J. Kepler im Prag. Eine Studie*, Prague, 1872. Tyge Brahe's tomb in the Tyn Church, near the University and Town Hall, shows an effigy in red marble representing Brahe in armor holding his globe and compasses. A photograph of this is reproduced in L. P. M. Leger, *Prague* (Les villes d'art célèbres), Paris, 1907, p. 9. See also F. H. H. V. Lützow, *The Story of Prague*, London, second edition, 1907, p. 104.

Among the publications which appeared on the three hundredth anniversary of T. Brahe's death were the following: (*a*) "Tyge Brahe in Bohemian literature" (in Bohemian), by L. Peprný, *Časopis pro pěst. matem.*, vol. 30, 1901, pp. 209–222—a list, with indications of contents, of Czech publications discussing Brahe and his work; (*b*) *Prager Tychoniana zur bevorstehenden Säkularfeier der Erinnerung an das vor 300 Jahren erfolgte Ableben des Reformators der beobachtenden Astronomie Tycho Brahe gesammelt*, by F. J. Studnička, Prague, 1901; 71 pp. + chromo-portrait + 7 reproductions—concerning some of Brahe's books, manuscripts, instruments, etc., now in Prague.

White Mountain in 1620 gave the desired pretext to inflict upon the whole Czech nation unheard-of cruelty. After the thirty years war there did not remain in Bohemia more than one-sixth of the old population; property of the native people was confiscated, and the finest and most intellectual Czechs wandered like beggars in foreign lands. Of course the development of the sciences would thus be interrupted and the participation of Czechs in international sciences greatly diminished.

The Czech Jan Marcus-Marci (1595–1667) [1] is worthy to be remembered. His writings on the squaring of the circle [2] and on physics, we know; he was the forerunner of Newton in the theory of the refraction of light.

The sciences in the seventeenth and eighteenth centuries found an asylum in the monasteries, especially among the Jesuits who took possession of the University of Prague. Among the native Jesuit Czechs were several mathematicians. The highest praise was given to Father Jakob Kreza (1648–1715) [3] who has left several mathematical works. Kreza, after having been professor in the Universities of Prague and Olomauc, was called to the University of Madrid, where he taught for fifteen years.

The eighteenth century produced many practical books, such as interest-tables and books of applied geometry. Of the practical manuals I would cite especially the one of Václav Veselý, to whom also is attributed an *Ars liberandi*, published in the year 1734.

Technical arts flourished in Bohemia. Already at the beginning of the eighteenth century a school of engineers was founded at Prague, not by the imperial government, but by the Diet of the Bohemian Kingdom. The Austrian government, relentlessly hostile to the Czech nation, wished to Germanize not only the offices but all the schools from the elementary to the university; for this reason Emperor Josef II excluded the Latin language at the end of the

[1] His life and work have been described by F. J. Studnička in a "Festvortrag," "Joannes Marcus Marci a Cronland, sein Leben und gelehrtes Wirken," *Jahresbericht der königl. böhm. Gesellschaft der Wissenschaften*, 1890, 32 pages. He is referred to as the "Bohemian Galileo" and a list is given of 16 of his works dealing with philosophy, geometry, physics, astronomy, and medicine. See also Marci a Kronland, *Liturgia mentis*, edited by J. J. W. Dobrzensky, Regensburg, 1678; F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, Prague, vol. 1, 1773; F. Hoefer, *Nouvelle biographie générale*, Paris, vol. 28, 1859; V. Láska, "Ueber Marcus Marci de Kronland," *Zeitschrift für Mathematik und Physik* (hist. liter. Abth.), vol. 35, 1890, pp. 1–3; Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2; D. E. Smith, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, 1917, p. 136; and *Enciclopedia Universel Illustrada Evropeo-Americana*, vol. 33, Barcelona, 192 0.

[2] *Labyrinthus in quo via ad circuli quadraturam pluribus modus exhibetur*, Prague, 1654. Compare the reference to Marci by Huygens, *Oeuvres Complètes de Christian Huygens*, vol. 12, 1910, pp. 97–98.

[3] There is a brief paragraph concerning "Jakob Kresa" in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 1. The name is spelled as in the text in M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 3, second edition, 1901, p. 12. It is here stated that while Kreza was in Madrid he translated into Spanish Euclid's Elements, books 1–6, 11, 12, which were published at Brussels in 1688; compare *Bibliotheca Mathematica*, series 3, vol. 12, p. 261, and P. Riccardi, *Saggio di una Bibliografia Euclidea*, Bologna, 1887, p. 50 of the first part. See A. Jemelka, ["On the life and works of the mathematician P. J. Kresa, S. J."] (in Bohemian), *Časopis pro pěst. matem.*, vol. 42, 1913, pp. 501–509.

eighteenth century and substituted for it the German language. A few years after the founding of the famous polytechnic school in Paris, the school of engineers in Prague was also changed by the Diet of the Bohemian Kingdom to a Polytechnic Institute of the type of the one in Paris. This was in 1806, that is, before the opening of the Polytechnic Institute in Vienna (1815).

In the eighteenth century, the leaders in the study of mathematics were Josef Stepling (1716–1778), author of many books on mathematics, astronomy and physics,[1] and Jan Tesánek (1728–1788)[2] who wrote a series of mathematical works, the commentaries on the celebrated work of Newton being noteworthy.[3]

At the end of the eighteenth century and at the beginning of the nineteenth there swept over Europe a wave of nationalism awakened by the grand ideas of the French revolution. Among the Czechs of the cultured class were idealists, who went to the huts of the peasants seeking the despised native language. They gave new life to the forgotten Czech literature, and in this way aroused national consciousness.

In the ranks of these idealists was a mathematician, professor at the University of Prague, Father Stanislav Vydra (1741–1804).[4] He wrote several mathematical books in Latin and German. In order to show that the Czechs had mathematical literature, he published in the year 1778 a *Historia matheseos in Bohemia et Moravia cultae*,[5] and in order to show that the language of the common people could serve also as a medium for the sciences, he wrote in Czech *Počâtkové arithmetiky* (Elements of Arithmetic) and an algebra which however was left incomplete.

In the nineteenth century the Czechs had the great task of establishing modern Czech mathematical literature, creating an adequate terminology and

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[1] Listed in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2. A life of Stepling by S. Vydra was published at Prague in 1779: *Vita ad modum reverendi ac magnifici viri J. Stepling.* See also F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, Prague, vol. 1, 1773; S. Vydra, *Historia matheseos in Bohemia et Moravia cultæ*, Prague, 1778; F. M. Pelzel, *Böhmische, mährische, und schlesische Gelehrte und Schriftsteller aus dem Order der Jesuiten*, Prague, 1786; J. G. Meusel, *Lexikon der von 1750 bis 1800 verstorbenen deutschen Schriftsteller*, Leipzig, vol. 13, 1815; *Oesterreichsche National-Encyklopädie*, Vienna, vol. 5, 1836.

[2] A common spelling of this name seems to be Tessánek. For authorities concerning his life, the works of Vydra, Pelzel (2), and Meusel, mentioned in the previous footnote, may be consulted. See also *Abhandlungen der königl. böhm. Gesellsch. d. Wiss.*, new series, vol. 4, 1837; F. J. Studnička, *Bericht über die mathematischen und naturwiss. Publicationen d. kön. Ger. d. Wissenschaften*, Prague, 1885, p. 11; *Vorlesungen über Geschichte der Mathematik*, vol. 4, ed. by M. Cantor, Leipzig, 1908; and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2.

[3] *Philosophiae naturalis principia mathematica auctore Isaaco Newtono illustrata commentationibus* . . ., Prague, 1780 and 1785.

[4] Compare "Stanislav Vydra," *Časopis pro pěst. matem.*, vol. 1, 1872, pp. 1–6, 49–54; A. Rybicka, "Stanislav Vydra," *Časopis pro pěst. matem.*, vol. 44, pp. 53–68, 103–119; *Vorlesungen über Geschichte der Mathematik*, vol. 4, ed. by Cantor, Leipzig, 1908; Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2; *Oesterreichsche National-Encyylopädie*, vol. 6, Vinena, 1837; and F. M. Pelzel, *Böhmische, mährische und schlesische Gelehrte und Schriftsteller aus dem Orden der Jesuiten*, Prague, 1786.

[5] This, together with Jan Smolík, *Mathematikové v Čechách*, is the principal source of material for this article in respect to early mathematics—Q. VETTER.

devoting themselves to original research. For this end it was well there were several distinguished foreigners as professors at the University and Polytechnic Institute of Prague. We may name the Milanese Bernard Bolzano,[1] also Professor Wilhelm Fiedler,[2] Carl Küpper[3] and Jakob E. C. Durège.[4] A fierce struggle had ever to be maintained against the unjust Germanization of the school system. Individual cities founded Czech secondary schools, without the help of the government and even against the government. Only after long political controversy did they succeed in dividing the Polytechnic in 1869 and the University in 1882 into two institutions, the Czech and the German, and only at the end of the last century was a second Czech Polytechnic founded at Brno. Not until today has the Austrian yoke been thrown off, and now it is possible to erect as many Czech schools as the nation needs. It is natural that with such conditions many Czech scientists of the older generation, educated in German schools, wrote in German in order to make themselves known in the international scientific world.

The Czech mathematicians applied themselves with all their might to the difficult problem concerning which I have already written. In the second half of the last century several excellent Czech mathematicians flourished. In the history of mathematics Jan Smolík deserves notice. It is a pity that he forsook this field to devote himself to numismatics, and of his book *Matematikové v Cechách* (Mathematicians of Bohemia) in 1863, he published only the first part which concludes with the beginnings of the eighteenth century.

In foreign countries perhaps the best known are Emil and Eduard Weyr, two brothers. Emil, the older (1848–1894), having taken a journey through Italy for the purpose of study, met almost all the best mathematicians of the time; especially noteworthy was his friendship with L. Cremona, with whom he studied at Milan. On his return home he was made professor at the Polytechnic Institute of Prague and later at the University of Vienna.[5] Emil Weyr wrote many

[1] I have not been able to verify that there was any "Milanese" mathematician of this name. Of course the catholic theologian and philosopher Bernhard Bolzano (1781–1848) was born, and died, at Prague and was professor of the philosophy of religion in the University there for many years. He is the author of a number of well-known mathematical writings.—After this note was in type Dr. Vetter informed me that he wished the clause to read: "We may name Bernard Bolzano of an Italian family" . . . ; Bolzano's father was a native of Northern Italy.

[2] Otto Wilhelm Fiedler, born in Saxony in 1832, was professor of geometry at the Polytechnic Institute of Prague. He left there for a similar position in the Polytechnikum at Zürich where he remained till his death in 1912. He is chiefly known to Americans as the genial translator and editor of the German editions of George Salmon's works, and as the author of *Cyklographie oder Construction der Aufgaben über Kreise und Kugeln*, Leipzig, 1882.

[3] Carl Joseph Küpper (1828–1900), professor of geometry at the Polytechnic Institute, Prague. An obituary notice (accompanied by a portrait) with a list of his writings, by E. Waelsch, appeared in *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 14, 1905, pp. 389–394. See also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 4.

[4] Apparently instead of the initial "E." should be H. since Jacob Heinrich Karl Durège (1821–1893) was professor in the Polytechnic Institute, Prague, before 1869 when he went to Vienna. He lived in the United States 1851–57 and his German work on the theory of functions of a complex variable has been made familiar through the English translation (Philadelphia, 1896) of the late G. E. Fisher (see this MONTHLY, *1920*, 239) and I. J. Schwatt.

[5] Emil Weyr was born in Prague and was appointed professor of mathematics in the Univer-

important books on geometry, not only in Czech and in German, but also in French and Italian; his works brought him international fame. Projective geometry is indebted to him for many important discoveries. To Italians it may be interesting to know that he translated into Czech the memoirs of L. Cremona.[1] Eduard Weyr (1852–1903)[2] was also professor at the Czech Polytechnic Institute of Prague and not only enriched Czech mathematical literature but published articles in Italian, French, and German periodicals and made a name for himself which was respected in foreign scientific circles. Analytical geometry and algebra were the objects of his study.

Both brothers were active in the "Jednota českých mathematiků a fysiků" (Union of Czech mathematicians and physicists), the center of Czech mathematical life; during half a century this union has published a journal, *Časopis pro pěstování matematiky a fysiky* (Journal for the study of mathematics and physics), and several scientific manuals and text-books for secondary schools.

The most prolific writer among Czech mathematicians was František Josef Studnička (1836–1903),[3] the first professor of mathematics in the Czech University of Prague, where he taught for twenty years. A group of books and many memoirs came from his pen. His interest in science was very great; he wrote several books on meteorology. He was greatly interested in Czech terminology and tried to fill the gaps in Czech scientific literature. For that purpose he compiled the first Czech manuals of the different mathematical theories. To him belongs the credit for advancing history of mathematics at this time; he not only wrote several historical articles and published the works of the early mathematicians but also awakened among his pupils an interest in history. Teaching in secondary schools also attracted his attention, as we see in his memoirs and text-books. His greatest work however was that done in research in determinants.[4] Descriptive geometry was studied in Bohemia[5] with success. Of the

sity of Vienna in 1875. For material regarding his life and work the following sources may be consulted: *Časopis pro pěst. matem.*, vol. 24, 1895, pp. 163–244, by A. Pánek; *Monatshefte für Mathematik und Physik*, Vienna, vol. 6, 1895, pp. 1–4, by G. Kohn (Italian translation in *Rendiconti del Circolo Matematico di Palermo*, vol. 9, pp. 260–262); *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 4, 1897, pp. 24–33, by G. Kohn; F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, pp. 365, 435 f.; *Ziva. Sbornik vědecký musea kralovství Českého*, Prague, 1894, 14 pages, by A. Sucharda; and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[1] By "memoirs" is meant, possibly, the Czech translation of "Introduzione ad una teoria geometria d. curve piane" (Bologna, 1862); and of "Sulle transformazione geometriche delle, figure piane" (Bologna, 1862–1865), the Czech translation of the latter appearing it 1872.

[2] A biography and chronological list of writings of Eduard Weyr, by K. Petr and J. Sobotka is given in *Časopis pro pěst. matem.*, vol. 34, 1905, pp. 457–516. See also Poggendorff's *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[3] A sketch of his life and work, by A. Pánek, is given in *Časopis pro pěst. matem.*, vol. 33, 1904, pp. 369–480; see also vol. 32, 1903, p. 297. A fairly complete list of his writings is given in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[4] For an account of this, see T. Muir, *The Theory of Determinants in the Historical Order of Development*, vol. 3, 1861–1880, London, 1920, various references in the index.

[5] Concerning descriptive geometry and its literature in Bohemia up to 1883 one may consult two works by Vaclav Lavička: (1) *Historie deskriptivní geometrie*, Prague, 1878; (2) *Deskriptiva geometrie ze stanoviště historickopaedagogického, Sešit prvý* (Descriptive geometry from the historical pedagogical standpoint, part 1), Pardubitz, 1883.

Czech professors of this subject at the Polytechnic Institute of Prague,[1] the earliest is Rudolf Skuherský (1821–1863).[2] After the division of the Institute, in the Czech section František Tilšer (1825–1913) [3] was the first professor of descriptive geometry. He worked on the theory of illumination and he is author of a special theory, the theory of "ikonognosie," [4] in which he tries to connect descriptive geometry with philosophy. Karel Pelz (1845–1908) [5] is rather well known in foreign mathematical circles. His works excel in clever solutions of construction problems. The most celebrated of his works are those on axonometric projection.

Of other mathematicians now dead, I shall cite only the professors of both the Czech polytechnic institutes at Prague and Brno, Gabriel Blážek,[6] Augustin Pánek (1843–1908),[7] Karel Zahradnik (1848–1916),[8] who translated into Czech from Italian the works of Bellavitis,[9] Václav Řehořovský (1849–1911),[10] Anton Sucharda (1854–1906)[11] and František Velísek,[12] (1877–1914), who was killed in the Great War.

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[1] For biographies of professors in this Institute the best work is A. V. Velflík, *Dějiny technikého učeni v Praze* (History of the Polytechnic Institute in Prague), vol. 1, 1906.—Q. Vetter.

[2] R. Shuherský was born in 1828, not 1821. He was professor at the Institute 1854–1863. For sketches of his life and work see F. J. Obenrauch, *Geschichte der darstellenden und projectiven Geometrie*, Brno, 1897, p. 353 f.; G. Loria, *Storia della Geometria Descrittiva dalle origini sino ai giorni nostri*, Milan, 1921, pp. 330, 340–341; and *Das ständisch-polytechnische Institut zu Prag. Programm zur fünfzigjährigen Erinnerungs-feier*. Ed. by C. Jelinek, Prag, 1856, pp. 250–251. Compare also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 2–3.

[3] A sketch of his life and work is given in G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, pp. 335–339, etc., and by B. Procházka in *Časopis pro pěst. matem.*, vol. 48, 1914, pp. 1–25. He was professor of geometry and astronomy at the Institute, 1864–1895. See also F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, p. 353 f., and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 3.

[4] "Grundlagen der Ikonognosie. I. Abteilung" by F. Tilšer, *Abhandlungen der königl. böhmischen Gesellschaft*, 1878, pp. 1–88.

[5] His work is discussed rather fully in G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, pp. 371–379, etc.; by J. Sobotka in *Časopis pro pěstov. matem.*, vol. 39, pp. 433–460; F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, p. 355 f. See also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3–4.

[6] Born 1842; appointed ordinary professor in the Bohemian Polytechnic Institute, Prague, 1872.

[7] Professor of higher mathematics at the Bohemian Polytechnic Institute 1896–1908. Editor, 1884–1908, of *Časopis pro pěst. matem.*, also of the mathematical part of the Bohemian encyclopædia *Ottův Slovnik Naučný*. For a sketch by K. Petr, with a portrait, see *Časopis pro pěst. matem.*, vol. 41, 1912, pp. 1–8.

[8] Karel Dragutin Zahradnik, a prolific writer of German and Bohemian mathematical papers, especially in the field of geometry. A critical biography with a list of his writings, by J. Vojtěch, appeared in *Časopis pro pěstov. matem.*, vol. 46, 1917, pp. 235–303.

[9] That is, G. Bellavitis, *Methoda equipollencé čili rovnic geometrických*, Prague, 1874. In the same year was published a French translation by C. A. Laisant.

[10] An obituary notice of V. K. Rehořovský, by J. Sobotka, appeared in *Časopis pro pěst. matem.*, vol. 42, 1913, pp. 129–145.

[11] An obituary notice by K. Rychlík appeared in *Časopis pro pěst. matem.*, vol. 51, 1922, pp. 247–248.

[12] Professor of mathematics in the Polytechnic Institute, Brno, 1900–1907; he died February 19, 1907, according to *Bibliotheca Mathematica*, series 3, vol. 7, p. 424. He was the constructor of various types of models of cubic surfaces (see W. Dyck, *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, Munich, 1892, pp. 299–300.) A biography by J. Sobotka appeared in *Časopis pro pěst. matem.*, vol. 37, 1908, pp. 353–359. A number of papers by Sucharda are listed in Poggendorff's *Biographisch-literarisches Handwörterbuch*, vols. 3–4. See also G. Loria, *Storia delle Geometria Descrittiva*, Milan, 1921, pp. 357, 534, 547.

Today we have a group of mathematicians in the Czech universities of Prague and Brno and in the polytechnic institutes of these cities. Professor Lerch [1] is known for his works on mathematical analysis. Václav Láska,[2] originally a geodesist and professor of applied mathematics and physics, is interested also in geophysics, and in the philosophy, history and teaching of mathematics. Professor Jan Sobotka[3] is a geometrician known also beyond the limits of Bohemia. Professor Karel Petr[4] has given his attention to analysis and arithmetical questions. Geometry is the specialty of Professors Bohumil Bydžovský, Jan Vojtěch, Ladislav Seifert, and Bohuslav Hostinsky who now is devoting himself to applied mathematics and theoretical physics. Professor Karel Rychlík gives his time to algebra, Professor Josef Beneš and Doctor Emil Schoenbaum to actuarial mathematics. The theory of functions is the special branch of Dozent Miloš Kössler. At the Polytechnic Institute of Prague are also Professors Matthias Vaněček [5] and Josef Klobouček, and Dozenten František Rádl and Karel Dusl. Professors Čenek Jarolímek [6] and Bedřich Procházka[7] have written several books

[1] Matyáš Lerch, born in Bohemia in 1860, professor of mathematics at the University of Freiburg, Switzerland, from 1896 till his appointment as professor in the Polytechnic Institute at Brno in 1906. Biographical notes may be found in *Annuario Biografico del Circolo Matematico di Palermo*, 1914, and in *Acta Mathematica 1882–1912, Table Générale des tomes 1–35*, Upsala, 1912, where there is also a portrait. For long lists of his writings see Royal Society of London, *Catalogue of Scientific Papers*, vol. 16; *Revue Semestrielle des Publications Mathématiques;* and Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4. The Academy of Sciences of the Institute of France awarded him in 1900 the grand prize for mathematical sciences. He died August 3, 1922.

[2] That is Václav Jan (=Wenzel Johann) Láska, professor of higher geodesy and astronomy, and director of the astronomical, meteorological and seismological observatory at the University of Lemberg, 1899–1911. He is best known to American mathematicians by his *Sammlung von Formeln der reinen und angewandten Mathematik*, Braunschweig, 1889–1894, 1090 pages. His other books and numerous papers written in German and Bohemian and published before 1904, and dealing with topics in mathematics, theoretical astronomy and mathematical geography, surveying, and geodesy, are listed in "Poggendorff," in Royal Society of London, *Catalogue of Scientific Papers*, vol. 16, and in *Revue Semestrielle des Publications Mathématiques*.

[3] Jan Sobotka was born in Bohemia in 1862. He was professor of descriptive geometry in the Technical Institute of Brno from 1899 till 1904 when he was appointed professor in the Bohemian University of Prague. See G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, various references in index; *Annuario Biografico del Circolo Matematico di Palermo*, 1914; Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4, and *Revue Semestrielle des Publications Mathématiques*. His *Deskriptivní Geometrii promítání parallelního*, vol. 1, was published in 1906 by the Society of Czech mathematicians as vol. 10 of its *Sbornik*. A biography and list of his writings, by F. Kadeřávek, appeared in *Časopis pro pěst. atem.* vol. 52, 1922, pp. 1–9 + portrait.

[4] K. Petr, professor of mathematics at the Bohemian University of Prague since 1903, was born in Bohemia in 1868. For a list of his papers see Royal Society of London, *Catalogue of Scientific Papers*, vol. 17; Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4; and *Revue Semestrielle des Publications Mathématiques*.

[5] Born 1859.

[6] Author of articles listed in Royal Society of London, *Catalogue of Scientific Papers*, vol. 16, and in *Revue Semestrielle des Publications Mathématiques*. His treatise on descriptive geometry was a three-volume work first published in 1875–1877 by the Jednota českých mathematiků. During the next thirty years it was almost the only text used in Bohemian Real Schools. It was translated into Bulgarian, by Šourek, for use in secondary schools. See G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921. The Christian name "Čenek" is the same as "Vincenc." Since the above was in type Dr. Vetter furnished me with the following information: Professor Jarolímek was born in 1846 and died in 1921. He was professor at the Bohemian Polytechnic

on descriptive geometry. Professors Miloslav Pelíšek[1] and František Kadeřávek (b. 1885), and Dozent Josef Kaunovsky (b. 1881) teach this branch also. At the University of Prague, philosophy of mathematics is taught by Professor Karel Vorovka[2] (b. 1879) and the history of mathematics by the author of this article (b. 1881).

SOME CURIOUS FALLACIES IN THE STUDY OF PROBABILITY.

By ROBERT E. MORITZ, University of Washington.

Part II.

The Petersburg paradox[3] discussed in Part I consists in the conclusion that under the following conditions Paul's expectation is, by elementary rules, infinite. A coin is tossed until head turns up. If this happens on the first trial Peter is to pay Paul 1 crown, if on the second trial 2 crowns, if on the third 4 crowns, and so on, the sum Peter is to pay Paul when head finally does turn up being double the sum he should have paid had head turned up on the immediately preceding trial.

It is not necessary to recount the earlier attempts that were made to explain the Petersburg paradox. This has been done by the historians of the science of probability, by Todhunter in his *History of the Theory of Probability* from Pascal to Laplace by Czuber in *Grunert's Archiv*, volume 67, and by Netto in the fourth volume of Cantor's *Geschichte der Mathematik*. To the modern reader many of these attempts seem puerile if not ridiculous, as for instance Cramer's attempt to explain the paradox by considering all sums greater than 2^{24} as practically equal, or D'Alembert's distinction between physical and metaphysical possibilities and his contention that some such expression as $1/[2^n(1 + \beta n^2)]$ or $1/(2^n + 2^{an})$ rather than $1/2^n$ must be accepted as the probability that head turns up for the first time on the nth trial, or Bequelin's conclusion that the oftener a chance event has happened the less likely it is to happen on the next trial, or Buffon's doctrine that any probability less than 10^{-4} must be considered absolutely zero.

It is, however, not without interest to consider some of the explanations offered by more recent writers.

Institute, 1907–1915. A biography by J. Sobotka, with two lists of writings, is given in *Časopis pro pěst. matem.*, vol. 45, pp. 439–449, and vol. 51, 1922, pp. 67–68.

[1] Author of articles listed in Royal Society of London, *Catalogue of Scientific Papers*, vol. 7, and in *Revue Semestrielle des Publications Mathématiques*. M. Pelíšek was born in 1855; see Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4. Professor Procházka's Christian name "Bedřich" is the same as "Friedrich."

[2] Heft 13 (89 pages, Vienna, 1914) of the report on mathematical instruction in Austria to the International Commission on Mathematical Instruction was by K. Vorovka, L. Červenka and V. Posejpal, with a preface by J. Sobotka. The report is entitled *Die Lehrbücher für Mathematik, darstellende Geometrie und Physik an den Mittelschulen mit böhmischer Unterrichtssprache*, and contains considerable historical material. Other reports of this series give an appreciable amount of information concerning mathematics in Bohemian institutions; for example: *Der Unterricht in der darstellenden Geometrie an den Technischen Hochschulen* by E. Müller (Heft 9); *Der mathematische Unterricht an den Universitäten* by R. v. Sterneck (Heft 7); *Der mathematische Unterricht an den Technischen Hochschulen* by E. Czuber (Heft 5). For other publications of Professor Vorovka see *Revue Semestrielle des Publications Mathématiques*.

[3] For numerous references, see E. Czuber's article *Wahrscheinlichkeitsrechnung* in *Encyclopädie der Mathematischen Wissenschaften*, I D 1, Section 17. Translated and amplified by J. Le Roux in *Encyclopédie des Sciences Mathématiques*, I 20.

Bachelier[1] discusses the paradox but makes no effort to explain it. "Common sense," he says, "can not be invoked to settle delicate mathematical questions. Common sense is unable to determine whether the area between a curve and its asymptote is finite or infinite, or whether a series is convergent or divergent. In the Petersburg problem the stake which Paul is to receive if he wins at any one trial is double the stake he would have received had he won at the preceding trial; if the multiplier had been 1.999 instead of 2 his expectation would have been finite, for the series

$$\frac{1}{2}+\frac{1.999}{2^2}+\frac{(1.999)^2}{2^3}+\frac{(1.999)^3}{2^4}+\cdots$$

is convergent. Common sense, however, is unable to distinguish the difference."

This of course does not explain why there is such a discrepancy between common sense and the results of logical reasoning. Nor is the illustration very fortunate, for the sum of the above series is 1000, and common sense would forbid Paul to venture 1000 crowns, or any other considerable sum, for what he might reasonably expect to win on accepting the conditions of the play. The paradox consists not in that the expectation is infinite, but in that it is larger than common sense can admit it to be.

Czuber[2] takes the view that the explanation of the paradox is to be sought in the conditions of the problem, which specify a play which may never end and which may require Peter to pay out a sum greater than any possible sum. These are mere words, he says, without a real meaning. If the problem is to have a real meaning it must be limited to a definite maximum number of trials, and this limitation must be such that Peter can actually meet all possible losses. When the problem is thus modified no objection can be raised against the resulting expectation of Paul.

Czuber then goes on to show that if the play were limited to n trials with the understanding that if head does not turn up during the first n trials Peter must give Paul 2^n crowns, since that is the least sum Paul could win if the play were continued indefinitely, Paul's expectation would be

$$\frac{1}{2}+\frac{2}{2^2}+\frac{2^2}{2^3}+\frac{2^3}{2^4}+\cdots+\frac{2^{n-1}}{2^n}+\frac{2^n}{2^n}=\frac{n}{2}+1.$$

Suppose that Peter had a million crowns, the maximum loss that he could pay would then be $2^{19} = 524{,}288$ crowns. It would therefore be necessary to limit the play to 19 tosses[3] and Paul's expectation would be only 10.5 crowns.

Now neither of the objections which Czuber raises, namely that the Petersburg problem is necessarily meaningless because it deals with a play that may never end, and that may require Peter to pay out a sum greater than he can possibly possess, objections which, by the way, were previously raised by both

[1] L. Bachelier, *Calcul des Probabilités*, vol. 1, Paris, 1912, p. 26.

[2] E. Czuber, *Wahrscheinlichkeitsrechnung*, Leipzig, 1903, p. 203; also in *Grunert's Archiv*, vol. 67.

[3] Czuber says 20, but that is evidently an error.

Poisson and Buffon, are valid objections. For nothing can prevent us from assuming that Peter has unlimited wealth, or that instead of playing for real money the players play for stage money, or counters, or marks of ink on paper to symbolize money. Nor does it follow that the expectation must be meaningless because of the fact that the play may never end, for there are many problems which impose this condition in which the resulting expectation is perfectly compatible with common sense. Let us assume that Peter is to pay Paul one crown as soon as head turns up for the first time. Then, clearly, this play also may never end, and Paul's expectation is

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = 1$$

which is in perfect agreement with common sense. The conditions tó which Czuber objects do not, therefore, necessarily imply a paradox.

De Morgan in his admirable *Essay on Probabilities* (London, 1838), like Condorcet before him, held that the Petersburg problem offered really no paradox and tried to show that the conclusion that Paul's expectation is infinite is consistent with the results of experience. He cited the results of Buffon's experiment which consisted of 2,048 sets of tosses of a coin. In his *Formal Logic* he gave the results of another 2,048 sets of trials made by one of his own pupils. In the combined number of 4,096 sets of trials head showed on the first toss in 2,109 sets, on the second in 1,001 sets, on the third toss in 480 sets, and so on. The complete results are exhibited in the following table.

	Results of Trials.		Theoretical Results.	
	Games Won.	Amount Won.	Games Won.	Amount Won.
On 1st toss	2,109	2,109	2,048	2,048
" 2d "	1,001	2,002	1,024	2,048
" 3d "	480	1,920	512	2,048
" 4th "	236	1,888	256	2,048
" 5th "	127	2,032	128	2,048
" 6th "	67	2,144	64	2,048
" 7th "	42	2,688	32	2,048
" 8th "	17	2,176	16	2,048
" 9th "	11	2,816	8	2,048
" 10th "	3	1,536	4	2,048
" 11th "	1	1,024	2	2,048
" 12th "	0	0,000	1	2,048
" 13th "	0	0,000		
" 14th "	1	8,192		
" 15th "	0	0,000		
" 16th "	1	32,768		
Total	4,096	63,295	4,096	24,576
Average per game		15.4		6

According to the theory, Paul in playing 4,096 games could have expected to win an average of 6 crowns per game; the actual trials gave him an average of 15.4 crowns per game. Buffon's 2,048 sets of trials would have given 5 crowns

per game as against the theoretical result of 5.5 crowns per game. De Morgan, in discussing the results of Buffon's trials, calls attention to the significant fact that the rarer occurrences contributed most to the total amount. "If Buffon had tried a thousand times as many games, the results would not only have given more, *but more per game.* The larger net would have caught, not only more fish, but more varieties of fish; in two million of sets, we might have expected to have seen cases in which head did not appear till the twentieth throw. . . . Thus the reader may readily perceive that the player might continue until he had realized not only any given sum, but any given sum *per game;* a result which is indicated by the application of our rule, when it tells us that the mathematical expectation of the player upon a single game is infinite."

Theoretically, as is obvious by inspecting the two right-hand columns in the foregoing exhibit, 2^n games should yield on an average $n/2$ crowns per game. To secure an average of 18 crowns per game would therefore require 2^{36} games, a number of games exceeding the number of seconds in the Christian era. Theoretically the games in which head turns up for the first time on the twelfth toss contribute exactly the same sum toward the total amount as the games in which head turns up at the first trial. De Morgan, therefore, seems to have failed to grasp the real significance of his own observation to the effect that the rarer occurrences contributed more to the total amount won than did the more frequent occurrences, and overlooked completely the even more significant fact that, contrary to the general rule, the average (15.6) attained from 4,096 sets of trials deviated more from the theoretical average (6) than did the average (5) resulting from Buffon's 2,048 sets from the corresponding theoretical average (5.5).[1] This fact points to a greater and greater deviation from the theoretical average as we increase the number of trials. If this should be the case, the mathematical expectation, as computed by the general formula derived for a finite number of contingent prospects, would be meaningless when applied to the Petersburg problem.

It is a cause of no little surprise that this observation has escaped the attention of all commentators on this subject. The mathematical expectation of a contingent prospect is by definition a mean value. It is not the value of any single hazard but the average value of a sufficiently large number of equal hazards. The mathematical expectation of one chance out of a thousand to secure a billion dollars is one million dollars, but this does not mean that anyone in his senses would pay a million dollars for a single chance of winning the billion dollars; a million dollars is the equivalent of such a chance only if the risk can be repeated sufficiently often to secure an average. If on repeating a risk sufficiently often, or it may be indefinitely, an average can not be secured, the formula for the mathematical expectation loses its significance. Now this is exactly what happens in the Petersburg problem. Let us suppose that n is a number so large that

[1] In his *Budget of Paradoxes* (Original Edition, p. 170) De Morgan cites the results of two other sets of 2,048 trials each. The combination of these results with those already cited, that is, of 8,192 trials in all, would yield an average of 14.6 crowns per game as against a theoretical average of 6.5 crowns per game.

n games will secure an average. Every game that is played involves the prospect of head not showing up till the nth toss of the coin, the probability of this prospect is $1/2^n$; this particular prospect will therefore be realized on an average but once in 2^n games. But this is contrary to the assumption that n games will secure an average. The foregoing reasoning is valid even if n is assumed equal to ∞, for each game involves events whose probability is $1/2^\infty$, and such events will occur on an average but once in 2^∞ games. The conjecture, occasioned by Buffon's and De Morgan's experimental results, that the formula for the mathematical expectation when applied to the Petersburg problem leads to a meaningless result is thus verified.

Whitworth[1] ignores the infinite result given by the formula for the mathematical expectation and proceeds to determine the unknown sum x which Paul might risk on the play on the condition that if the play were repeated each time on a scale proportional to his capital at that time, he would in the long run be left neither richer nor poorer. His procedure is in substance as follows:

Let x denote the sum which Paul with an initial capital of C may pay for a chance p of winning a prize P. If he wins, his capital will be $C + P - x$, if he loses $C - x$, so that the effect of winning is to multiply his capital by the factor $(C + P - x)/C$, that of losing to multiply it by $(C - x)/C$.

If Paul repeats the venture n times, n being large enough to secure an average, he will win pn times and lose qn times, where $p + q = 1$. On the condition that the sum he risks in any one venture is proportional to the capital he then has, the result of the n operations will be to multiply his initial capital by the factor

$$\left(\frac{C + P - x}{C}\right)^{pn}\left(\frac{C - x}{C}\right)^{qn}.$$

Since now the result of the n operations is to leave Paul neither richer nor poorer, this factor must equal unity, hence also

$$\left(\frac{C + P - x}{C}\right)^{p}\left(\frac{C - x}{C}\right)^{q} = 1.$$

This equation immediately suggests the corresponding equation in which x denotes the sum which Paul may pay for the chance of winning one of k prizes $P_1, P_2, \cdots, P_k$, with the respective probabilities $p_1, p_2, \cdots, p_k$ of winning them, where $p_1 + p_2 + \cdots + p_k = 1$. The condition that in the long run Paul is neither richer nor poorer becomes

$$\left(\frac{C + P_1 - x}{C}\right)^{p_1}\left(\frac{C + P_2 - x}{C}\right)^{p_2}\cdots\left(\frac{C + P_k - x}{C}\right)^{p_k} = 1.$$

This equation does not admit of a general solution, but if *x is small as compared with C* it may be approximated to any desired degree of accuracy. As

[1] W. A. Whitworth, 5th edition, Cambridge, 1901, p. 243.

the result of such approximation, Whitworth gives without proof

$$x = \frac{\left(1+\frac{P_1}{C}\right)^{p_1}\left(1+\frac{P_2}{C}\right)^{p_2}\left(1+\frac{P_3}{C}\right)^{p_3}\cdots - 1}{\frac{p_1}{C+P_1}+\frac{p_2}{C+P_2}+\frac{p_3}{C+P_3}+\cdots}.^{1}$$

In the Petersburg problem the prizes are $1, 2, 2^2, 2^3, \cdots$, and the respective chances of winning them are $(1/2), (1/2)^2, (1/2)^3, (1/2)^4, \cdots$, so that in this case the equation for determining x becomes

$$\left(\frac{C+1-x}{C}\right)^{1/2}\left(\frac{C+2-x}{C}\right)^{1/4}\left(\frac{C+4-x}{C}\right)^{1/8}\cdots = 1,$$

and the approximate solution

$$x = \frac{\left(1+\frac{1}{C}\right)^{1/2}\left(1+\frac{2}{C}\right)^{1/4}\left(1+\frac{4}{C}\right)^{1/8}\cdots - 1}{\frac{1}{2(C+1)}+\frac{1}{4(C+2)}+\frac{1}{8(C+4)}\cdots}.$$

From this formula Whitworth finds that when $C = 8$, $x =$ about 3.8; when $C = 32$, $x =$ about 4; and when $C = 1{,}024$, $x =$ about 6, results which are in remarkable close agreement with those obtained from Bernoulli's hypothesis of moral expectation.[2]

However much Whitworth's highly ingenious solution may compel our admiration, it can not be considered a satisfactory solution of the Petersburg problem for at least three reasons.

First, it introduces a new condition into the problem, namely, that the price which Paul may pay for the advantage which Peter offers him is to be such that if the play were repeated each time on a scale proportionate to his funds at that time, he would in the long run come out even. Now it can be readily shown that any player who repeats a play, which is otherwise fair, on a scale proportionate to his capital, is sure to lose in the long run, therefore to stipulate that the x which Paul is to pay for his advantage is to be such as to leave him neither richer nor poorer if he repeats the play in this manner, is equivalent to asking odds in favor of Paul.

Suppose a player tosses a coin staking each time one half of what he possesses on the chance of heads turning up. If he wins on the first toss, he will have increased his funds by one half; if he tosses again and loses, he will have diminished

[1] This seems to be incorrect. The first approximation to x is

$$\frac{\log_e\left[\left(1+\frac{P_1}{C}\right)^{p_1}\left(1+\frac{P_2}{C}\right)^{p_2}\cdots\right]}{\frac{p_1}{C+P_1}+\frac{p_2}{C+P_2}+\cdots}.$$

[2] The correct approximation formula given in the preceding footnote yields the values $x = 2.85, 3.45, 6.28$ for $C = 8, 32, 1{,}024$ respectively.

his funds by one half, so that he will have left three fourths of what he originally possessed. If he loses the first toss, he will have left one half of what he had; if he wins the second toss, he will increase this half by one half of itself, so that again he will have left only three fourths of what he originally possessed. Suppose now that he continues to play until he has secured an average, that is, until he has won as often as he has lost, say $2n$ games in all. He will then have lost n games and won n games. Each game he loses will diminish his fortune by one half, each game he wins will increase it by one half. The result is that in the end he will have his original fortune multiplied by the factor $(1/2)^n(3/2)^n$, that is by $(3/4)^n$. If $n = 10$, $(3/4)^n$ is less than one sixteenth, so that, although the game is fair in every other respect and the number of games sufficiently large to secure an average, the player will have lost more than 15/16 of his original fortune.

It is commonly supposed that a player must necessarily win in the long run provided the odds are in his favor. This itself is a fallacy, as the following illustration will show. Suppose that a player plays for even stakes, always staking one half of his funds on a hazard in which the odds are 3 to 2 in his favor, that is, such that, in the long run, he will win 3 times out of 5. Suppose that he plays $5n$ games, n being sufficiently large to secure an average. He will then have won $3n$ games and lost $2n$ games. Each game he wins increases his fund by one half, each game he loses diminishes his fortune by one half. If his initial fund be denoted by C, his fund after playing $5n$ games will be

$$(1/2)^{2n}(3/2)^{3n}\cdot C = (27/32)^n\cdot C$$

which is certainly less than C. If $n = 10$, $(27/32)^n$ is less than 1/5, so that though the odds are 3 to 2 in favor of the player, he will have lost more than 4/5 of his original fortune.

In the second place it is obvious that the condition which Whitworth imposes may be replaced by any one of an infinite number of similar conditions and we should thus be led to an infinite number of different solutions of the problem. Instead of repeating the play on a scale proportionate to his funds at the time, the price Paul should pay might be determined on the condition that, if he repeated the play on some increasing or decreasing scale, he should come out even in the long run, the only essential consideration being that the scale be such that he may continue the play indefinitely without exhausting his fund.

Third, granting that the value of x fairly represents Paul's advantage, it remains to find x. Whitworth approximates the value of x on the assumption that x is small as compared with C, that is, so small that the second and higher powers of the quantity x/C, which enter into the expansion of the infinite product, may be neglected. This begs the question. In other words, Whitworth finds that x is small as compared with C on the assumption that x is small as compared with C.

The view which has been advanced in this paper, that the rule for the mathe-

matical expectation gives a meaningless result when applied to the Petersburg problem, explains the paradox but does not solve the problem. Indeed it may be safely asserted that the problem does not admit of solution without the aid of assumptions not warranted by the expressed conditions of the problem.

THE ORIGIN OF THE SYMBOLS FOR "DEGREES, MINUTES AND SECONDS."

By FLORIAN CAJORI, University of California.

At the present time there is a conflict of views regarding the origin of the symbol $^\circ$ for "degrees" in angular measure. Edward O. von Lippmann[1] gives $\overset{\circ}{\mathrm{M}}$ as the symbol for the Greek word μοῖραι standing for parts, including parts of circular periphery, and then adds, "woraus wohl das Zeichen $^\circ$ für Kreisgrad entstand" (from which doubtless originated the sign $^\circ$ for circular degree). On the other hand, J. Tropfke[2] assigns the origin of $^\circ$, not to Greek antiquity, but to the sixteenth century, and connects it, not with Greek symbolism, but with the exponential concept as applied to different units in the sexagesimal system. Tropfke cites L. Schoener (1586) as the earliest author he could find who used exponents for geometric series, thus:

$$\begin{array}{ccccccc} \text{III} & \text{II} & \text{I} & 0 & \text{I} & \text{II} & \text{III} \\ 216000. & 3600. & 60. & 1 & \frac{1}{60} & \frac{1}{3600} & \frac{1}{216000} \end{array}$$

and for sexagesimal integers and fractions, thus:

$$\begin{array}{cccccc} \text{"IIae} & \text{Iae} & 0 & \text{I} & \text{II} & \text{III"} \\ 3. & 15. & 7. & 50. & 34. & 23. \end{array}$$

As between Lippmann and Tropfke, the researches I have made decidedly favor the general conclusions of Tropfke, but I have found earlier dates than his for the first appearance of the signs $^\circ\ '\ ''$.

In Athelard of Bath's translation into Latin (twelfth century) of certain Arabic astronomical tables, the names signa, gradus, minutæ, secundæ, etc., are abbreviated. The contractions are not always the same, but the more common ones are "Sig.," "Gr.," "Min.," "Sec."[3] No symbolism is used here in the designation of angular measure. Neither here nor elsewhere have I found indications of notations in support of Lippmann's explanation of the origin of the sign $^\circ$. In the Alfonsion Tables[4] one finds marked by $49^{\ddot{2}}\ 32^{\ddot{1}}\ 15^{\ddot{m}}\ 4^{\ddot{2}}$ the sexagesimals $49 \times 60 + 32 + 15 \times \frac{1}{60} + 4 \times \frac{1}{60^2}$. It was during the six-

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[1] E. O. von Lippmann, *Entstehung und Ausbreitung der Alchemie*, Berlin, 1919, p. 353.

[2] Johannes Tropfke, *Geschichte der Elementar-Mathematik*, 2. ed., Leipzig, vol. 1, 1921, p. 43.

[3] See H. Suter, *Die Astronomischen Tafeln des Muḥammed ibn Mūsā Al-Khwārizmī in der Bearbeitung des Maslama ibn Aḥmed Al-Madjriti und der latein. Uebersetzung des Athelard von Bath.* Copenhagen, 1914, p. xxiv.

[4] A. Wegener in *Bibliotheca mathematica*, series 3, vol. 6, 1905, p. 179.

teenth century that the modern notation for degrees, minutes and seconds took form.

Oronce Fine[1] in 1535 used for them the marks *grad*, $\bar{m}$, $\bar{2}$, $\bar{3}$, ; Gemma Frisius[2] in 1540 wrote

Integr.	Mi.	2.	3.	4.	
36.	30.	24	50	15	for our modern $36^\circ\ 30'\ 24''\ 50'''\ 15^{iv}$,

but in the revised 1569 edition of his book published in Paris there is an appendix on astronomical fractions, due to J. Peletier (dated 1558), where one finds,

"Integra,	Mi vel	$\bar{1}$,	$\bar{2}$,	$\bar{3}$,	$\bar{4}$,	$\bar{5}$,	$\bar{6}$,	$\bar{7}$,	$\bar{8}$,	etc."
$\circ$		1,	2,	3,	4,	5,	6,	7,	8.	

This is the first appearance that I have found of $^\circ$ for integra or degrees. It is explained that the denomination of the product of two such denominate numbers is obtained by combining the denominations of the factors; minutes times seconds give thirds, because $1 + 2 = 3$. The denomination $^\circ$ for integers or degrees is necessary to impart generality to this mode of procedure. "Integers when multiplied by seconds make seconds, when multiplied by thirds make thirds" (folio 62, 76). It is possible that Peletier is the originator of the $^\circ$ for degrees. But nowhere in this book have I been able to find the modern angular notation $^\circ\ '\ ''$ used in writing angles. The $^\circ$ is used only in multiplication. The angle of 12 minutes and 20 seconds is written "S.o.$\bar{g}$.0 $\bar{m}$.12.$\bar{2}$.20." (folio 76).

Twelve years later (1670) one finds in a book of Johann Caramuel[3] the signs $^\circ\ '\ ''\ '''\ ^{iv}$ used in designating angles. In 1571 Erasmus Reinhold[4] gave an elaborate explanation of sexagesimal fractions as applied to angular measure and wrote "$^\circ 63\ '13\ ''53$," also "$62^\circ\ 54'\ 18''$." The positions of the $^\circ\ '\ ''$ are slightly different in the two examples. This notation was adopted by Tycho Brahe[5] who in his comments of 1573 on his *Nova Stella* writes $75^\circ\ 5'$, etc.

As pointed out by Tropfke, the notation $^\circ\ '\ ''$ was used with only minute variations by L. Schoener (1586), Paul Reesen (1587), Raymarus Ursus (1588), Barth. Pitiscus (1600), Herwart von Hohenburg (1610), Peter Crüger (1612), Albert Girard (1626). The present writer has found this notation also in Rhaeticus[6] (1596), Kepler[7] (1604), and W. Oughtred.[8] But it did not become universal. In later years many authors designated degrees by "Grad." or "Gr.," or "G."; minutes by "Min." or "M."; seconds by "Sec." or "S."

[1] Orontius Finaeus *Arithmetica practica*, Paris, 1535, p. 46. We quote from Tropfke, *op. cit.*, p. 43.

[2] Gemma Frisius, *Arithmeticae practicae methodus facilis*, Strasbourg, 1559, p. 57 v°. From Tropfke, *op. cit.*, p. 43.

[3] Joannis Caramvelis, *Mathesis Bicepts Vetus, et Nova*, Companiae, 1670, p. 61.

[4] E. Reinhold, *Prvtenicae tabvlae coelestivm motvvm*, Tübingen, 1571, folio 15.

[5] Tycho Brahe, *Opera omnia*, ed. I. L. E. Dryer, vol. 1, Copenhagen, 1913, p. 137.

[6] *Opus Palatinvm de triangvlis a Georgio Joachimo Rhetico coeptvm.* 1696, p. 3 ff.

[7] J. Kepler, *Ad vitellionem paralipomena quibus astronomiae pars optica traditur.* Frankfurt, 1604, p. 103, 139, 237.

[8] William Oughtred in *Clavis Mathematicae*, 1631; Anonymous Appendix to E. Wright's 1618 translation of John Napier's *Descriptio.*

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

Not long ago Professor H. S. Uhler described in this MONTHLY (*1921*, 447) a plan for the multiplication of large numbers without undue expenditure of labor. The principal feature of the method was the preparation of nine strips of cardboard containing the products of the multiplicand by the digits 1, 2, ⋯, 9, from which these products might be obtained and inserted in their usual places, as often as they occurred, by mere copying. The saving of labor would be due to the fact that with a large multiplier the same partial products would appear a number of times. Since the appearance of Professor Uhler's note, several communications on the same subject have come in to this department.

It is evident that, if we were under the unfortunate necessity of carrying out multiplications involving thousands of figures, we might find it worth while to prepared in advance the products of the multiplicand by numbers up to 99, or higher. Actually, however, we may spare ourselves this task by use of a calculating machine. The notes printed below by D. N. Lehmer, J. P. Ballantine, and D'A. W. Thompson are concerned with the problem as modified by this consideration.

In the second discussion, Professor Bradley presents in a convenient form the solution of the celebrated problem on triangles with integral sides and area.

I*a*. ON THE MULTIPLICATION OF LARGE NUMBERS.

By D. N. LEHMER, University of California.

In certain researches in the theory of cubic irrationalities it has been necessary to compute determinants of the third order whose elements are of the order of twenty digits. Since I was interested in the factors of the result, it was necessary to obtain the product to the very last digit. I have found for this purpose a multiplying machine to be indispensable even when the factors are far larger than the capacity of the machine. By a slight modification of the method of cross-multiplication which is described in *Science*, volume 16, July, 1902, page 71, I was able to perform the multiplication given by Professor Uhler in this MONTHLY, *1921*, p. 447, in half an hour, using an eight place Monroe Calculating Machine. For such a machine I separate the numbers into periods of eight digits, and write one over the other thus:

23140692	63277926	90057290	86367948	54738026	61062426	00211600
43213918	26377224	97744177	37171728	01127572	81098106	33085400
9999999	66551256	78226922	68470128	58244096	21798502	61794556
	33448743	21773076	31529870	41755902	78201496	38205832
		1	1	1	1	1
10000000	00000000	00000000	00000000	00000000	00000000	00000389

The above is all the work that is put down in the actual computation. I obtain first the product

$$23140692 \times 43213918 = 9999999\ 66551256$$

which I set down in the third line. Then since the machine can compute $AB + CD$ without taking anything out of the machine, I obtain $23140692 \times 26377224 + 63277926 \times 43213918 = 33448743\ 21773076$. This result I put down in the fourth row, eight places to the right of the first product in the third line. I then obtain on the machine the sums $23140692 \times 97744177 + 63277926 \times 26377224 + 90057290 \times 43213918 = 78226922\ 68470128$ and set this result down eight places to the right of the preceding product sum. The machine then gives me, $23140692 \times 37171728 + 63277926 \times 97744177 + 90057290 \times 26377224 + 86367948 \times 43213918 = 1\ 31529870\ 41755902$. This I put down eight places to the right as before, the 1 being put in an additional row. The process is carried on in this way, eight new digits being added to the product at every step. The final addition is then made. The result differs a little at the end from Professor Uhler's, because he used digits beyond the fifty-two places which he has written down.

The method might be used with Crelle's *Rechentafeln,* only the numbers would be separated into periods of three, and the additions would have to be made "by hand." It is nevertheless a much better method for most purposes than the old methods.

Ib. Note on the Multiplication of Long Decimals.

By J. P. Ballantine, Chicago, Ill.

We will suppose a calculating machine (such as the Monroe) which will carry 6, 8 or 10 digits on the keyboard, and twice as many on the carriage, denoting these numbers by μ and 2μ. We will denote by a any positive integer of μ or fewer digits, and will understand by a_1a_0 the number $a_1 \cdot 10^\mu + a_0$. Thus if a number contained 2μ digits at the left of the decimal point and 6μ at the right, it could be written in the form $a_1a_0 \cdot a_{-1}a_{-2}a_{-3}a_{-4}a_{-5}a_{-6} = \sum_{i=-6}^{1} a_i 10^{\mu i}$.

The multiplication of two numbers follows the ordinary laws of multiplication of two polynomials, thus:

$$A = a_0 \cdot a_{-1}a_{-2}a_{-3}, \qquad A' = a_0' \cdot a_{-1}'a_{-2}'a_{-3}',$$
$$A \times A' = (a_0 \times a_0') \cdot (a_0 \times a_{-1}' + a_{-1} \times a_0')(a_0 \times a_{-2}' + a_{-1} \times a_{-1}' + a_{-2} \times a_0') \cdots.$$

The combinations, of which the expression $(a_0 \times a_{-2}' + a_{-1} \times a_{-1}' + a_{-2} \times a_0')$ is typical, can be computed on the machine very readily. The products $a_0 \times a_{-2}'$, $a_{-1} \times a_{-1}'$, and $a_{-2} \times a_0'$ are taken, and if the carriage is not cleared between the operations, they are automatically added. The result will be a number of 2μ digits, and hence can be written in the form $a_2''a_1''$. The a_1'' is written down as part of the product $A \times A'$ (in the proper place), while the a_2''

is carried, that is, entered in the carriage of the machine to be taken as part of the next combination to the left.

In a precisely similar fashion it is possible to perform any of the familiar processes, such as long division, square root, and even Horner's method. It is as if the radix were 10^{μ}, and instead of our memorizing the multiplication table up to $10^{\mu} \times 10^{\mu}$, the entries in this table were read from the machine.

To give a comparison between the longhand method and the machine method of multiplying, there are $\frac{1}{8}$ as many possibilities of error when an eight-place machine is used as when the work is done by hand; and, as for time, while Professor Uhler spent an afternoon on his multiplication, it takes by machine less than 20 minutes.

Ic. On the Multiplication of Large Numbers.

By D'Arcy W. Thompson, St. Andrews University.

The method of multiplying large numbers described by Professor H. S. Uhler is one which I happened to hit upon several years ago, and which I have used frequently. But it may be further simplified.

First, it is not necessary to *write* out the figures on coördinate paper, nor to transfer them to strips of cardboard. For the typewriter is itself an automatic "coördinator," and to typewrite the figures is at once to space them properly.

Secondly, if one performs the necessary summation by means of a Brunsviga, *no preliminary multiplication* is necessary; one does all that in the process of summation, by giving at each step the proper number of turns to the handle of the machine. So one has merely to type the multiplicand again and again, each time one space further to the right, and then place the multiplier vertically at the right-hand side and follow its indications as one turns the handle. Professor Uhler's example then is typed as follows:

```
4 3 2 1 3 9 1 8 2 · · · · · × 2
  4 3 2 1 3 9 1 8 · · · · · × 3
    4 3 2 1 3 9 1 · · · · · × 1
      4 3 2 1 3 9 · · · · · × 4
        · · · · · · · · · · · ·
```

Lastly, as a further improvement, the multiplier is typed (vertically) on a separate piece of paper, which serves as a movable rider, to be slid along the multiplicand as block after block of the latter is successively dealt with by the Brunsviga; and this rider has a slot which reveals just so broad a block of figures as may correspond to the capacity of the machine. Using this slotted rider, one does away with the necessity of blocking out the multiplicand by columnar interspacing, as in Professor Uhler's example. The rows of figures become thereby easier to write, and much easier to check, for the eye runs over identical figures in diagonal lines.

Professor Uhler says that his long multiplication of 50 digits by 50 was done in an afternoon and evening. With an assistant to dictate the numbers, I think it could be done very easily in an hour.

	9	1	8	2	2
	3	9	1	8	3
	1	3	9	1	1
	2	1	3	9	4
....					...

Diagram of "rider."

II. Rational Oblique Triangles.

By H. C. Bradley, Massachusetts Institute of Technology.

In this Monthly, *1921*, page 246, Professor Dickson gives Euler's results for the sides of a rational oblique triangle. Omitting the arbitrary constant multiplier, these are

$$x = \frac{m^2 + n^2}{mn}, \qquad y = \frac{M^2 + N^2}{MN}, \qquad z = \frac{(mN \pm nM)(mM \mp nN)}{mnMN},$$

where m and n are relatively prime integers, not both odd, and where M and N are relatively prime integers, not both odd.

These formulæ are not well adapted for finding integral solutions. However, let us multiply each by the factor mMn^2N^2. This gives

$$x = Mn(m^2N^2 + n^2N^2), \qquad y = mN(M^2n^2 + n^2N^2),$$
$$z = (mN \pm Mn)(mMnN \mp n^2N^2).$$

Now let $mN = p$, $Mn = q$, $nN = r$, and we have the simple three parameter solution

$$x = q(p^2 + r^2), \qquad y = p(q^2 + r^2), \qquad z = (p \pm q)(pq \mp r^2).$$

If from these three sides we find the area by the formula $A = \sqrt{s(s-x)(s-y)(s-z)}$, the result is

$$A = pqrz.$$

These results are the same as in Carmichael's *Diophantine Analysis* (New York, 1915), page 12, except that he does not give the double sign in the value of z. While this omission causes no loss in generality, it makes solutions in small integers less readily obtainable.

In using these formulæ, a few points may be noted. If we take $p = q$, only isosceles triangles are obtained. If we take either $p = r$, or $q = r$, only right triangles will result. Hence for general oblique triangles, all sides unequal, take $p \neq q \neq r$, though any two may have a common factor. Even then the equations occasionally give a right or an isosceles triangle. A few interesting cases are appended.

x	y	z	A
4	13	15	24
9	10	17	36
3	25	26	36
7	15	20	42
6	25	29	60
11	13	20	66
5	29	30	72
10	17	21	84
13	14	15	84
12	17	25	90

RECENT PUBLICATIONS.

REVIEWS.

The Cambridge Colloquium, 1916. Part 2: Analysis Situs. By OSWALD VEBLEN. (American Mathematical Society Colloquium Lectures, volume 5, part 2.) New York, 1922. 150 pp. Price $2.00 (unbound).

Extracts from Preface: "The Cambridge Colloquium lectures on Analysis Situs were intended as an introduction to the problem of discovering the n-dimensional manifolds and characterizing them by means of invariants. For the present publication the material of the lectures has been thoroughly revised and is presented in a more formal way. It thus constitutes something like a systematic treatise on the elements of Analysis Situs. The author does not, however, imagine that it is in any sense a definitive treatment. For the subject is still in such a state that the best welcome which can be offered to any comprehensive treatment is to wish it a speedy obsolescence.

"The definition of a manifold which has been used is that which proceeds from the consideration of a generalized polyhedron consisting of n-dimensional cells. The relations among the cells are described by means of matrices of integers and the properties of the manifolds are obtained by operations with the matrices. The most important of these matrices were introduced by H. Poincaré, to whom we owe most of our knowledge of n-dimensional manifolds for the cases in which $n > 2$. But it is also found convenient to employ certain more elementary matrices of incidence whose elements are reduced modulo 2, and from which the Poincaré matrices can be derived.

"The operations upon the matrices lead to combinatorial results which are independent of the particular way in which a manifold is divided into cells and therefore lead to theorems of Analysis Situs. . . .

"It will be seen that, aside from this one question which has to be dealt with in order to give significance to the combinatorial treatment, we leave out of consideration all the work that has been done on the point-set problems of Analysis Situs and on its foundation in terms of axioms or definitions other than those actually used in the text. We have also been obliged by lack of space to leave out all reference to the applications. . . ."

Contents—Chapter I: Linear graphs, 1–33; II: Two-dimensional complexes and manifolds, 34–72; III: Complexes and manifolds of N dimensions, 73–99; IV: Orientable manifolds, 100–124; V: The fundamental group and certain unsolved problems, 125–150.

No review, in the usual sense, of this work will be appropriate in this MONTHLY. Many of the readers of the MONTHLY will not be interested in a critical estimate of the importance, originality, or accuracy of this significant series of lectures. A few remarks to supplement the statements in the above given extracts from the "Author's Preface" will suffice here.

The study of Analysis Situs is the study of properties which remain invariant under continuous transformations. These may be roughly divided into "im Kleinen" and "im Grossen" properties, that is, properties which have reference to the infinitesimal character of the figure under consideration, and those which derive their significance from the figure as a whole. The distinction is a qualitative one which may be difficult to apply in certain cases, but in many familiar instances, from every mathematical study employing notions of continuity, this separation corresponds to an actual division of subject matter and of methods of attack. One might call these, respectively, the differential and the integral properties. In but one dimension, the major part of the theory of sets of points may be properly counted under Analysis Situs (of the differential sort), while the classification of different types of closed one-dimensional continua, with

respect to internal properties alone, is trivial in the extreme. The handling of sets of points in the plane, a workable definition of a "curve," considerations of equi-continuity, and so forth, present all kinds of unexpected problems and even paradoxes. Space-filling curves, curves of content, the proof that a simple closed curve divides the plane into two regions, and similar topics rightly suggest a host of delicate questions of rigor. All of these fall outside of the scope of these lectures.

The fact that there are actually questions of Analysis Situs in the integral sense, in two dimensions, is usually brought to vivid consciousness for the first time by the exhibition of a twisted paper strip of one continuous edge and one continuous surface. Since such "applications" are largely missing from the present work, and since the concrete problems are at once translated into equivalent problems in finite combinatorial analysis, the merely casual reader may find difficulty in seeing the connection between most of the topics here discussed and continuous transformations. The study is really one of matrices, the permissible operations being those suggested by the geometry of the problem, while little further mention is made of the geometrical interpretation of the several steps. By this procedure, dependence upon spatial preconceptions is largely eliminated and the subject acquires a degree of mathematical rigor, with flexibility of manipulation, that is a refreshing surprise to most mathematicians who approach this study for the first time.

Except for the applications to Riemann surfaces, whose study served to introduce the entire subject on a systematic basis, the topic has won as yet but few investigators in America. This is perhaps due to the recentness of most of the developments. While serviceable expositions of much of the theory have been provided by Dehn, Heegard, and Poincaré, there is not a little that is new in this exposition, and one need hardly point out that it is the only work in English to attempt to cover this ground. The subject is basic, important unsolved problems are still open and much machinery is now available. What greater incentive can an adventurous mathematician ask to induce him to read a series of lectures which, like these, are simple to read, are self-contained, and are developed in a progressive logical manner?

ALBERT A. BENNETT.

Précis d'Arithmétique. By J. POIRÉE. Paris, Gauthier-Villars, 1921. 8vo. 5 + 64 pages. Price 7.50 francs.

Author's Preface (translated): "In composing this work, we have not sought to give verbal formulation to the rules of arithmetic, which are found furthermore in all of the treatises. We have been interested particularly in explaining the reason and mechanism of each operation. We have shown how the consideration of sets of objects awakes in us the notion of integral number and how the comparison of magnitudes of the same sort leads to the notion of fraction. The calculation of a fraction to a given degree of approximation permits us to define what is called *limit of a sequence of numbers*. The study of the square root leads us to consider a certain class of these numbers called *irrationals*, of which analysis offers so many examples. If we have spoken of progressions it is because these serve in algebra for the definition of logarithms. We have also shown that progressions enter in certain problems of arithmetic. Finally, the last chapter,

devoted to the examination of the remainder of certain series, will familiarize the reader with the Gaussian notation and will facilitate for him the later study of the theory of numbers which is only sketched here. To close, let us add that different types of arithmetical problems are studied as applications of proportions."

Preface by C. CAMICHEL, Director of the Electrotechnical Institute of the University of Toulouse: "Elementary arithmetic is an excellent introduction to the study of mathematics. There one finds in concrete form models of all types of reasoning from the most simple to the most delicate of analysis. However, this part of mathematics is usually neglected by students. Perhaps they are not entirely to blame. Aiming at extreme rigor and desirous of leaving no difficulty concealed, authors often insist on points whose importance is not well understood by beginners, and which prevent these from seeing the essential parts of the reasoning. Monsieur Poirée has shown how to avoid this danger and has condensed into a few pages the whole object of the classical courses; he has even added a most interesting chapter introducing the reader to the elements of number theory.

"Suppressing all that was not absolutely indispensable, although observing rigor throughout, the author passes rapidly over numeration, the fundamental operations, divisibility, and prime numbers, making the logical order of the theorems clearly evident. Fractions offer him the occasion for introducing and rendering precise the notion of limit, which beginners find so difficult to understand. Similarly, while studying the square root, he introduces, simply and naturally, the irrational numbers. All this is accompanied by numerical examples indispensable to beginners for securing an understanding of the theories.

"Assuredly a treatise so conceived will render the greatest service to all who would understand arithmetic. Like his preceding work, this new book by Monsieur Poirée is a 'real success.' "

Contents—Preface; Chapter I: Numeration, 1–2; II: Fundamental operations, 3–4; III: Tests for divisibility by 2, 5, 4, 3, 9, 11. Casting out nines, 5–6. IV: Greatest common divisor, least common multiple, 7–10; V: Prime numbers, 11–17; VI: Fractions, 18–27; VII: Proportions. Various problems. The metrical system, 28–37; VIII: Square roots. Square roots to the nearest tenth, hundredth, etc. Definition of an irrational number, 38–42; IX: Arithmetic and geometric progressions. Applications, 43–46; X: Introduction to the theory of numbers, 47–62; Table of contents, 63–64.

NOTES ON RECENT PUBLICATIONS.

MORITZ PASCH'S "Die Begriffswelt des Mathematikers in der Vorhalle der Geometrie" has been reprinted from the *Annalen der Philosophie* in pamphlet form (Leipzig, F. Meiner, 1922, 4 + 45 pages; price 50 marks in Germany).

In *Monatshefte für Mathematik und Physik*, Vienna, volume 32, there is a sketch by EMIL MÜLLER (pages 281–293) of the life and work of GUSTAV KOHN, 1859–1921; compare *1922*, 232. There are over fifty titles in the list of his scientific writings.

In the *Quarterly Journal of Pure and Applied Mathematics*, volume 49, number 3, October, 1922, there is an article (pages 226–283) "Abstract definitions of the symmetric and alternating groups and certain other permutation groups" by R. D. CARMICHAEL.

The concluding part (pages 377–524) of the fourth volume of the *Opera Omnia* of Tycho Brahe (1546–1601) was published in 1922 (compare *1920*, 421; *1921*, 314). Volumes I–VI, except for the second part of volume V, have been published since 1912. It will be recalled that this sumptuous publication is being issued by the Danish Society of Language and Literature under the editorship of Professor J. L. E. Dreyer of Oxford.

The mathematical bibliographer and librarian will welcome *Catologo crono-*

logico e alfabetico, per Autori e per Materie, delle Edizioni Hoepli 1872–1922, con Introduzione di Michele Scherillo (Milano, Hoepli, January, 1922, 71 + 404 pages). The exact month in which each one of their five thousand works was published is here indicated; also the dates of different editions. Unfortunately the edition of the *Catologo* was exhausted within six months of its publication.

There is a short memoir on THEODOR REYE (1838–1919), by C. Segre, in *Atti della reale Accademia Nazionale dei Lincei, Rendiconti,* volume 311, pages 268–272, April 2, 1922; on pages 398–404, May 7, there is also a memoir on CAMILE JORDAN (1838–1922), with a list of his principal publications, by L. Bianchi; a brief notice of MAX NOETHER (1844–1921) and his work, by G. Castelnuovo, follows on pages 404–407. Compare this MONTHLY, *1920,* 90; *1922,* 138–139, 233.

The last number of *Nouvelles Annales de Mathématiques,* volume 79, was published in December, 1920; the first number of volume 80 appeared in October, 1922, edited by R. Bricard, H. Villat (the new editor of *Journal de Mathématiques Pures et Appliquées*), and J. Pérès. The periodical is designed, as formerly, to serve candidates in special schools, and those preparing for the examinations of the licence and agrégation. The subscription price is 35 francs a year in the postal union.

In *The Encyclopædia Britannica,* volume 31, 1922, the second of the new volumes, the article on "Mathematics" occupies pages 874–880. The subsections and their authors are as follows: "Mathematical logic and the foundations of mathematics" by JEAN NICOD, agrégé de philosophie; "Theory of numbers" by G. H. HARDY; "Theory of series" by G. H. HARDY; "Theory of functions" by G. H. HARDY; "Geometry" by H. F. BAKER. The article on "Nomography" by R. H. HEZLET occupies pages 1139–1144. In volume 30, 1922, "Aeronautics" and "Ballistics" are extensive articles. The article on "Relativity," volume 33, pages 261–267, is by J. H. JEANS.

Doctor ALFRED ERRERA, secretary of the Belgian Mathematical Society, received the degree of "doctor in physical and mathematical sciences" at the University of Brussels in 1920. His thesis, published at Ixelles, by G. Bothy, in 1921 (66 pages + 1 plate), is entitled *Du Coloriage des Cartes et de quelques Questions d'Analyse Situs.* It contains a fairly complete survey of work done on this problem. In the "Bibliographie" the following Americans are referred to: J. W. ALEXANDER, G. D. BIRKHOFF, H. R. BRAHANA, P. FRANKLIN, N. J. LENNES, ISABEL MADDISON, W. E. STORY, O. VEBLEN. In *Revue de l'Université de Bruxelles,* April–May, 1922, Dr. Errera has a 13-page article entitled, "L'origine et les problèmes de l'analysis situs," and in *Mathesis,* February, 1922, he has a 6-page article, "Analysis situs. Une démonstration du théorème de Petersen."

In this MONTHLY, *1921,* 452, we have discussed the question of the existing copies of Brother Juan Diez's *Sumario Compendioso,* 1556, the first work on mathematics published in the New World. In *Bibliotheca Americana,* catalogue no. 429, published by Maggs Bros., London, in 1922, one of these copies, formerly

in the National Library at Madrid, was offered for sale at £ 120. Of the 103 folios which it contained, 2, 41, and 48 are missing. This seems to be the copy (original vellum with tabs) of which we have already given a reproduction of the title page, *1921*, 11. Maggs Bros. state, on page 48 of their catalogue, that the copy belonging to the British Museum "is non-existent, as we have convinced ourselves by personal research"; nevertheless this is the copy of which there is a rotograph copy at Michigan (*1921*, 452).

In *Journal für die reine und angewandte Mathematik*, volume 152, Heft 1–2, published September 22, 1922, there is a page in memory of K. H. A. SCHWARZ (compare *1922*, 232–233) whose first scientific paper was published in volume 63 when he was a student scarcely twenty years old. This was followed by ten other memoirs in the same journal. He was a member of its editorial committee for volumes 125–151, inclusive. In two papers by RUDOLF STURM, in volume 152, the following problems are discussed: (*a*) The largest tetrahedron with faces of given areas (pages 90–98); (*b*) Through a given point in the plane of two lines, draw a line such that the part cut off by the given lines shall be a minimum; (*c*) Given three planes, and a point or a line, respectively supports of a bundle or a sheaf of planes; to find that one of these planes from which the three given planes cut off the triangle of least area in the solid angle containing the point.

Section 3, December, 1920, of *Publications of the Clark University Library*, volume 6, 1922, contains a "List of degrees granted at Clark University and Clark College 1889–1920." The following 24 men received the degree of Ph.D. in mathematics: J. W. A. YOUNG (1892), W. H. METZLER (1893), T. F. HOLGATE (1893), J. E. HILL (1895), L. W. DOWLING (1895), T. F. NICHOLS (1895), W. G. BULLARD (1896), F. C. FERRY (1898), E. W. RETTGER (1898), J. S. FRENCH (1899), F. B. WILLIAMS (1900), S. E. SLOCUM (1900), H. C. MORENO (1900), H. G. KEPPEL (1901), J. N. VAN DER VRIES (1901), J. N. GATES (1904), R. B. ALLEN (1905), H. L. SLOBIN (1908), R. K. MORLEY (1910), SOLOMON LEFSCHETZ (1911), W. J. MONTGOMERY (1911), J. A. BULLARD (1914), PEYSAH LEYZERAH (1916), SAMUEL ZELDIN (1917). The titles of the theses, and also, in the case of 16, the places of publication, are given. It is noted that a seventeenth was privately printed. The theses of Hill, Dowling, Nichols, and W. G. Bullard were published in *The Mathematical Review* founded by Professor W. E. STORY, of Clark University, and published in Worcester. It has been generally thought that only two numbers of this *Review* were published: I, July, 1896, pages 11–96; II, April, 1896, pages 97–192. It appears, however, that a third number was published in 1899, and that it contained (pages 193–208) the thesis "On the general classification of plane quartic curves" by Professor W. G. Bullard, now of Syracuse University—Section 4, April, 1921, of the *Publications* contains a report of the inauguration of President Atwood of Clark University. Among the published greetings are those of A. P. WILLS, professor of mathematical physics at Columbia University, and of FRANK MORLEY, professor of mathematics in the Johns Hopkins University.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3004. Proposed by W. B. FORD, University of Michigan.

Is there a plane curve such that two tangents whose lengths are in the constant ratio $n : 1$ $(n \neq 1)$ may be drawn to it from any point in its plane? If so, discuss its properties.

3005. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Given the recurrence formulæ: $x_{n+1} = x_n(x_n + y_n)$, $y_{n+1} = y_n(2x_n + y_n)$ with the initial values $x_1 = 1$, $y_1 = 1$, it is desired to find a good approximation to $\dfrac{x_n}{2x_n + y_n}$ for large values of n. In particular, for values of n from $n = 20$ to $n = 30$.

Note.—This question arose in connection with a problem in genetics.

3006. Proposed by S. A. COREY, Des Moines, Iowa.

The formula

$$\begin{aligned} m\int f'(mx)dx = x\{c &+ f'(0) + f'(x) + f'(2x) + f'(3x) + f'(4x) + \cdots + f'[(m-1)x] \\ &+ \tfrac{1}{1440}\{965f'(mx) - 462f'[(m+1)x] + 336f'[(m+2)x] \\ &\qquad - 146f'[(m+3)x] + 27f'[(m+4)x\}\}, \end{aligned}$$

gives the exact value of the integral when the fifth and all higher derivatives are zero. Find an expression for the remainder term in the general case.

3007. Proposed by NORMAN ANNING, University of Michigan.

Given two opaque spheres, radii r_1 and r_2, at a fixed distance $d(> r_1 + r_2)$ apart; locate the points on either from which the maximum surface of the other can be seen.

3008. Proposed by P. R. RIDER, Washington University.

The altitude of a right circular cone is a, the radius of its base is b, and its slant height is c. A string is wrapped n times about the cone, starting at the vertex and ending at the base, in such a manner that for any complete circuit the vertical rise (the cone being supposed to rest on its base) is the same. A bird at the vertex takes the end of the string in its beak and flies around the cone, unwinding the string, keeping it taut and always tangent to the curve of the string as it lies around the cone. Find an expression for the distance that the bird has flown when the string is completely unwound, (a) if it starts at the vertex, (b) if it starts at the base.

3009. Proposed by J. G. COFFIN, New York City.

Rectangular pieces of cardboard of the same dimensions are piled so that they overhang to the greatest extent possible; what curve do the edges touch? how great a distance between first and last piece can be obtained? and what are the properties of the material volume thus produced?

It is assumed that the solver will be led to consider an infinite number of infinitely thin layers.

3010. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Find an expression for the volume of the pedal tetrahedron, with respect to the tetrahedron of reference, of a point whose perpendicular distances from the sides are (x_1, x_2, x_3, x_4); from this expression show that the locus of points the feet of whose perpendiculars on the faces of the

tetrahedron of reference are coplanar is Steiner's cubic surface,

$$\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} = 0,$$

where the A's are the areas of the faces of the tetrahedron of reference.

SOLUTIONS.

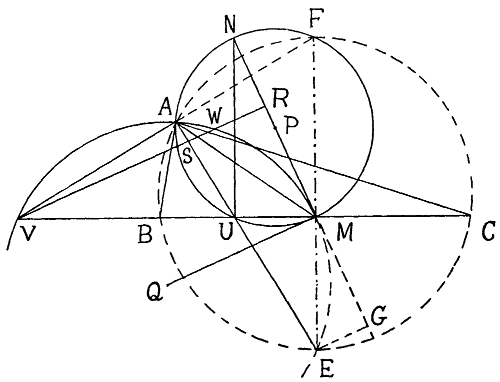
2903 [1921, 277]. Proposed by A. A. BENNETT, University of Texas.

Given the base of a triangle in position and length, the length of the median to the base, and the difference of the base angles; find a simple ruler and compasses construction for the triangle.

Two Solutions by Nathan Altshiller-Court, University of Oklahoma.

I. Analysis. Let ABC be the required triangle with $\angle B \geqq \angle C$; let AM be the given median, and let AU and AV be the interior and exterior bisectors of the angle A. From the triangle ABU we have: $\angle AUM = \angle B + \frac{1}{2}\angle A = \frac{1}{2}\angle B + \frac{1}{2}(\angle A + \angle B)$. But $\frac{1}{2}(\angle A + \angle B + \angle C) = 90°$; hence $\angle AUM = 90° + \frac{1}{2}(\angle B - \angle C) = 90° + d$, where $2d$ is the given difference of the two base angles of the triangle. Thus the given median AM subtends at U an angle of known magnitude; therefore, the point U lies on a circle (P), which may readily be constructed.

From the triangle AUV we have: $\angle AVU = \angle AUM - \angle VAU = 90° + d - 90° = d$. Thus again the given median AM subtends at the point V an angle of known magnitude, and, therefore, V lies on a circle (Q), which may readily be constructed.



Let R be the foot of the perpendicular dropped from the point V upon the diameter MPN of the circle (P). The two right triangles MRV and MNU, having the acute angle M in common, are similar; hence, $MR \cdot MN = MU \cdot MV$. Now the product of the last two segments is known. In fact, the points U, V divide the base BC harmonically, and since M is the mid-point of BC, we have:[1] $MU \cdot MV = \overline{MB}^2 = a^2$, if $2a$ denotes the given length of the base BC of the triangle ABC. Thus $MR \cdot MN = a^2$. The diameter MN is known; hence this equality determines the segment MR and the point R, so that the perpendicular RV may be constructed. Thus the point V lies on the circle (Q) and on a given line RV. (The reader who is familiar with the method of inversion will readily notice that the perpendicular RV is the inverse of the circle (P) with respect to the center of inversion M, the power of inversion being equal to a^2.)

Construction. Let $2a$, m, $2d$ denote, respectively, the magnitudes of the given base, of the given median, and of the given difference of the base angles.

Draw a right angle PMQ, and a line MA so that $\angle AMP = d$. Lay off $MA = m$, and let the perpendicular bisector of MA meet the lines MP, MQ in the points P, Q, respectively. Draw the two circles having P and Q for centers, and PA, QA for radii. On the diameter MPN lay off the segment MR such that $MR \cdot MN = a^2$. At the point R erect a perpendicular to MN, meeting the circle (Q) beyond A in the point V (see discussion below). On the line MV lay off $MB = MC = a$. The triangle ABC satisfies the conditions of the problem.

Proof. The triangle ABC has the required base and the required median by construction. It remains to show that $\angle B - \angle C = 2d$.

The line MN being drawn perpendicular to QM touches the circle (Q) at M. The inscribed angle AVM is equal to the angle formed by the chord AM and the tangent to (Q) at M. But $\angle AMN = d$, by construction; hence, $\angle AVM = d$. In a similar way, it may be shown that the angle AUM, where U is the second point of intersection of MV with circle (P), is equal to $90° + d$. Consequently, the $\angle UAV = \angle AUM - \angle AVM$ is a right angle.

[1] John W. Russell, *A Sequel to Elementary Geometry*, Oxford, 1913, p. 54.

By construction $MR \cdot MN = a^2$, and from the similar right triangles MRV and MNU we have $MU \cdot NV = MN \cdot MR$; hence $MU \cdot MV = a^2$. Therefore, the four points B, C, U, V are harmonic.[1] On the other hand UAV is a right angle, as shown above; consequently, the lines AU, AV are the bisectors of the angle BAC.[2]

From the triangle ABU we have:

$$\angle AUM = \angle UAB + \angle ABC. \tag{1}$$

Since $\angle AUM = 90° + d = \frac{1}{2}(\angle A + \angle B + \angle C) + d$, and $\angle UAB = \frac{1}{2} \angle A$, (1) becomes $\frac{1}{2}(\angle A + \angle B + \angle C) + d = \frac{1}{2} \angle A + \angle B$; whence, $\frac{1}{2}(\angle B - \angle C) = d$.

DISCUSSION. The difference $2d$ of the two base angles must necessarily be smaller than 180°. If this condition is fulfilled, the two circles (P), (Q) can always be constructed, and the solution of the problem depends upon the finding of the point V.

In the case of the figure, the perpendicular VR meets the circle (Q) in another point W, and the two points V, W lie on opposite sides of the median AM. However, the line MW, lying between AM and MN, cannot be taken for the line of the base of the required triangle, because the triangle constructed on MW would have for the difference of its base angles not $2d$, but its supplement. If S is the point of intersection of VR with (P), we have from the right triangle MNS, $MS^2 = MN \cdot MR = a^2$; hence, $MS = a$. Thus the two points of intersection of (Q) with VR will lie on opposite sides of AM, when $a < m$. The problem will have one solution. The angle at A will be acute.

If $a = m$, the triangle ABC will have a right angle at A.

If $a > m$, and MR is smaller than the radius of (Q), the perpendicular VR will meet (Q) in two points, which will lie on the same side of AM. The problem will have two solutions, if $d < 45°$. The angle at A will be obtuse. If MR is equal to the radius of (Q), the two solutions will coincide. If $d \geqq 45°$ the two points will lie between A and M and there will be no solution.

If $a > m$, and MR is larger than the radius of (Q), the problem will have no solutions.

If A' is the symmetric of A with respect to the perpendicular bisector of the base BC, the triangle $A'BC$ satisfies the conditions of the problem.

If the base is given in position, the above construction determines the angle between the base and the median, and the construction of the required triangle is easily completed.

If, in the analysis of the problem, instead of dropping from the point V of (Q) a perpendicular upon the diameter MN of (P), we had dropped from the point U of (P) a perpendicular upon MQ, the rôle of the two circles (P), (Q) in the construction of the triangle ABC would have been interchanged.

II. Let ABC be the required triangle, and EF the diameter of the circumcircle perpendicular to the base BC. The angles ABC, ACB, AEF have for their respective measures half the arcs AFC, AB, AF. But arc AFC − arc AB = 2 arc AF, hence $\angle AEF = \frac{1}{2}(\angle B - \angle C) = d$, if $2d$ is the given difference of the base angles of the triangle.

From the right triangle AEF we have $\angle AFE = 90° - d$. Thus the given median AM subtends at the points E and F angles of given magnitude; hence, the points E, F lie on two circles (Q), (P), which may readily be constructed. Let G be the foot of the perpendicular EG from E to the diameter MPN of the circle (P). From the similar right triangles MEG and MFN, we have: $MG \cdot MN = ME \cdot MF$. But from the circumcircle of ABC we have: $ME \cdot MF = MB \cdot MC = a^2$; if $2a$ is the given length of the base BC, hence $MG \cdot MN = a^2$. Since MN is known, the length MG, and therefore also the point G, may be determined. Thus the point E lies on the circle (Q) and on the perpendicular EG to a given line MN at a given point G. The base BC of the required triangle is perpendicular to the known line ME.

It is easy to see that the lines AE, AF are the bisectors of the angle A; that is, they are identical with the lines AU, AV considered in the first solution, and that the circles (P), (Q) are the same as considered above.

III. DISCUSSION BY OTTO DUNKEL, Washington University.

In a triangle ABC with BC fixed and equal to $2a$, let θ be the angle at C and $\theta + 2d$ the angle at B. Then the rays BA and CA will describe equal pencils and the locus of A will be a conic passing through B and C, there being no self-corresponding ray. When $\theta = 0$, BA becomes tangent at B, making the angle $2d$ with BC; when $\theta = -2d$, CA is a tangent at C. Since these

[1] Russell, *loc. cit.*, p. 55.

[2] Russell, *loc. cit.*, p. 58.

two tangents are parallel, BC is a diameter and its mid-point M is the center. When $\theta = 90° - d$, CA and BA are parallel, and when $\theta = -d$, they are parallel again. Hence the conic is a rectangular hyperbola, its asymptotes making angles $90° - d$ and $-d$ with MB. Since B is regarded as the larger of the two angles it is only necessary to consider that part of the branch through B which lies above MB, for this gives the extreme variations of B from $2d$ to $90° + d$. If I is the vertex of the hyperbola on the branch through B, and if the x-axis is taken along MI and the origin at M, the equation of the curve is $x^2 - y^2 = MI^2$. As MB is equal to a, the coördinates of B are $a \cos (45° - d)$ and $a \sin (45° - d)$, and therefore $MI = \sqrt{\sin 2d}$.

Now if m is the length of the median the circle with center at M and radius m will cut this part of the hyperbola in the desired point A. This leads to the following cases:

$m < a\sqrt{\sin 2d}$	no solution
$m = a\sqrt{\sin 2d}$ and $d < 45°$	one solution
$m = a\sqrt{\sin 2d}$ and $d \geqq 45°$	no solution
$a > m > a\sqrt{\sin 2d}$ and $d < 45°$	two solutions
$a > m > a\sqrt{\sin 2d}$ and $d \geqq 45°$	no solution
$m \geqq a$	one solution.

The abscissa of the vertex is found to be $\sqrt{(m^2 + a^2 \sin 2d)/2}$ and this may be used for a construction, but such a construction would not be nearly as elegant as the author's.

In the figure above $MN = m/\cos d$, $MR = (a^2 \cos d)/m$, and $MQ = m/(2 \sin d)$. Therefore $MR \gtreqless MQ$ according as $a\sqrt{\sin 2d} \gtreqless m$.

Also solved by J. F. HOWARD, ARTHUR PELLETIER, and F. L. WILMER.

2917 [1921, 327].

A parabola is rolled upon a fixed right line. Find the locus of (a) its vertex and (b) its focus.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

If the parabola with vertex V and focus F has contact at C with the fixed line and if FV cuts that line in S, then from well-known properties of the parabola the triangle SFC is isosceles and the tangent at the vertex V passes through M, the middle point of SC. Hence MF is perpendicular to SC. Since C is the instantaneous center of rotation, the tangent to the locus of F is perpendicular to FC. Let this tangent cut SC in T and set $\tau = \angle MTF = \angle VFM$, $MF = y$, $VF = a$. Then

$$y = a \sec \tau, \tag{1}$$

which is characteristic of the catenary.

To deduce from (1) the ordinary equation of the curve, we have $dy = a \sec \tau \tan \tau \, d\tau$ and $dy = \tan \tau \, dx$ and hence $dx = a \sec \tau \, d\tau$ or

$$x = a \log (\sec \tau + \tan \tau) = -a \log (\sec \tau - \tan \tau). \tag{2}$$

By eliminating $\tan \tau$ from the two expressions for x and using (1) we find

$$2y = a(e^{x/a} + e^{-x/a}). \tag{3}$$

In order to obtain the parametric equations of the path of V, we have merely to subtract from the right hand sides of (2) and (1) $a \sin \tau$ and $a \cos \tau$, respectively, giving for the required locus

$$x = a[\log (\sec \tau + \tan \tau) - \sin \tau],$$
$$y = a \sin \tau \tan \tau.$$

Also solved by GEORGE AGINS, T. L. BENNETT, T. M. BLAKSLEE and WILLIAM HOOVER.

2919 [1921, 327]. Proposed by V. M. SPUNAR, Chicago, Ill.

An equilateral hyperbola which touches a conic and is concentric with it is called a hyperbolic tangent to the conic. Being given two hyperbolic tangents to a conic, the arc of any third hyperbolic tangent which is intercepted by the first two subtends a constant angle at either focus of the given conic.

I. Solution by William Hoover, Columbus, Ohio.

This problem has added interest on account of its history. In *Liouville's Journal*, volume 13, 1848, p. 209, a letter of William Roberts to the journal dated Dublin, November 10, 1847, appears in which he obtains a large number of theorems by use of the transformation in polar coördinates

$$r = R^n, \qquad \omega = n\Omega, \tag{1}$$

where n is any number. This transformation is conformal, he points out, for $rd\omega/dr = Rd\Omega/dR$ If we apply (1) to the straight lines of the plane,

$$r \cos (\omega - \alpha) = p, \tag{2}$$

we obtain the set of curves,

$$R^n \cos (n\Omega - \alpha) = p, \tag{3}$$

which, for a given α, is a family of curves having an axis inclined at the angle α/n to the initial line. If $p = 0$ the curves in (2) and (3) are straight lines through the pole. If $p \neq 0$ and $n = 2$, (3) is a system of equilateral hyperbolas with centers at the pole; if $n = 1/2$, confocal parabolas. Since (1) is a conformal transformation, the curves of the two families given by $\alpha = \alpha_1$ and $\alpha = \alpha_2$ cut each other under the constant angle $\alpha_2 - \alpha_1$, which is n times the angle between their respective axes.

The transformation (1) for $n = 2$ applied to the central conics

$$r(1 - e \cos \omega) = l \tag{4}$$

gives the central conics $R^2(1 - e \cos 2\Omega) = l$, for which the focus of the former goes into the center of the latter. For convenience of statement Roberts introduces the term hyperbolic tangent defined in the problem. Now it is known that, if the two points in which two fixed tangents to a conic are cut by a third tangent be joined by straight lines to a focus, the latter two lines include a constant angle.[1] If the conic is central, this theorem goes over by (1) and the facts above into the theorem of the problem.

Another theorem given by Roberts is as follows: Being given a central conic and a concentric equilateral hyperbola, if from any point on the latter are drawn two hyperbolic tangents to the first conic, the concentric equilateral hyperbola passing through the two points of contact will pass through a fixed point. This results easily by passing from well known theorems regarding poles and polars of a conic by (1) for $n = 2$ to the corresponding theorems of the transformed figure.

He also states that a system of homofocal conics transforms into a system of homofocal conics.

II. Remarks by Otto Dunkel, Washington University.

It should be observed that the real proof in the above consists in transforming the theorem to be proved into known theorems, or by showing that the reverse process covers all cases. Either procedure is easy here.

The transformation used here is merely a special case of transformation in polar coördinates which satisfy the condition of conformality:

$$\frac{r}{R}\frac{\partial \omega}{\partial \Omega} = \frac{\partial r}{\partial R}, \qquad rR\frac{\partial \omega}{\partial R} = -\frac{\partial r}{\partial \Omega}.$$

2920 [1921, 392]. Proposed by N. P. PANDYA, Sojitra, India.

Construct a triangle, having given the base, the angle between the base and the median on it, and the difference of the remaining two sides.

I. Solution by Arthur Pelletier, Montreal, Canada.

Let $c = AB$ be the given base, d the difference between the remaining two sides, $d < c$, DF the position of the median making the acute angle α with DB. Describe a circle with center A and radius d, and construct the point B' symmetrical to B with respect to DF. Pass a circle through B and B' tangent to the above circle (a well-known problem). The center C of this circle is evidently the remaining vertex of the required triangle.

[1] See Reye, *Die Geometrie der Lage*, 1909, part 1, p. 165, where the theorem is attributed to Poncelet.

II. Notes by Otto Dunkel, Washington University.

Two circles can be passed through B and B' tangent to the circle with center A, one tangent externally and the other tangent internally. In this particular problem the center C' of the second tangent circle gives a triangle BAC' which may be easily shown to be congruent to the triangle ABC.

The point B' lies on a circle with AB as a diameter and $\angle BAB' = \alpha$. From this it follows at once that the construction is not possible if $c \cos \alpha \leqq d$.

The tangent circles C and C' may be constructed as follows: Pass any circle through B and B' cutting the circle A in two points and let their common chord cut BB' in I. Then the circle constructed with IA as a diameter cuts the circle A in G and G' which are the points of tangency of the required circles.

Also solved by A. R. Nauer, J. B. Reynolds, Marcus Skarstedt, J. H. Weaver and F. L. Wilmer.

2927 [1921, 393]. Proposed by PHILIP FRANKLIN, Harvard University.

Prove that the only positive integral values greater than unity which satisfy the equation $3^x - 2^y = \pm 1$ are $x = 2$, $y = 3$. (Cf. Carmichael, *Diophantine Analysis*, 1915, p. 116, exercise 69.)

Solution by the Proposer.

(*a*) The equation, $3^x - 2^y = 1$, implies

$$2^y + 1 = (4 - 1)^x.$$

Since y is greater than one, the left and consequently the right member of this equation must be congruent to 1, modulo 4, and x must be even. We may therefore write

$$2^y + 1 = (2m + 1)^2 = 4m(m + 1) + 1.$$

This shows that m must be unity, since otherwise $m(m + 1)$ would contain an odd factor. Hence $(x, y) = (2, 3)$ is the only pair of integers each greater than unity satisfying (*a*).

(*b*) The equation, $3^x - 2^y = -1$, implies

$$3^x + 1 = (3 - 1)^y.$$

By taking the remainders, modulo 3, we show that y is even. We may thus write

$$3^x + 1 = (3m \pm 1)^2 = 3m(3m \pm 2) + 1.$$

The only value of m which makes $m(3m \pm 2)$ a power of 3 is unity, and we must take the minus sign. As this requires x to be unity, it does not lead to any further solutions of the kind specified.

Also solved by F. L. Wilmer.

2929 [1921, 466]. Proposed by R. E. GAINES, University of Richmond.

Denote by A, O and B, respectively, the points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ on the curve $y^2 = x^3 - x$, and let P be a variable point on the curve. Let PA and PO meet the curve again in Q and R, respectively, and let BQ and BP meet AR in M and N, respectively. Prove that QN is perpendicular to PM.

Solution by the Proposer.

The coördinates of P and Q must satisfy the equation of the curve and $\lambda y = x + 1$ and hence also the equation $y - \lambda x(x - 1) = 0$. If we take B as origin, the last two equations become $y - \lambda x' - \lambda x'^2 = 0$ and $\lambda y - x' = 2$. Making the first equation homogeneous by aid of the second we have

$$\lambda(y^2 - x'^2) - (\lambda^2 + 1)x'y = 0$$

as the equation of BQ and BP and it follows at once that they are perpendicular.

In a similar manner, by finding the intersections of RP with the curve and taking A as the origin we show that AR and AP are perpendicular. In the triangle MNP the two altitudes MB and PA meet in Q and hence NQ must be the third altitude, *i.e.*, perpendicular to MP.

Also solved by E. F. ALLEN, T. M. BLAKSLEE, R. M. MATHEWS, E. J. OGLESBY, ARTHUR PELLETIER and J. B. REYNOLDS.

2930 [1921, 467]. Proposed by R. E. GAINES, University of Richmond.

If in reducing p/q (p and q integers, $q > p$) to a decimal the remainder $q - p$ ever appears, then the fraction will give a repeating decimal the number of digits in whose repetend will be exactly twice the number of digits in the quotient already obtained, and the remaining digits may, without further division, be obtained by subtracting the quotient already found from a succession of 9's.

SOLUTION BY E. M. BERRY, Purdue University.

Suppose $\frac{p}{q} = \cdot d_1d_2d_3 \cdots d_n + \frac{q-p}{q} 10^{-n}$, where d_i is the digit in the ith decimal place. Then

$$\frac{q-p}{q} = 1 - \frac{p}{q} = 1 - \cdot d_1d_2d_3 \cdots d_n - 10^{-n} + \frac{p}{q} 10^{-n}$$
$$= \cdot(9 - d_1)(9 - d_2)(9 - d_3) \cdots (9 - d_n) + \frac{p}{q} 10^{-n},$$

since $1 - 10^{-n} = .999 \cdots$ to n decimal places. Thus

$$\frac{p}{q} = \cdot d_1d_2d_3 \cdots d_n(9 - d_1)(9 - d_2)(9 - d_3) \cdots (9 - d_n) + \frac{p}{q} 10^{-2n}$$

where $(9 - d_i)$ is the digit in the $(n + i)$th decimal place. Hence the theorem is proved.

Also solved by MICHAEL GOLDBERG, R. M. MATHEWS, ARTHUR PELLETIER, and the PROPOSER.

2931 [1921, 467]. Proposed by R. C. ARCHIBALD, Brown University.

From the equality $\sec(\pi/14) + \sec(3\pi/14) - \sec(5\pi/14) = 0$ find a relation between the lengths of the side and diagonals of a regular inscribed heptagon.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let z be the side of the regular heptagon inscribed in a circle of radius r, and let x and y be the longer and shorter diagonals. Then $x = 2r \sin 3\pi/7 = 2r \cos \pi/14$ or $\sec \pi/14 = 2r/x$. Similarly, $\sec 3\pi/14 = 2r/y$, $\sec 5\pi/14 = 2r/z$. Substituting these values in the given equality, we have

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}.$$

Hence the side is one half the harmonic mean of the two diagonals. If x and y are taken as two sides of a triangle including an angle of 120°, z is the length of the bisector of this angle.

Also solved by ARTHUR PELLETIER and T. M. BLAKSLEE.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Assistant Professor C. E. WILDER, of Northwestern University, has been appointed assistant professor of mathematics at Dartmouth College.

Assistant Professor C. E. MELVILLE, of Clark University, has been promoted to a full professorship.

Miss MARTHA F. CHADBOURNE has been appointed instructor of mathematics at Wheaton College, Norton, Mass.

Dr. C. R. ADAMS, of Harvard University, has been appointed instructor of mathematics at Brown University for the year 1923–1924.

At Trinity College, Hartford, Conn., Mr. F. J. BURKETT, of the University of Pittsburgh, has been appointed assistant professor of mathematics, and Assistant Professor H. M. DADOURIAN has been promoted to an associate professorship of physics.

Assistant Professor MARY E. WELLS, of Vassar College, has been promoted to an associate professorship of mathematics.

At Princeton University, Assistant Professor J. H. M. WEDDERBURN has been promoted to an associate professorship of mathematics, and Dr. EINAR HILLE and Mr. G. B. BRIGGS have been appointed instructors.

Assistant Professor C. C. Bramble, of the U. S. Naval Academy, has been promoted to an associate professorship of mathematics.

Mr. F. H. MURRAY (see *1921*, 191) has been appointed instructor of mathematics at the University of West Virginia.

Mr. F. L. BROWN, of Northwestern University, has been appointed assistant professor of mathematics at the University of Virginia.

At the University of North Carolina, Assistant Professor E. T. BROWNE, of Trinity College, Hartford, Conn., has been appointed assistant professor of mathematics, and Assistant Professor J. W. LASLEY, JR. has been promoted to an associate professorship.

Professor A. T. DELURY, of the University of Toronto, has been appointed Dean of the Faculty of Arts.

At Queen's University, Kingston, Ontario, Associate Professor C. F. GUMMER has been promoted to a full professorship, and Dr. NORMAN MILLER has been promoted to an associate professorship.

Mr. WALTER DENSTON, formerly of the Imperial Naval Engineering College, Kronstadt, Russia, has been appointed assistant professor of mathematics at Kenyon College.

Associate Professor LOUIS BRAND, of the University of Cincinnati, has been promoted to a full professorship of mathematics.

At Purdue University, Professor WILLIAM MARSHALL has been appointed head of the department of mathematics. Assistant Professor R. B. STONE, who is also Registrar, has been promoted to an associate professorship, and Dr. W. E. EDINGTON, Mr. W. H. LYONS and Mr. W. J. WAGNER have been appointed instructors.

Dr. C. P. SOUSLEY is professor and head of the department of mathematics at Rose Polytechnic Institute, Terre Haute (not associate professor, as listed in the Register).

At Indiana University, Associate Professor U. S. HANNA has been promoted to a full professorship, and Assistant Professor CORA B. HENNEL to an associate professorship of mathematics.

At Northwestern University, Professor F. E. WOOD of the Michigan Agricultural College has been appointed assistant professor of mathematics, and Mr. R. L. JACKSON and Mr. J. D. VASS have been appointed instructors.

Assistant Professor W. L. MISER, of the Armour Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. Mayme I. Logsdon has been appointed instructor of mathematics at the University of Chicago.

Dr. C. C. Wylie is associate in astronomy and acting head of the department at the University of Illinois.

At the University of Iowa, Dr. W. H. Wilson has been appointed assistant professor of mathematics, Dr. Roscoe Woods has been appointed associate, and Mr. R. E. Kennon has been appointed instructor.

At Washington University, St. Louis, Assistant Professor P. R. Rider has been promoted to an associate professorship of mathematics, and Mr. Theodore Doll, of Northwestern University, has been appointed assistant professor.

Mr. V. B. Hinsch, of the Missouri School of Mines, has been promoted to an assistant professorship of mathematics.

Professor E. B. Stouffer, of the University of Kansas, has been appointed acting dean of the Graduate School.

Assistant Professor E. D. Meachem, of the University of Oklahoma, has been promoted to an associate professorship of mathematics.

Professor J. E. Burnam, of Simmons College, Abilene, Texas, has leave of absence for the present college year and is studying as a fellow at the University of Texas.

Mr. R. G. Lubben has been appointed instructor of mathematics at the University of Texas.

Professor J. N. Donohue, of Notre Dame University, has been appointed professor of mathematics at Columbia University, Portland, Oregon.

Dr. Nina M. Alderton has been promoted to an assistant professorship of mathematics at Mills College (California).

At the University of California, Dr. B. C. Wong, Dr. Elsie McFarland, and, at the Southern Branch, Mr. Paul Daus, have been appointed instructors of mathematics.

Professor A. Sommerfeld delivered a course of lectures at the Bureau of Standards, Washington, early in March, 1923, on the quantum theory and related subjects.

Professor C. Kostka of Insterburg, died Dec. 28, 1921, at the age of seventy-five years.

Professor Ernest Lebon, of the Lycée Charlemagne, Paris, died February 12, 1922, in his seventy-sixth year.

Dr. A. R. Willis, of the department of mathematical physics at the Royal College of Science, died June 23, 1922, at the age of seventy-two years.

The Fall Meeting of the New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland was held at Hunter College, New York City, December 15, 1922. The program included "The great problem of algebra" by D. E. Smith; a debate on the subject "Resolved, that the mathematics of the first year of the New York City standard high-school course should be confined and held to algebra," in which E. H. Koch and Fletcher Durrell

spoke for the affirmative; and "A proposed new course for senior high-school mathematics" by G. R. MIRICK.

The following awards of Nobel prizes in physics have been announced: for 1921, to Professor ALBERT EINSTEIN, for his theory of relativity and his general work in physics; for 1922, to Professor NIELS BOHR, for his researches in the structure of the atom and in radiation.

The Paris Academy of Sciences has awarded its Lalande Medal to Professor H. N. RUSSELL, of Princeton University, and its Janssen Medal to Dr. CARL STORMER, professor of pure mathematics at the University of Christiania, for his work on the aurora borealis.

The Royal Society of London has awarded its Sylvester Medal to Professor TULLIO LEVI CIVITA, for his researches in geometry and mechanics.

HANS BATTERMANN, professor of astronomy at the University of Königsberg and director of the University Observatory, died June 15, 1922. He was born in 1860. "In addition to valuable work with the transit circle, Professor Battermann carried out valuable investigations on the heliometer, and made accurate determination of the solar parallax from a long series of lunar occultations."

EDMOND HERBERT GROVE HILLS, author of many papers on astronomical and allied subjects, died October 2, 1922. He was born at Winchester, England, August 1, 1864. After a course at the Royal Military Academy, Woolwich, he received a commission in the Royal Military Engineers in 1884. In 1893 he was elected a fellow of the Royal Astronomical Society (president, 1913–1915) and he took part in various eclipse expeditions, the last that of 1914 in Kieff, Russia, whence he was recalled for military service. He became a colonel in 1914 and a brigadier-general in 1918. His notable work in the topographical section of the war office was recognized by the order Commander of St. Michael and St. George, conferred on him in 1902. He was elected a fellow of the Royal Society in 1911, and made a Commander of the Order of the British Empire in 1919.

CHARLES MICHIE SMITH, government astronomer at Madras, 1891–1911, and director of the Kadaikánal and Madras observatories 1899–1911, died September 27, 1922. He was born in Keig, Aberdeenshire, Scotland, July 13, 1854, was educated at Aberdeen and Edinburgh Universities, and was appointed professor of physical science in Madras Christian College in 1877. Apart from publications of the observatories of which he was a director (these included, for example, *New Madras General Catalogue of 5303 Stars*, 1892, 314 pages) most of his papers appeared in the *Proceedings* and *Transactions of the Royal Society of Edinburgh*. He was made a companion of the Order of the Indian Empire in 1910.

HENRY TRESAWNA GERRANS, fellow, vice-provost, and lecturer in mathematics at Worcester College, Oxford, died June 20, 1921. He was born at Plymouth, England, August 23, 1858. As a student at Christ Church, Oxford, he won first class in mathematics in the "mods." and "greats." He was appointed a fellow of Worcester in 1882 and later occupied numerous executive positions in connection with the University including that of being a delegate of the Oxford University Press since 1896. He was a member of many mathematical and scientific societies. As editor of the fifth edition of the second volume of G. M. Minchin's *Treatise on Statics* (Oxford, 1915) he made large additions to the work in connection with problems for solution. "He was a musician, and as much at home in German as in English. Thirty years ago his rooms on Sunday evening used to be crowded with foreigners, with all of whom, it was said, he could converse fluently." At one time, "he considered that continental holiday ill-spent in which he did not master some unlearnt language." A portrait and sketch of Mr. Gerrans appeared in *The Periodical*, Oxford, September, 1921, page 95.

CARGILL GILSTON KNOTT, lecturer on applied mathematics at the University of Edinburgh since 1892, and general secretary of the Royal Society of Edinburgh, died October 26, 1922. He was born at Penicuik, Scotland, June 30, 1856, and became a graduate of the University of Edinburgh. During 1883–1891 he was professor of physics at the University of Tokyo, Japan, and was decorated with the Fourth Order of the Rising Sun in 1891. He was honorary secretary of the Napier Tercentenary Celebration, Edinburgh, 1914, and editor of its sumptuous *Memorial Volume*, London, 1915, which contained papers by more than a score of contributors including the American authors, F. Cajori, A. Martin, and D. E. Smith. He was the editor of *Edinburgh's Place in Scientific Progress*, Edinburgh, 1921, and he wrote the chapter on "Mathematics and natural philosophy" (compare *1921*, 456). His third edition of Kelland and Tait's *Quaternions*, London, 1904, and his elaborate *Life and Scientific Work of Peter Guthrie Tait*, Cambridge, 1911, are well known.

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

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Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Summer Meeting of the Association, Vassar College, September 5–6, 1923

Eighth Annual Meeting, University of Cincinnati, December, 1923

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Knox College, Galesburg, May 4–5

IOWA, Cornell College, Mount Vernon, April 27–28

KANSAS, Topeka, January 20

KENTUCKY, University of Kentucky, Lexington, April

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Baltimore, May 12

MINNESOTA, St. Paul, May 27

MISSOURI, University of Missouri, Columbia, November 30–December 1

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April

SOUTHEASTERN, Agnes Scott College, Decatur, March 10

TEXAS, Houston, December 1–2

The American Mathematical Monthly

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SEVENTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of Harvard University, the seventh annual meeting of the Mathematical Association of America was held at Harvard on Thursday and Friday, December 28 and 29, 1922, in affiliation with The American Association for the Advancement of Science, and in conjunction with the annual meeting of the American Mathematical Society. There were 216 at the sessions, including the following 148 members of the Association:

R. C. Archibald, Brown University.
C. S. Atchison, Washington and Jefferson College.
F. H. Bailey, Massachusetts Institute of Technology.
L. A. Bauer, Dept. of Terrestrial Magnetism, Washington, D. C.
Ralph Beatley, Harvard Graduate School of Education.
W. J. Berry, Brooklyn Polytechnic Institute.
O. F. H. Bert, Washington and Jefferson College.
E. G. Bill, Dartmouth College.
G. D. Birkhoff, Harvard University.
G. A. Bliss, University of Chicago.
S. L. Boothroyd, Cornell University.
H. C. Bradley, Massachusetts Institute of Technology.
F. E. Brasch, Chicago, Illinois.
B. H. Brown, Dartmouth College.
Eleanor Brown, Hanover, N. H.
R. E. Bruce, Boston University.
N. R. Bryan, University of Maine.
Margaret Buchanan, West Virginia University.
W. G. Bullard, Syracuse University.
R. W. Burgess, Brown University.
W. R. Burwell, Brown University.
W. D. Cairns, Oberlin College.
Florian Cajori, University of California.
B. H. Camp, Wesleyan University.
Mildred E. Carlen, Brown University.
F. E. Carr, Oberlin College.
B. E. Carter, Colby College.
J. W. Clawson, Ursinus College.
J. L. Coolidge, Harvard University.
Lennie P. Copeland, Wellesley College.
J. A. Cragwall, Wabash College.
W. L. Crum, Yale University.
C. H. Currier, Brown University.
W. W. Denton, University of Michigan.
C. E. Dimick, U. S. Coast Guard Academy.
E. L. Dodd, University of Texas.
W. F. Downey, English High School, Boston.
L. H. Dube, Ottawa University.
L. P. Eisenhart, Princeton University.
L. C. Emmons, Michigan Agricultural College.
T. C. Esty, Amherst College.
G. C. Evans, Rice Institute.
G. W. Evans, Charlestown High School.
Floyd Field, Georgia School of Technology.
J. C. Fields, University of Toronto.
C. H. Forsyth, Dartmouth College.
Tomlinson Fort, University of Alabama.
Philip Franklin, Harvard University.
C. A. Garabedian, Harvard University.
R. W. Gardner, Eastern Nazarene College.
O. T. Geckeler, Carnegie Institute of Technology.
R. E. Gilman, Brown University.
O. E. Glenn, University of Pennsylvania.
J. W. Glover, University of Michigan.
Alice B. Gould, Boston.
W. C. Graustein, Harvard University.
C. F. Gummer, Queen's University.
C. E. Haigler, Wentworth Institute, Boston.
W. M. Hamilton, U. S. Naval Observatory.
E. S. Hammond, Bowdoin College.
J. G. Hardy, Williams College.
J. N. Hart, University of Maine.
Olive C. Hazlett, Mt. Holyoke College.
H. C. Hicks, Brown University.
H. L. Hodgkins, George Washington University.
E. V. Huntington, Harvard University.
W. A. Hurwitz, Cornell University.
M. H. Ingraham, University of Chicago.
O. D. Kellogg, Harvard University.
A. E. Kennelly, Harvard University.
W. D. Lambert, U. S. Coast and Geodetic Survey.
Gillie A. Larew, Randolph-Macon Woman's College.
H. E. A. Lazott, South High School, Worcester.
D. D. Leib, Connecticut College for Women.
Joseph Lipka, Massachusetts Institute of Technology.
C. C. MacDuffee, Princeton University.

MARTHA P. MCGAVOCK, Wellesley College.
H. P. MANNING, Brown University.
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F. M. PEDERSEN, College of the City of New York.
ANNA J. PELL, Bryn Mawr College.
L. R. PERKINS, Middlebury College.
E. C. PHILLIPS, Woodstock College.
H. B. PHILLIPS, Massachusetts Institute of Technology.
A. D. PITCHER, Western Reserve University.
W. R. RANSOM, Tufts College.
R. G. D. RICHARDSON, Brown University.
L. H. RICE, Massachusetts Institute of Technology.
H. L. RIETZ, University of Iowa.
E. D. ROE, JR., Syracuse University.
W. H. ROEVER, Washington University.
GEORGE RUTLEDGE, Massachusetts Institute of Technology.
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MARIAN M. TORREY, University of West Virginia.
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JESSICA M. YOUNG, Washington University.
J. W. YOUNG, Dartmouth College.
S. D. ZELDIN, Massachusetts Institute of Technology.

The various meetings of the American Association for the Advancement of Science were held throughout the week, mainly at the buildings of the Massachusetts Institute of Technology, beginning Tuesday evening with the address of the retiring President, Professor E. H. Moore, on the subject "What is a Number System?" The manifold programs in the meetings of the various Sections of the American Association and of the affiliated Societies furnished a wealth of opportunity for exercising one's choice. The majority of mathematicians stayed in comfortable rooms in the Harvard dormitories and were thus within easy reach of the places of meeting, and of the Harvard Union and other convenient dining halls. While the formal programs of both mathematical

organizations were held at Harvard University, a close coöperation between the staff of Harvard and of the Massachusetts Institute of Technology gave the attending members a chance to enjoy the hospitality of both institutions. It was thus possible between the two morning sessions of the Society to enjoy the Symposium on Space and Time, under the auspices of the Society and Section A, Wednesday afternoon at the Technology buildings, at which time Professors G. D. BIRKHOFF, P. W. BRIDGMAN, and HARLOW SHAPLEY, all of Harvard University, spoke respectively, on "The logic of Space and Time," "The physical meaning of Space and Time," and "The astronomical measures of Space and Time." On Wednesday evening various groups attended the dinner for visiting women, the trip to the Harvard Astronomical Observatory, a private dinner for Harvard men, the Sigma Xi dinner, and a smoker in the Harvard Union. On Thursday which was "Harvard Day," for the American Association and the affiliated societies, a complimentary luncheon was served by Harvard University at Memorial Hall. A pleasant occasion at the close of each afternoon session was the serving of tea at Phillips Brooks house by the ladies of the Harvard Faculty, under the chairmanship of Mrs. Graustein; this, with the other provisions made by the ladies for the entertainment of the visitors, was recognized in a resolution adopted at the closing session, on motion by Professor Clara Smith. At this same session Professor Floyd Field offered a motion, which was heartily adopted, recognizing the unusual interest aroused by the programs, and expressing the thanks of the Association to the departments at Harvard and "Tech" for the hearty reception and careful arrangements; to Professors Coolidge, Kellogg and Lipka of the Committee on Arrangements for their efficiency, courtesy, and unfailing good humor in their busy task; to Professors Huntington, Copeland and Currier of the Program Committee for the presentation of the program of such a wide range of interest to all, and to the officers of the Association for their administration of the affairs of the Association for the past year.

The joint dinner of the two organizations and Section A was held Thursday evening at the Walker Memorial Building of the Institute of Technology, with 135 persons present, Professor David Eugene Smith being the genial toastmaster. Introducing four men, as he said, to give an account of the three periods of mathematical history in America, he called first upon Professor Cajori, who spoke on the early period of mathematics in America, describing what was probably the first controversy on a scientific subject, and describing the earliest permanent observatory, established at Bogota in 1803. Professor Van Vleck spoke on the "middle ages" in American mathematics, telling how this central period began to make strong provision for the existence of suitable journals, of acceptable centers of influence and, most of all, of an atmosphere favorable for mathematical study; how these ends were provided through the *American Journal*, as the first able mathematical publication, Johns Hopkins University as the first strong center, followed by the establishment of those at Clark University and at Chicago, the establishment of the New York Mathematical Society in 1891 and its extension in 1894 into the American Mathematical Society; he appealed

especially to the younger men in American mathematics to see to it that it shall be possible for us to say:—"The best is yet to be." Professor Eisenhart spoke of the large and influential work done by the American Mathematical Society and its journals, and of the difficult financial problem that is now being faced in conducting the Transactions. Professor Slaught, called upon as a representative of the Association, spoke of the founding of the Mathematical Association of America, as lying within the more recent period of American mathematics, of the widened opportunity on the one hand for edifying receptivity in matters of collegiate mathematics, and on the other hand for fruitful activity represented, for example, in the pages of the MONTHLY, in the summer and annual meetings, and in at least ten section meetings each year, stressing above all the opportunity to help larger and larger numbers of our teachers and students to begin to mount the ladder of research through the furnishing of the lower rounds of this ladder. Professor Smith expressed very happily the sentiments of those present toward "Old Harvard" and "Tech," and this was reinforced by the visitors by a rising vote.

Professor G. A. Miller presided at the joint session on Thursday afternoon. President Archibald presided for most of the Friday morning session, calling Professor Eisenhart, one of the newly elected vice-presidents, to the chair for the rest of the day.

The following program was given, abstracts of part of these papers being given with numbers corresponding to the numbers in the list of titles.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION.

(1) "Reduction of singularities of plane curves by birational transformation" by Professor G. A. BLISS, University of Chicago, retiring president of the American Mathematical Society.

(2) "The grafting of the theory of limits on the calculus of Leibniz" by Professor FLORIAN CAJORI, University of California, representing the Mathematical Association of America.

(3) "Geometry and physics" by Professor OSWALD VEBLEN, Princeton University, retiring vice-president of Section A.

1. President Bliss's address is to be printed in the *Bulletin of the American Mathematical Society.*

2. Professor Cajori's address will appear shortly in this MONTHLY.

3. Professor Veblen's address appeared in *Science* for February 2, 1923.

SESSION OF THE ASSOCIATION.

(4) "Period of the bifilar pendulum for finite amplitudes" by Professor H. S. UHLER, Yale University.

(5) "Skew squares" by Professor W. H. ECHOLS, University of Virginia.

(6) "On the averaging of grades" by Professor C. F. GUMMER, Queen's University.

(7) "Mathematics at Oxford and the Ph.D. degree" by Professor W. R. BURWELL, Brown University.

(8) "Some unsolved problems in the theory of sampling" by Professor B. H. CAMP, Wesleyan University.

(9) "Some unsolved problems in solid geometry" by Professor J. L. COOLIDGE, Harvard University.

4. The primary object of Professor Uhler's investigation was to derive a convergent series for the period of the bifilar pendulum corresponding to any feasible finite amplitude of vibration. The material is conveniently divided in three parts. In part (*a*) attention is called to the unsatisfactory state of the literature of the problem. Then a set of rigorous equations are given which lead to a hyperelliptic integral as the formula for the period. In part (*b*) this integral is expanded into two related series which depend upon a fundamental series the law of development of which is evident. It is shown that the second and third approximation terms of the two final series may be either independently or simultaneously positive, or zero, or negative. Numerical illustrations are then presented. In part (*c*) the domain of convergence of the series is discussed and shown to depend upon points lying between certain straight and parabolic lines in a Cartesian diagram.

This paper appeared in the *Journal of the Optical Society of America* for March, 1923.

Professor Slaught stated that Professor W. D. MacMillan has treated a similar problem where the rod is suspended in the manner of a bifilar pendulum, but where the motion is not limited to mere twisting, the rod being put in motion from any position in which it may be drawn off from its vertical position of rest. The motion of the rod when released is of an amazing nature.

Professor Huntington entered a plea for the construction of a table of elliptic integrals for all values of the parameter, saying that such a table would be of great usefulness in this and similar problems.

5. In the absence of Professor Echols, due to illness in the family, Professor Graustein gave a very acceptable presentation of the paper. This will be printed in an early issue of the MONTHLY.

In the discussion Professors Young and Clara Smith criticized the use of the term "skew square," which usually suggests a quadrilateral not lying in one plane; and Professor Archibald stated that several French writers had used the term pseudo-square for this kind of figure.

6. Examining boards often wish to arrange in some kind of order of merit candidates who have been examined in a number of different subjects; and in several other connections the idea of "average grade" is more or less directly present. Of the various functions $f(x_1, x_2, \cdots, x_n)$ that may be arbitrarily selected to represent such an average for grades $x_1, x_2, \cdots, x_n$, of equal weight, it may reasonably be demanded that (*a*) $f(x_1, x_2, \cdots)$ be symmetric, (*b*) $f(x_1, x_2, \cdots, x_n) = x$, (*c*) $f(kx_1, kx_2, \cdots) = kf(x_1, x_2, \cdots)$ for positive k, (*d*) continuous first partial derivatives exist and remain positive for all positive x's.

The simplest solution meeting these requirements would seem to be that in which f is the quotient of a homogeneous quadratic by a homogeneous linear form, the formula being $f = a + cd/a$, where a is the arithmetic mean, d the mean square of the deviation from a, and c a constant open to choice, which will be taken negative if it is desired to put a premium on equality of performance. To satisfy the latter part of (d) c must lie between $-1/(n-1)$ and $(n-1)/(2n-1)$.

A formula in some ways more convenient is available if (d) is so far relaxed as to allow the derivatives to become discontinuous and non-existent at certain places, the discontinuity being preferably not too abrupt; namely, $f = m_1y_1 + m_2y_2 + \cdots + m_ny_n$, where y_i is the ith greatest x and the m's are constants. If we agree that a additional marks equally divided among the r best subjects shall be counterbalanced by the loss of b marks equally divided between the $n - r$ lowest, whatever r may be, we are able to determine the m's in terms of the arbitrary ratio a/b. The result may be written $f = y_n + s_{n-1}(y_{n-1} - y_n) + \cdots + s_1(y_1 - y_2)$, where $s_i = m_1 + m_2 + \cdots + m_i = ib/(na - ia + ib)$. This formula is very easy to use. To favor equality a must be greater than b. Various modifications of the method suggest themselves.

In the discussion of Professor Gummer's paper, Professor Dodd referred to Schimmack's axioms in the *Mathematische Annalen*, volume 68, 1910, pp. 125–132, pointing out that the assumption of the existence of the derivatives rules out the use of the median in making an average. He referred also to the treatment of a somewhat similar question in a paper by Daniell in the *American Journal of Mathematics*, volume 42, 1920, 222–236.

7. Professor Burwell's paper will appear in an early issue of the MONTHLY.

8. Professor Camp confined his attention to simple sampling from distributions involving but one variate. The general question to be answered is: How good is a sample? The answers may be discussed under three heads, as follows. (a) Sampling for a mean value. What is the probability that the mean of the sample will lie within a prescribed amount of the mean of the parent distribution? The solution of this problem has been published only for the special case where the parent distribution is normal, or sufficiently near to normal to warrant the assumption that the "curve of means" is normal. It is, however, probable that the general case may be solved by a combination of some of the theory of Charlier with certain relations developed by the Pearsonian school. (b) Sampling for a standard deviation or other statistic. Here again only the special case where the parent distribution is normal has been adequately treated. (c) Sampling for a frequency curve. What is the probability that the curve found by the sample will lie within a prescribed belt of the ideal? The classic answer to this question is given by K. Pearson's χ test. Edgeworth's criticism of the random use of this test is important and may be found in his article on "Probability" in the *Encyclopædia Britannica*.

The relatively simple questions considered above suggest more difficult and more practical ones, for which, save in highly specialized instances, there exist no answers. If one starts with a sample, instead of with a parent distribution,

one has the problem of inverse probability, called by K. Pearson "the fundamental problem of practical statistics," with the various special cases (*a*), (*b*), (*c*), as above. Finally there is the broad field of generalization to functions of more than one variable, involving the properties of correlation surfaces rather than of frequency curves.

Professor Rietz called attention to the fact that in the *Journal of the Royal Statistical Society* for January, 1922, Yule and Fisher presented a modification of Pearson's criterion of goodness of fit of theory and observation by showing that the usual n with which to enter the tables of Elderton should be one more than the number of degrees of freedom of the frequency system.

9. The paper by Professor Coolidge adds to the series of valuable papers which have already appeared on the programs of the Association as giving suggestions as to possible problems in research in familiar fields. This paper will appear in full in an early issue of the MONTHLY.

SYMPOSIUM ON MATHEMATICAL STATISTICS.

(10) "The subject matter of a course in mathematical statistics" by Professor H. L. RIETZ, University of Iowa.

(11) "Time series of economic statistics: Their fluctuation and correlation" by W. M. PERSONS, professor of economics, Harvard University, and editor of the *Review of Economic Statistics*.

(12) "Some fundamental concepts of the calculus of mass variation and their relation to practical problems" by ARNE FISHER, author of *The Mathematical Theory of Probabilities and Its Application to Frequency Curves and Statistical Methods*.

10. Professor Rietz's outline of a course in mathematical statistics will be printed in a later issue of the MONTHLY.

11. Time series differ in important respects from other series of economic statistics. The items of a time series (1) must be defined for selected time units, (2) are ordered in time, and (3) during adjacent time-intervals are affected by the same or similar influences. These peculiarities give rise to characteristic types of fluctuations and to special correlation problems.

Four types of variations are to be found in most time series. These are: the long-time movement or secular trend; the seasonal variation; the cyclical movement connected with the ebb and flow of business; and irregular fluctuations resulting from panics, strikes, etc. In order to isolate the cyclical movement, so far as may be, it is necessary to measure the secular trend and seasonal variation. In Professor Persons' paper attention was directed to the problems of measuring seasonal variation and of finding the significant relations between series corrected for secular trend and seasonal variation. Monthly rates on commercial paper, 1866–1922, are used to illustrate the methods of handling the problems.

The obvious method of finding seasonal variation is to eliminate secular trend and then take arithmetic means of corresponding months (Januaries,

Februaries, etc.). This method works well for long series or parts of series *which are not subject to large irregular fluctuations*, but even in these cases it has the defect that the *degree of regularity* of the supposed seasonal movement is not set forth. The method works badly for the great majority of economic series, because such series contain many non-seasonal fluctuations, highly irregular both as to size and time of occurrence, which greatly distort the arithmetic means of the items. In other words, the actual data with which the economist has to deal do not follow the ideal functional relations, which would lead to the use of the arithmetic means of all the items for corresponding months for determining the seasonal variation. (The assumptions made by W. L. Hart, for instance, in his paper on "The method of monthly means for determination of a seasonal variation," *Journal of the American Statistical Association*, September, 1922, cannot be retained in the great majority of cases.)

A method of measuring seasonal variation which has been found to work well in practice (see the *Review of Economic Statistics*, January 1919, pp. 18–31) is the following: find the link relatives of the original items; tabulate the link relatives for January, February, etc., as frequency distributions; find the medians (or, if preferred, the averages of the 3, 4 or more central items) of the link relatives; chain the medians (or selected average of central items) to secure a fixed base; distribute the discrepancy between the beginning and end of the chain relatives among all the members of the chain (the distribution may be either geometric or arithmetic).

The chief advantages of the method just outlined are: (*a*) the use of frequency tables of link relatives enables one to ascertain the degree of regularity of seasonal changes; (*b*) the use of the median (or average of central items) minimizes the influence of non-seasonal, irregular fluctuations; (*c*) the entire procedure makes it possible to utilize series of items covering long intervals, not entirely homogeneous, and subject to varying cyclical and irregular fluctuations.

Comparison of the indices of seasonal variation computed by the link-relative-median method, and the arithmetic average method for rates on commercial paper for the intervals 1866–73, 1874–89, 1890–99, and 1900–13 shows: significant divergence for intervals with marked irregular disturbances, such as 1866–73 and 1890–99; and practically identical results *when the years of great disturbance, such as 1873, 1890, 1893, 1896, 1898, 1907*, and *1914 are excluded from the arithmetic averages.*

The frequent occurrence of large irregular fluctuations in economic data greatly affects the Pearsonian coefficients of correlation for pairs of series, even when the items are corrected for seasonal variation and secular trend. For instance, for Bradstreet's prices and pig-iron production, bimonthly, 1903–1914, the coefficient of correlation for concurrent items (65 in number) is + .73, and omitting 4 items in 1908 is only + .66; for Bradstreet's prices and New York clearings six months later the coefficient is + .60, and omitting 4 items is only + .50. Coefficients of correlation possess different significance for economic data than for individual measurements and great caution must be used in the

interpretation of such coefficients. Comparison of the graphs of economic series, duly corrected for secular trend and seasonal variation and expressed in units of standard deviation, gives much more information concerning existing relationships than is given by coefficients of correlation. The irregular fluctuations of time series of economic statistics not only must be considered and allowed for in computing indices of seasonal variation and in interpreting coefficients of correlation but also in applying periodogram analysis to such series.

A question by Professor Burgess brought the answer that the foregoing is a method of "red-handed empiricism," used with the purpose of trying to separate the terms of the first three types and then to analyze the irregular fluctuations.

12. Mr. Fisher in his address emphasized the fact that the dominant feature of our material world is that of variation. The old religious philosophies of our Aryan ancestors emphasized the presence of inequality and variation in nature; without variation evolution would be impossible according to the gospel as preached by the great Buddha. Modern science and especially the far-reaching studies inaugurated by Charles Darwin have again brought the question of variation to the forefront. Darwin was the first to point out the definite presence of variation among a large group of similar objects (a species). While Darwin was content to establish the paramount feature of variation in animal and plant life and to outline its general philosophical contours, he did not attempt to subject it to purely quantitative measurement.

This latter aspect of the problem of variation is primarily of a statistical, *i.e.*, of a purely mathematical nature. Statistics, according to the speaker, might properly be defined as the calculus of mass variations. The question then arose how this particular calculus should be formulated and taught. Should it be presented as a purely empirical discipline, or should it be made subject to a more rigorous mathematical method of attack and like all mathematical epistemology be made to rest on a purely *a priori* foundation?

The empirical method of investigation is, however, very limited in scope and apparently rests upon the fallacious doctrine of John Stuart Mill that it was possible to derive the corpus of human knowledge by what Mill termed an "*inductio per simplicem enumerationem.*" This method has especially been used by the German school of statistics under von Mayr and his disciples. Until comparatively recent years this school has been the only one followed in statistical work here in America. Its chief exponents are found among such writers as Mayo-Smith, Willcox, Hoffman, Dublinsky and Secrist. Unfortunately, in the opinion of the speaker, most of such writings often degenerate into mere verbiage. One recent American writer in his text on statistics spends more than fifty pages in the discussion of such simple statistical characteristics as the mean and the dispersion. By the employment of elementary mathematical theorems from the calculus of probabilities the same amount of information could be compressed into a few simple formulas, not occupying more than a single book page. All that such authors spent pages upon pages in telling to their readers could be seen in a mere glance at the empirical tables themselves.

The reason for this is chiefly to be found in the fact that the method of investigation as employed by the authors referred to above is really no method at all. The fallacy of these authors is to be found in the fact that they mistake a mere collection of observations (statistical data) for a statistical analysis.

The keen mind of the great Greek scholars (Aryan scholars) always laid stress upon their doctrine that it was the relation *between* the facts which was the important thing to unearth. Modern astronomy offers a striking illustration of this statement. For centuries the Egyptians, the Babylonians, the Greeks, the Hindus, and the Arabs had made mass observations on the celestial objects. An immense mass of statistical data had been gathered and later on augmented and arranged in tabular form by the Danish astronomer Tycho Brahe. Yet the mechanical laws governing our universe were not discovered before Kepler. Newton and Laplace had subjected these statistical data to a mathematical investigation. It was then found that apparently complicated phenomena in reality were subject to simple mathematical laws. Our present day so-called sciences of economics and sociology are still waiting for their Newtons and Laplaces before they really are worthy of the name of science.

What made astronomical science possible was the mathematical tools as furnished by the differential and the integral calculus, a mathematical discipline which rests upon a purely *a priori* foundation, and which made it possible to subject the collected astronomical statistics to a quantitative analysis. Have we now a similar calculus in the case of mass variation? While the speaker was of the opinion that we do not possess a complete set of tools and many new implements have still to be forged, he would, on the other hand, emphasize that we have at least several good tools in the theory of probabilities. Statistical analysis and mathematical statistics must necessarily, in his opinion, be an application and an extension of the theory of probability.

In this particular mathematical discipline the great work of Laplace remains still the greatest contribution. Laplace's theory has become especially useful through the additional researches by the Scandinavian statisticians and mathematicians Gram, Thiele, Jörgensen, Charlier, Wicksell and Westergaard. The speaker contrasted these methods with those introduced by the English mathematician and biometrician Karl Pearson.

Pearson's theory of frequency distribution is derived from the differential equation

$$\frac{d \log y}{dx} = \frac{x + a}{b_0 + b_1 x + b_2 x^2}.$$

This equation gives rise to a great variety of frequency curves. It is, however, more or less a graduation formula based upon rather empirical foundations, one disadvantage of Pearson's formula being that it does not allow of more than four constants, which often is insufficient to represent frequency distributions.

The Scandinavian authors who follow Laplace have based their theory of frequency curves on the following integral equation:

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \frac{\lambda_3\omega^3}{3!} + \cdots} e^{xi\omega} d\omega,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the semi-invariants of Thiele, defined by the relation

$$e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \frac{\lambda_3\omega^3}{3!} + \cdots} \int_{-\infty}^{+\infty} \varphi(x)dx = \int_{-\infty}^{+\infty} e^{x\omega}\varphi(x)dx.$$

If all semi-invariants above λ_2 vanish, we have

$$\varphi(x) = \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\frac{(x-\lambda_1)^2}{2\lambda_2}}$$

or the usual normal probability curve of Laplace.

As shown by Gram and Charlier any frequency function which, together with its derivatives, vanishes for $x = -\infty$ and $x = +\infty$ can be represented by the following expansion:

$$F(x) = \varphi(x) + c_3\varphi'''(x) + c_4\varphi''''(x) + \cdots,$$

where

$$c_3 = -\frac{\lambda_3}{3!}, \qquad c_4 = \frac{\lambda_4}{4!}, \qquad \cdots.$$

The speaker also mentioned the expansion of frequency series by means of the Poisson-Charlier series and the logarithmic transformation as discussed in his own works on probability. As a very general expansion the following function introduced by Bachelier:

$$F(x, t) = \frac{1}{\sqrt{2\pi}\,\varphi(t)} e^{-\frac{[x-\varphi(t)]_2}{2\varphi(t)^2}}$$

was mentioned as being very powerful.

Mr. Fisher claimed that this function was especially useful in economic statistics and afforded a far more useful tool of investigation than the somewhat looser mathematical methods employed by the Harvard Bureau of Economics. The Bachelier and Gram methods might, for instance, be used to solve the following problem: What is the probability that a certain stock or bond will be quoted at a price x at time t on the stock exchange? The speaker showed an actual application he himself had made in the matter of forecasting three months in advance the weekly quotations of a certain gilt-edge stock on the Copenhagen Stock Exchange. During the year of 1922 the lowest value of this stock had been 196 and the highest value 243. The greatest difference between any weekly forecast and the prices actually quoted had been 4 per cent. for one of the first weeks of March. The total accumulated percentage differences from January to December 15, 1922, were less than one tenth of one per cent.

The speaker criticized the investigations by various economists of the so-

called business cycles as being the work of mathematical dilettantes. He mentioned an eminent economist as having said at a recent statistical meeting "that the economists would like to have the mathematicians express their formulas in such a manner that the man in the street could understand what they (the mathematicians) were talking about." The danger of such a spurious argument was that it might equally well be applied to the economists themselves. The immortal answer of Euclid to the Ptolemæan emperor: "There is no royal road in mathematics" was in the opinion of the speaker as true today as it was in the days of antiquity and applied equally well to our present-day analysis in economics and sociology.

Professor J. R. Musselman, of Johns Hopkins University, described the elementary and advanced course in mathematical statistics as given at Johns Hopkins, the former being offered to those who have had the elements of calculus and analytic geometry, and including the topics of permutations and combinations, elementary probability, the use and limitations of the theory of the correlation coefficient, the theory of sampling and curve fitting; time is also available to include the theory of least squares.

Professor Franz Boas, of the department of anthropology, Columbia University, spoke of the great need of instruction in this field for students in various departments of biology. We try too often to treat the theory of averages, standard deviation, correlation and similar topics in a purely formal way; we need rather to give the student of biometry and analogous fields a clear understanding of the subject, and the ability to come to a clear grasp of just what the problems under consideration actually are. This aspect of the subject is hardly recognized in the existing literature.

Professor Theodore H. Brown, of the School of Business, Columbia University, stated that he continually met with New York business men who claim on various grounds to be statisticians. Too frequently men are put to work on statistical matters without an adequate, or at least a ready, hold of the necessary mathematics. There is a large class of students who are hungry for training in this line, and who might be reached through properly prepared texts.

Dr. Herbert A. Toops, of the Institute of Educational Research, Teachers College, Columbia University, pointed out the need that educators have for statistical work, and stated that they ordinarily depend in a groping way on difficult texts in statistics. The Institute aims to develop new methods of research. Presuming a basis of college algebra and a high degree of intelligence, the Institute does, through shortened methods, enable their students to carry on tabulations and similar operations, and also carries out extended work with a few who by further study shall be able to go back to the original sources. It is quite important in this field of statistics to develop really practical statisticians, to give, for example, practice in computing a percentile from both ends in order to check the result, in the use of computing machines, etc.

Professor Huntington called attention to the fact that the National Research Council has appointed a Committee on the mathematical analysis of statistics,

of which Professor Rietz is the chairman, and said that a handbook will be published by this committee within a year, through one of the well-known publishing houses, a book to which those present at this program, and others interested in the subject, will look forward with eagerness.

Meeting of the Board of Trustees of the Association.

Ten members of the Board were present at the meeting.

The following thirty-one persons and one institution, on applications duly certified, were elected to membership:

To individual membership.

Olive Atwood, A.B. (Carleton Coll.). Instr., Agricultural Coll., Agricultural College, N. Dak.
Eleanor Brown, (Mrs. B. H.), Ph.D. (Radcliffe). Hanover, N. H.
R. D. Burdick, A.M. (Columbia). Instr., Coll. of City of New York, New York, N. Y.
H. W. Chandler, M.S. (Iowa). Instr., Univ. of Minnesota, Minneapolis, Minn.
E. E. Colyer, A.M. (Kansas). Prof., Fort Hays Kansas State Normal School, Hays, Kansas.
W. H. Cramblet, Ph.D. (Yale). Prof., Bethany Coll., Bethany, W. Va.
L. A. V. DeCleene, (Rev.), A.M. (Catholic Univ.). Graduate student, Catholic Univ., Washington, D. C.
Lalit Mohan Dey. Govt. official, Calcutta, Shambazar P.O., India.
Laura Duerner, B.S. (Oregon). Instr., N. Dak. Agric. Coll., Fargo, N. Dak.
J. T. Erwin, A.M. (Vanderbilt). Prof., George Washington Univ., Washington, D. C.
H. W. Ficken, B.S. (Coll. City of N. Y.). Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
S. E. Field, A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.
R. W. Gardner, B.S. (Olivet). Head of dept. of math., Eastern Nazarene College, Wollaston, Mass.
O. T. Geckeler, A.B. (Indiana). Asso. prof. and acting head of dept., Carnegie Inst. of Tech., Pittsburgh, Pa.
Martha Hildebrandt, Ph.B. (Chicago). Teacher, Proviso High School, Maywood, Ill.
F. W. John, M.E. (Cornell). Instr., Washington Square Coll., N. Y. Univ., New York, N. Y.
H. E. A. Lazott, A.B. (Boston Univ.). Instr., South High School, Worcester, Mass.
E. A. LeLacheur, A.B. (Valparaiso). Computer, U. S. Coast and Geodetic Survey, Washington, D. C.
W. V. Lovitt, Ph.D. (Chicago). Prof., Colorado Coll., Colorado Springs, Colo.
W. E. MacDonald, A.M. (Harvard). Prof., Canton Christian Coll., Canton, China.
William Marshall, Ph.D. (Zurich). Prof., Purdue Univ., Lafayette, Ind.
A. E. Meder, Jr., A.B. (Columbia). Asst., Columbia Univ., New York, N. Y.
Helen Moon, M.S. (Iowa). Graduate student, Univ. of Iowa, Iowa City, Ia.
L. T. Moore, B.S. (Emory Univ.). Asst., Johns Hopkins Univ., Baltimore, Md.
R. H. Mortimore, A.B. (Iowa). Instr., Graceland Coll., Lamoni, Ia.
P. H. Nygaard, A.B. (St. Olaf). Teaching Asst., Univ. of Minn., Minneapolis, Minn.
Margaret C. Packer, A.M. (Brown). Instr., math. and physics, Hood College, Frederick, Md.
N. A. Pattillo, Ph.D. (Johns Hopkins). Dean and prof. of math., Randolph-Macon Woman's Coll., Lynchburg, Va.
J. E. Redden, A.B. (Furman Univ.), B.S. in C.E. (Clemson Coll.). Asst. prof., John Tarleton Agric. Coll., Stephenville, Tex.
Arthur Tilly. Instr., Washington Square Coll., N. Y. Univ., New York, N. Y.
Mary Ruth Whaley, A.B. (Smith). Instr., St. Agnes School, Albany, N. Y.

To Institutional Membership.

Western College for Women, Oxford, Ohio, Prof. Harriet E. Schoonmaker, Official representative.

The following have been appointed associate editors of the MONTHLY for the year 1923:

N. H. ANNING	B. F. FINKEL	R. B. MCCLENON
R. W. BURGESS	TOMLINSON FORT	C. N. MILLS
E. L. DODD	D. C. GILLESPIE	F. B. MURNAGHAN
OTTO DUNKEL	C. F. GUMMER	D. E. SMITH

Professor H. L. RIETZ was appointed to be the representative of the Association in the Division of Physical Sciences, National Research Council, for the three-year term beginning June 11, 1923, in succession to Professor E. R. Hedrick.

It was voted to hold the annual meeting next December at the University of Cincinnati in affiliation with the American Association. The incoming President was empowered to appoint the necessary committees for the summer meeting, and an announcement of the place and time will be made as early as possible. Professor A. D. PITCHER and the Secretary were appointed as representatives of the Association on the Council of the American Association for the year 1923.

Professor R. C. ARCHIBALD was appointed to the vacancy for the term ending January 1925, caused by the election of Professor Eisenhart as a vice-president.

The Trustees appointed the following persons for the year 1923:
Secretary-Treasurer: W. D. CAIRNS.
Librarian: L. C. KARPINSKI.
Members on Committee on Official Journal: J. L. COOLIDGE, W. B. FORD, and H. E. SLAUGHT.
Assistant Secretary: C. H. YEATON.
Assistant Librarian: MARY E. SINCLAIR.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership by the Board. He reported also the death of the following members:

FREDERICK ANDEREGG, Professor of mathematics, Emeritus, Oberlin College (October 9, 1922);
W. W. BEMAN, Professor of mathematics, University of Michigan (January 18, 1922);
C. L. BOUTON, Associate professor of mathematics, Harvard University (February 20, 1922);
C. A. FISCHER, Professor of mathematics and astronomy, Trinity College, Hartford, Conn. (December 7, 1922);
ANGELO HALL, Professor of mathematics, U. S. Naval Academy (April 14, 1922);
JAMES MCMAHON, Professor of mathematics, Emeritus, Cornell University (June 1, 1922);
C. M. NOLAND, Professor of mathematics, Howard Payne College, Brownwood, Texas (October 2, 1922);
B. F. SIMONSON, Professor of mathematics, Upper Iowa University (February 27, 1922).

The election of officers for the year 1923 was conducted by mail, and in person at this meeting, as provided in the By-Laws. The tellers reported the result of the balloting as follows:

For President: R. D. Carmichael, 223 votes; H. L. Rietz, 179 votes.

For Vice-Presidents: A. A. Bennett, 173 votes; A. B. Chace, 256 votes; L. P. Eisenhart, 186 votes; B. F. Finkel, 176 votes.

For additional members of the Board of Trustees (to serve until January, 1926): C. F. Gummer, 223 votes; Dunham Jackson, 226 votes; N. J. Lennes, 116 votes; E. H. Moore, 277 votes; R. E. Moritz, 130 votes; Clara E. Smith, 217 votes; Oswald Veblen, 188 votes; J. W. Young, 215 votes.

The following were accordingly declared elected:

President: R. D. CARMICHAEL, University of Illinois.

Vice-Presidents: A. B. CHACE, Brown University; L. P. EISENHART, Princeton University.

Additional members of the Board of Trustees: C. F. GUMMER, Queen's University; DUNHAM JACKSON, University of Minnesota; E. H. MOORE, University of Chicago; CLARA E. SMITH, Wellesley College.

The Secretary-Treasurer made his financial report for the year, giving an account of all the business transacted for the Association, up to December 5th, 1922. This report has been audited by a committee consisting of Professors A. D. PITCHER, C. E. WILDER, and H. E. SLAUGHT. The financial report is printed in full below:

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 5, 1922.

RECEIPTS.

Balance Dec. 6, 1921		$ 4,013.98	Sale copies of MONTHLY	53.00	
1920 indiv. dues	$ 15.00		Sale of old typewriter	15.00	
1920 instit. dues	5.00		Advertising	341.50	
1921 indiv. dues	234.89		Refund from Editor-in-Chief's office	25.00	
1921 instit. dues	38.25		Sale of Register	245.50	
1921 subscriptions	10.50		Interest State Svgs. Bk.	100.12	
1922 indiv. dues	4,822.45		Interest Peoples Bk.	90.39	
1922 instit. dues	586.00		Interest Liberty Bds.	34.38	
1922 subscriptions	734.41				
Contribution to 1922 expenses	413.00		Total 1922 receipts		$ 7,966.39
Initiation fees	202.00				

EXPENDITURES.

Publisher's bills (Oct.–Dec. '21, Feb., Apr., May '22)		$3,169.87	Part expense Register	72.50	
Manager's office		26.53	Printing	272.48	
Editor-in-Chief's office		956.66	Toronto meeting	63.75	
Secretary-Treasurer's office:			Rochester meeting	52.78	
Postage	$205.19		Paid copies of MONTHLY	24.70	
Bond	5.00		Refund on subscriptions	23.05	
Safety deposit (2 yrs.)	8.00		Library expense	54.90	
Office supplies	28.60		Paid to sections from initiation fee	38.91	
Typewriter	87.64				
Express, tel., etc.	41.23				$1,333.98
Clerical work	355.25				

Total assets to the end of 1922 business	$11,980.37
Total expenditures	5,587.04
Balance to the end of 1922 business	$ 6,393.33
Received on 1923 and later business	406.40
Book balance Dec. 5, 1922	$ 6,799.73

Annals subvention	100.00
Total expenditures	$5,587.04
Cash on hand	34.13
Checking account	372.12
State Savgs. Co. account	2,796.06
Peoples Bkg. Co. account	2,597.42
Liberty Bond	500.00
U. S. Treasury Note	500.00
Bank balance Dec. 5, 1922	$6,799.73

As a separate fund, there was received on the Carus Fund $1,200.00, which, with the interest accruing at 4 per cent., compounded quarterly, amounts at present to $1,236.36

When the accounts were closed on December 5, 1922, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1922 the following items:

Bills Receivable.

1922 indiv. dues unpaid	$ 90.00
1922 instit. dues unpaid	42.00
Advertising	230.00
Interest Liberty Bond	10.00
Interest U. S. Treasury Note	11.00
	$383.00

Bills Payable.

(Either paid or estimated.)

Publisher's bills (7 issues @ $500.00)	$3,500.00
Register	590.00
Manager's office	30.00
Editor-in-Chief's office	150.00
Other editor's postage	20.00
Committee on Membership	55.00
Secretary-treasurer's office	185.00
Annals subvention for 1922	200.00
Init. fees due to sections	100.00
Printing annual ballots, programs, etc.	150.00
	$4,980.00

If to the balance on 1922 business shown in this report, $6,393.33, there be added the amount of bills receivable, $383.00, and there be subtracted the estimated amount of bills payable, $4,980.00, there results an estimated final balance on 1922 business of approximately $1,800.00. The corresponding estimated final balance one year ago on 1921 business was $1,530.00. That a surplus rather than deficit will evidently result from the year's business is due to the generous gift of $400.00 made by Chancellor Chace of Brown University, a contribution much appreciated by the Board of Trustees and by all the members of the Association. Your officers are continually endeavoring to keep the expenses of the Association as low as possible consistent with effective work, and it is their hope that the members will support the Association by continuing their own membership and by urging others to become members.

ANNOUNCEMENT FOR THE ASSOCIATION.

The summer meeting of the Association will be held by invitation at Vassar College on Wednesday afternoon and Thursday, September 5 and 6, 1923, in conjunction with the summer meeting of the American Mathematical Society.

The following persons and institutions, on applications duly certified, have been elected to membership in the Association since the annual meeting:

To individual membership.

J. W. ALEXANDER, Ph.D. (Princeton). Asst. prof., Princeton Univ., Princeton, N. J.
Sr. M. ANGELICA, Ph.D. (Fordham). Head of dept. of math., St. Joseph Coll., Brooklyn, N. Y.
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NINTH REGULAR MEETING OF THE KANSAS SECTION.

The ninth regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, January 20, 1923, in connection with a meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, the first of which was a joint session with the Kansas Association. Professor W. H. Garrett presided at the first session and Professor A. E. White at the second.

The attendance was sixty-two, including the following twenty-three members of the Association:

Florence Black, W. H. Garrett, W. A. Harshbarger, T. B. Henry, Emma Hyde, S. Lefschetz, C. F. Lewis, T. Lindquist, O. B. Loewen, Anna Marm, U. G. Mitchell, Thirza Mossman, P. Pretz (institutional representative), B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, H. G. Titt, Eula Weeks, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor LINDQUIST; Vice-Chairman, Professor TITT; Secretary-Treasurer, Professor MITCHELL.

The following papers were presented:

(1) "The development of the junior high school movement in Kansas and its effect on the efficiency of mathematics instruction in the seventh, eighth and ninth grades" by Professor T. LINDQUIST.

(2) "The National Committee's report on the reorganization of secondary mathematics" by Dr. EULA A. WEEKS.

(3) "Some peculiar limiting functions and their graphs" by Professor G. W. SMITH.

(4) "The teaching of unified mathematics" by Professor P. PRETZ.

(5) "The area of a cone having an elliptical base" by Miss THIRZA MOSSMAN.

(6) "A new method of determining sufficient conditions for real roots of equations" by Miss WEALTHY BABCOCK, (by invitation).

(7) "A map of sinh z" by Professor T. B. HENRY.

At the joint session there was a general discussion of the topic "Should the State Board of Education recognize the existence of the junior high school system in the state?" and it was voted to appoint a committee of three to work for the standardization of junior high school mathematics in the state.

After the presentation of each paper there was a general discussion. Abstracts of the papers are given below, the numbers corresponding to numbers in the list of titles:

1. Professor Lindquist brought out the following facts: (*a*) the first junior high schools in Kansas were organized in 1911; (*b*) of 61 cities with population above 2000 which replied to a questionnaire in 1921, 22 reported using the 6–3–3 plan, 21 the 6–2–4 plan and 18 no junior high school organization; (*c*) schools using the 6–3–3 plan reported 9200 junior high school pupils and those using the 6–2–4 reported 5000; (*d*) this indicates that 64 per cent. of the junior high school pupils in Kansas are in schools which have a three-year junior high school course.

Since a number of cities known to have junior high schools failed to answer the questionnaire, 20,000 is a conservative estimate of the number of pupils attending junior high schools in Kansas in 1921; (*e*) nearly all of the teachers in the junior high schools of the state have had at least two years of college training and about one half hold bachelor's degrees.

2. Dr. Weeks gave a general résumé of the work of the National Committee and emphasized especially the need for reading, discussion and criticism by teachers of the work of the Committee, if its reports are to be most valuable.

3. Professor Smith showed that expressions can be set up, the limit of which represents a certain function $f(x)$ for $0 \leqq x < a$ and another function $g(x)$ for $x > a$. By properly introducing $\sqrt{x^2 - b^2}$ and $\sqrt{c^2 - x^2}$ he set up expressions which have real values only in the interval $b \leqq x \leqq c$. Many examples were given and for some of these several of the approximation curves were plotted.

4. Professor Pretz stated that the teaching of unified mathematics is an issue in the field of education that must be met and judged on its merits by the teachers of mathematics themselves. On the basis of his own experience and the experience and opinions of others he believed that unified courses can be taught successfully in both secondary schools and colleges. He advocated the teaching of unified mathematics, chiefly because he believed the unified scheme to be more broadly scientific than other plans. The proper preparation of teachers of unified courses was emphasized.

5. The lateral area of a cone whose base is an ellipse can be expressed as an elliptic integral. If the vertex of the cone is over the principal axis of the ellipse, the area is computable. This area is in general represented by an elliptic integral, but in the special cases of the right circular cone, the cone of zero altitude, and the cone whose vertex lies on a certain hyperbola in a plane vertical to the base, the area integral is non-elliptic.

6. Miss Babcock outlined a method of determining conditions to be placed upon the coefficients of an equation such that the roots of the equation would be real. The general symmetric determinant of order equal to the degree of the equation was expressed in terms of its coaxial minors of the first and second orders, and these minors then expressed in terms of the coefficients of the equation, so that, as a result, the equation was expressed in the form of a symmetric determinant. Conditions to be placed upon the coefficients of the equation, sufficient to make the roots real, may be determined from this symmetrical determinant form of expression. These conditions were given for equations of the third and fourth degrees.

7. Professor Henry presented a comprehensive discussion of the conformal representation of hyperbolic functions of a complex variable, with graphic representations of the results obtained.

U. G. Mitchell, *Secretary-Treasurer.*

HISTORICAL-MATHEMATICAL PARIS.

By DAVID EUGENE SMITH, Columbia University.

I. Ile de la Cité and the Voltaire-Châtelet Paris.

The World War has naturally turned the steps of many of our advanced students to the paths their intellectual ancestors trod soon after the American and French revolutions, namely, to Paris. There will still be large numbers who go to Germany and England, and many who go to Italy, but for some years to come it is probable that Paris will attract American students more than it ever has in the past and more than any other single city of Europe. For these students, and for the more casual visitor of mathematical tastes as well, this article has been prepared in the hope that new interest may be added to their sojourn in what is, all things considered, the most attractive city of the world. Having spent much time there during repeated visits spread over a period of more than forty years, I have naturally come to know a considerable number of the places of mathematical interest, and my collection of autograph letters of those who have made the science what it has become in the last three or four centuries has supplied considerable information as to where the various writers and their correspondents lived and labored and died. I have also been aided by such works as those of the learned M. Cain (for example, his *Promenades dans Paris*) and by the more detailed but less well-written work of the Marquis de Rochegude (*Promenades dans toutes les Rues de Paris*), but I have naturally selected only a few of the many spots of historic interest that anyone could readily find if he should attempt such a pretentious piece of work as a book upon the subject. In many cases the houses mentioned are still standing, as, for example, two on the Rue de Bac and one on the Rue de Lille, but in any event the mere location has enough interest to make some reference to it worth while.

Beginning with the most ancient part of the city,—the Lutetia of Cæsar's time, now a part of the Ile de la Cité, upon which Notre-Dame stands,—we may turn to that little gem of Gothic architecture, the Sainte-Chapelle in the ancient Palais de Justice. It was constructed in 1245–1248 by Saint Louis as a fitting receptacle for the Crown of Thorns and a portion of the True Cross, and it seems to have had as one of its canons Rollandus, whose general treatise on mathematics, written *c.* 1425, has come down to us in manuscript copy.[1]

On the north side of the Palais de Justice, and entered from the Quai de l'Horloge, is the Conciergerie, the most ancient prison of the city. It was here that Jean Silvain Bailly[2] was confined before his execution in the Reign of Terror. While he was still in favor of the revolutionists he was mayor of Paris, although carrying on his studies in mathematical astronomy, and lived in the Hôtel de la Mairie which stood where is now No. 14 of the Rue des Capucines, a street which runs from the Rue de la Paix to the Boulevard des Capucines. He also lived for a time at No. 21 (old numbering) of the Rue de Chaillot, once the principal street

[1] This copy is now in the library of Mr. George A. Plimpton, New York City.

[2] Born in Paris, Sept. 15, 1736; guillotined Nov. 12, 1793.

of the village of Chaillot, the Colloelum of the 11th century, but now a residential section southeast of the Arc de Triomphe de l'Étoile, so called to distinguish it from the Arc de Triomphe du Carrousel. It was on July 17, 1791, that he and Lafayette directed the charge on a mob which had demanded the surrender of the king. After his execution (1793) in the Place de la Concorde[1] his body was buried in the ancient cemetery (1659–1865) of the Madeleine, now (since 1865) the Square Louis-XVI, on the Rue Pasquier, a little south of the Gare Saint-Lazare.

If we rank Voltaire[2] in our guild because of his work on the philosophy of Newton,[3] we shall naturally find many spots in Paris connected with his name, and portraits and statues in great number and often of much excellence. The present Rue Molière, running from the Avenue de l'Opéra to the Rue Richelieu, for example, was once the Rue Traversière, and at the old number 25 was a house which was rented to the Marquise du Châtelet,[4] and there Voltaire lived for some time, setting up a little theatre for his plays. Around the corner, at No. 8 of the Rue de Richelieu, the street on which the Bibliothèque Nationale fronts, was the café of Charlotte Bourette, who was known as the Muse Limonadière, and whom Voltaire esteemed for her wit. Farther up the Rue de Richelieu, at No. 102, stood a house which Voltaire owned and in which his niece, Mme. Denis, lived after the death of the Marquise du Châtelet. Next door, at No. 100, stood the house of Voltaire's friend, Mme. de St. Julien, whom he often visited. Voltaire also lived (1732 and 1733) at what is now No. 20, Rue de Valois, in the same vicinity, east of the Palais Royal. Not far from here, at No. 161, Rue Saint-Honoré, is the Café de la Régence, which I well recall as still prominent in the artistic life of Paris when I was a boy. Its predecessor stood a little to the east, at the Place du Palais Royal, and was frequented by Voltaire as well as by Benjamin Franklin, Diderot, Napoleon, and other makers of history. Over on the Ile Saint-Louis, at No. 2, on the Rue Saint-Louis-en-l'Ile, is the hôtel (mansion) of Nicolas Lambert de Thorigny, sometime president of the Cour des Comptes, built in 1680. The Marquise du Châtelet lived there for a time, and Voltaire was, as usual, a guest of the house. His sister, Mme. Mignot, mother of Mme. Denis (to whom Voltaire was greatly attached), lived at No. 133, Rue Saint-Antoine, a continuation of the Rue de Rivoli and leading into the Place de la Bastille. Although the Bastille has long since ceased to exist,[5] when the wanderer stands upon its ancient site he may reflect that Voltaire was twice imprisoned there,[6] for his rash utterances on the rights of man. Voltaire was baptized (1694) in the church of Saint-André-des-Arcs, which was built in 1210.

[1] The guillotine was at the entrance to the Champs Elysées.

[2] François Marie Arouet, who took the name of Voltaire (anagram on Arouet le jeune = Arovet l. i.); born at Paris, November 21, 1694; died at Paris, May 30, 1778.

[3] *Élémens de la philosophie de Neuton*, Amsterdam, 1738. Compare this MONTHLY, *1921*, 303–305.

[4] Gabrielle Émilie Le Tonnelier de Breteuil, born at Paris, December 17, 1706; died at Commercy, September 10, 1749. She spelled her married name in the full form,—Chastelet.

[5] The upper part of the Pont de la Concorde was built from the stones of the Bastille.

[6] In 1717 and 1726, more than sixty years before its destruction.

It stood on the present Place Saint-André-des-Arts,[1] near the Point Saint-Michel, and was demolished about 1800. In 1793 it became the Temple de la Révolution. Voltaire once worked as a clerk in the office of Maître Alain, No. 1, Rue des Grands-Degrés, so called from the steps leading down to the quai, and he became a mason in the lodge of the Neuf Soeurs which stood at No. 80 of the Rue Bonaparte; but the atmosphere of the Quartier Latin was perhaps not so well suited to his maturer years, although he lived for a time in Rue Mazarine and in 1743 was living at No. 23, Rue Fontaine Molière. He died in the house of the Marquise de Villette, at No. 27, Quai Voltaire, as an inscription states. The present name of the quai, formerly the Quai des Théatins, was given in memory of this event, as was that of the Rue Voltaire which branches off at No. 211. His final resting-place is appropriately in the Panthéon, the Valhalla of France.

As to busts, bas-reliefs, and statues of Voltaire, Paris has been over-generous. Houdon's bust in the Comédie Française is the best known, but the statue by Caillé (1885) on the Quai Malaquais is also familiar to every visitor to the bookstalls on the Rive Gauche.

As to the Marquise du Châtelet[2] and her family, Tonnelier de Breteuil, there are various interesting spots connected with each. The family owned a hôtel at No. 14, Rue Portefoin, a little to the southeast of the Conservatoire des Arts et Métiers. They also owned (1760) a place at No. 56, Rue des Francs-Bourgeois, near the Palais des Archives Nationales, and somewhat earlier (1728) one at No. 4, Place des Vosges, on the same street. In 1752 the marquise was living at No. 18 of the same Place.

II. The Quartier Latin.

Returning in our wanderings to the Quartier Latin, and to names more mathematical, at No. 1, Rue de la Sorbonne, Hermite[3] died in 1901, and on the walls of the Église de la Sorbonne is his *médaillon*. At No. 2 of the Rue Rollin, which opens on the west side of Rue Monge, Pascal[4] died, as an inscription states, at the house of his sister, Marguerite Périer, who afterward wrote his biography. Descartes lived at No. 14 of the same street. A little to the south, parallel to and east of Rue Monge, is the Rue de la Clef where, at No. 38, Monge[5] lived for a time. He also lived in Rue de Dragon, a little to the west of Saint-Germain-des-Prés and to the south of Boulevard Saint-Germain. One of the letters[6] written from Linz, when he was with Napoleon on the Austrian campaign, is addressed to Madame Monge at "Rue neuve Belle chasse No. 3," the Rue de Bellechasse (formerly Belle Chasse). The part between the Rue Saint-Domi-

[1] The change from *arcs* to *arts* is relatively recent.

[2] Compare this Monthly, *1921*, 368–369.

[3] Charles Hermite (1822–1901), who proved the transcendence of *e*.

[4] Blaise Pascal, born at Clermont-Ferrand in 1623; died at Paris in 1662.

[5] Gaspard Monge, born at Beaune in 1746; died at Paris in 1818. He is known chiefly for his work in descriptive geometry.

[6] Whenever such letters are mentioned it is to be understood that they are at present in t author's collection.

nique and the Rue de Grenelle was formerly called Rue Neuve Belle Chasse. It seems, from the old maps, that this house stood at the corner of the present Rue Saint-Dominique. Monge also lived for a time at Nos. 7–9, Rue Bonaparte, as stated later.

To the west of Rue Monge, and nearly parallel with it, is the Rue d'Ulm which runs from the Panthéon south to Rue Claude Bernard. At No. 43, Rue d'Ulm, is the École Normale Supérieure, founded in 1795 and occupying the present building since 1847,—an institution with which have been connected many mathematicians of prominence, Jules Tannery being one of the last of those who have now passed away. At the next corner to the north and east of this school the Rue Lhomond continues the line of the Rue des Fosses Saint-Jacques. It was formerly called the Rue des Postes, and a letter was written from this street by Pierre Bouguer [1] on October 31, 1750. The first street to the north is now known as the Rue de l'Estrapade, known a century ago as the Rue de la Vieille-Estrapade, so called from the *estrapado* or *strappado* punishment there inflicted upon soldiers in early times. At No. 11, in a mansion still standing, there lodged Georges Marie Raymond,[2] professor of mathematics in Geneva, whose contributions to algebra appeared frequently in Gergonne's *Annales*.

At No. 9 of the Quai Malaquais, which extends westward from the Palais de l'Institut, Legendre[3] lived for four years (1809–1813). The building stands next to the École des Beaux Arts, on what was formerly part of the grounds of the ancient Abbey of Saint-Germain-des-Prés. Before this he lived at No. 12, Rue Condé, a street running north from the Palais du Luxembourg, one of his letters having been written from there in 1804.

Near the Pont Saint-Michel, and west of the Boulevard, is the Rue Saint-André-des-Arts, dating from 1179,—formerly Saint-André-des-Arcs. At No. 52 of this street, a fine old mansion at the corner of Rue des Grands Augustins, Joseph-Louis-François Bertrand [4] was born in 1822. The next parallel street to the north is Rue Christine where, in a dignified old mansion at No. 2, Laplace [5] lived in 1802. It leads to the east into the narrow Rue de Savoie, with numerous old houses, where, at No. 13 as an inscription states, Sophie Germain [6] died on June 27, 1831.

The Rue Mazarine runs southward from the Institut, and leads into the Rue de l'Ancienne Comédie where, at the oldest café in Paris, the Procope (No. 13), founded in 1689, d'Alembert, Voltaire, and many others among the intellectuals gathered. The café still stands, shorn of its ancient prestige, as is also the case with the old Comédie Française across the street, at No. 14.

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[1] Born in 1698; died at Paris in 1758. He was engaged with Condamine and others on the figure of the earth. He contributed to the geometry of curvilinear figures.

[2] Born in 1769; died in 1839. Delambre wrote a letter to him at this address in January, 1822.

[3] Adrien-Marie Legendre, born at Toulouse in 1752; died at Paris in 1833.

[4] He was professor of mathematical physics in the Collège de France and died in 1900.

[5] Pierre Simon Laplace, born in 1749; died in 1827. Known primarily for his work in celestial mechanics.

[6] She was born in 1776. She is known for her work on the theory of elastic surfaces.

Puissant [1] lived in the Rue Mazarine in 1826, as is shown by a letter written by Bouvard [2] to him in 1826, and Gaston Darboux[3] lived at No. 3 of the same street,—his official residence as secrétaire perpétuel de l'Académie des Sciences. Where the Rue Mazarine becomes the Rue de l'Ancienne Comédie the Rue de Buci runs to the west, and at No. 19 Mme. Denis, niece of Voltaire, was joint proprietor of a residence. In the Rue de Seine, nearly parallel to the Rue Mazarine and the best place in Paris for the collector of early mathematical portraits, Legendre lived at one time and the widow of the unfortunate Condorcet [4] died in 1822. She had lived before that (1812) in the Rue de Penthière, north of the Champs Elysées. At No. 4 of the southern part of Rue Mazarine, there called the Rue de Tournon, J. L. F. Bertrand died in 1900, and at No. 12, in an elaborate old mansion near the entrance to the Palais du Luxembourg, Cauchy [5] and Leverrier [6] both lived.

Not far to the east of the Rue de Tournon is the Place de l'Odéon where the Café Voltaire was much frequented by Voltaire and his friends. From here to the Boulevard Saint-Michel there runs the Rue Racine in which, at No. 30 as an inscription states, there died Auguste Comte (1798–1857), the founder of positivism, a writer on the philosophy of mathematics, and the editor of some of Pascal's works.

In Rue Bonaparte, the next street to the west of the Rue de Seine, Monge [7] lived in the hôtel of the Marquis de Persen (Nos. 7–9),—a building now given over to commercial uses. To the west of this street, and extending to the south from the church of Saint Sulpice, at No. 15 of the narrow Rue Servandoni with its ancient structures, the widow of François Vernet, the sculptor,[8] concealed Condorcet while he was writing his *Esquisse du progrès de l'esprit humain* (1793), the last work undertaken by him before he poisoned himself to escape the guillotine. The old house still stands, bearing a proud inscription of the incident, but looking dilapidated enough to conceal with perfect safety any unfortunate seeker after oblivion. Just before this, Condorcet lived a short distance away, at No. 71, Rue de Lille,[9]—from 1640 to 1792 known as the Rue de Bourbon, and again by the same name from 1814 to 1830. Twenty years before this time Monge wrote a letter to Condorcet, addressing it to him "Chez M^{r} le M^{is} d'Essé, Rue de Bourbon, St. germain," but no number was given. It was then one of the most fashionable parts of Paris and, being near the Quai d'Orsay, is still a diplomatic center. The well-known Rue du Bac crosses this street, and at No. 26 Baron

[1] Louis Puissant, born in 1769; died at Paris in 1843. He wrote extensively on algebra, geometry, and geodesy.

[2] Alexis Bouvard (1767–1843) the astronomer.

[3] Born at Nîmes, 1842; died at Paris, February 23, 1917.

[4] Marie-Jean-Antoine-Nicolas Caritat, Marquis de Condorcet, born at Nemours in 1730; died near Paris in 1793. He poisoned himself to escape the guillotine.

[5] Augustin-Louis Cauchy, born at Metz in 1788; died at Paris in 1867.

[6] Urbain-Jean-Joseph Leverrier (Le Verrier, 1811–1877) lived there in 1853.

[7] See page 109.

[8] He died in 1784.

[9] The fine mansion may be seen by entering the court.

Charles Dupin[1] died in 1873. The house still stands, and at No. 108, as an inscription states, Laplace died in 1827. It also crosses, farther south, the Rue de Sèvres where, in a mansion formerly standing at No. 16, Madame de Récamier spent the last thirty years of her life and held a notable salon at which all the academicians were received. Among those who frequented her house were Arago and Ampère, and it is of some interest to know that Victor Hugo was baptized there. The building was the ancient Abbaye aux Bois and was demolished in 1908. A few minutes' walk to the southwest of this locality, and near the Gare Montparnasse, is the modern Rue Littré where, at No. 5, Émile Lemoine[2] (1840–1912) lived.

Few who visit this scholastic and artistic part of Paris fail to enter the Musée de Cluny, but probably not many of these recall the fact that the old palace which houses it has a mathematical as well as an archeological interest. Lalande (1732–1807),[3] however, lived there for a time, and Charles Messier (1730–1817), the astronomer, followed him and died there in 1817, the tower serving at that time as the Marine Observatory. It was not until 1833 that M. Du Sommerard (1779–1842) installed his great collection there, the government acquiring it ten years later.

The Panthéon, too, is not usually connected in thought with mathematics, for the inscription, "Aux grands hommes, la Patrie reconnaissante," is not ordinarily associated in the public mind with the mathematicians of the country. Nevertheless there is an indirect interest in recalling that it was here, in February 1851, that Foucault made his experiments relating to the pendulum and the rotation of the earth. He is not buried here, however, but in the cemetery of Montmartre.

Coming out from the Panthéon we face the Rue Soufflot. At No. 14 there stood, from 1217 to 1790, the convent of the Dominicans, or Jacobins, where Albertus Magnus[4] taught in the Middle Ages. Around to the right, as one leaves the Panthéon, stands the Église Saint-Étienne-du-Mont, dating from the 16th and 17th centuries, and it is here that Pascal was buried, as is recorded by an inscription on a pillar back of the chancel.[5] Descartes[6] was also buried here, but his remains were later (1819) transferred to Saint-Germain-des-Prés. In the Bibliothèque Saint-Geneviève nearby is the oldest known French algorism, a parchment manuscript of *c.* 1275, but of course the great collection of mathematical manuscripts is in the Bibliothèque Nationale.

The Sorbonne lies a little to the northeast of the library of Saint-Geneviève,—not the ancient building which I remember as a boy, and where generations of

[1] Born in 1784; died in 1873. Prominent because of his works on mechanics and differential geometry. See this MONTHLY, *1921*, 121.

[2] One of his letters is dated there October 13, 1894.

[3] See this MONTHLY, *1921*, 207.

[4] Count of Bollstädt and bishop of Regensburg. His works include a certain amount of astronomy and some mention of Pythagorean arithmetic. Saint Thomas Aquinas also taught there.

[5] See this MONTHLY, *1921*, 64.

[6] Born at La Haye, March 31, 1596; died at Stockholm, February 11, 1650.

mathematicians had taught since the Middle Ages, but the new and imposing edifice which has gradually replaced its ancestor during the last forty years. Across the Rue Saint-Jacques, to the east, is the Collège de France, where many of the greatest of the French mathematicians have taught. The present building was completed about 150 years ago. The square in front was formerly called the Place Cambray, and a letter of Lalande was written there on October 14, 1800, the year after the death of Montucla whose history of mathematics he edited in its second edition, the letter having probably been written at the collège.

Three letters of Jean-Nicolas-Pierre Hachette[1] were written from "Rue d'Enfer St Michel n° 31." The Boulevard Saint-Michel was opened during the Second Empire, and its present name dates only from 1867. It was formed by straightening and enlarging the ancient Rue de la Harpe, Rue d'Enfer, and other minor streets. The Rue d'Enfer was the portion lying south of the present corner of the Rue Soufflot. In late Roman times it was the Via Infera (voie inférieure, the lower route), a name corrupted as early as the 13th century to Rue d'Enfer (hell's street). The next street to the south of Rue Soufflot is Rue Royer-Collard, known before 1846 as Rue Saint-Dominique-d'Enfer. At No. 15 is found the Impasse Royer-Collard (opened in 1590), formerly called the Impasse Saint-Dominique. It was here that Hachette probably lived, for another of the letters bears the address "Impasse st. dominique d'enfer." Poisson[2] dates a letter from No. 20 of the same street, where he seems to have been living in 1814.

When Rue Soufflot was completed, in the middle of the 19th century, Rue Hiacinte (Hyacinthe) was closed. It ran from the eastern entrance of the Luxembourg gardens to the present Rue Saint-Jacques, and Hachette also lived here,—at No. 20 in 1814 and at No. 8 in 1830, as is shown by two other letters of his.

Down by the river the Palais de l'Institut, built in 1663, is of course closely connected with the history of mathematics, especially in the class of the Académie devoted to this subject and physics. D'Alembert,[3] for example, was the secrétaire perpétuel during the latter part of his life.

(To be concluded in the next issue.)

CYCLIC OPERATIONS ON DETERMINANTS.

By A. L. CANDY, University of Nebraska.

"A determinant is not altered in value by adding to all the elements of any column (or row) the same multiples of the corresponding elements of any number of other columns (or rows)."

This is a well-known theorem that is stated in substantially the same form in

[1] Jean-Nicolas-Hachette (1769–1834), well known for his works on algebra and geometry.

[2] Siméon-Denis, Baron Poisson (1781–1840). He wrote on probability, equations, and the calculus.

[3] Jean-Baptiste-le-Rond d'Alembert, born at Paris in 1717; died at Paris in 1783.

all textbooks on determinants. Furthermore, the determinant is not altered in value if more than one column (or row) be operated upon in this manner.

As far as I have examined the textbooks no author has given an explicit statement of this extension, although all make use of it in their illustrative examples. But what is of more importance, I believe no one has hitherto pointed out any exception to this general extension of the above theorem. There is, however, a very important exception,[1] some of the results of which I shall try to show. This exception may be stated as follows:

A cyclic operation upon the rows (or columns) of a determinant will change the value of the determinant, if the cycle is complete.

Let Δ_s represent the determinant formed by adding the rows of Δ cyclically s in a set. When necessary to distinguish the order n of Δ we may write ${}_n\Delta$ and ${}_n\Delta_s$.

For example, let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then by definition

$$\Delta_2 = \begin{vmatrix} a_1 + a_2, & b_1 + b_2, & c_1 + c_2 \\ a_2 + a_3, & b_2 + b_3, & c_2 + c_3 \\ a_3 + a_1, & b_3 + b_1, & c_3 + c_1 \end{vmatrix}.$$

Taking the sum of all the rows for a new first row, and taking out the factor 2, gives a determinant which can be easily reduced to Δ itself. Therefore, ${}_3\Delta_2 = 2 \cdot {}_3\Delta$. When $n = 4$ we have the determinant ${}_4\Delta_2$ equal to zero, since the sum of its first and third rows is the same as the sum of its second and fourth rows. In like manner it can be shown that for any value of n, ${}_n\Delta_2 = 0$ or $2{}_n\Delta$ according as n is even or odd.

A determinant of the form

$$\begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ k_4 & k_1 & k_2 & k_3 \\ k_3 & k_4 & k_1 & k_2 \\ k_2 & k_3 & k_4 & k_1 \end{vmatrix}$$

is called a *circulant*. The elements of the first row may have any values whatsoever, and a particular circulant will be completely known when the elements of the first row are given.

Let C_s represent a circulant in which the first s elements of the first row are 1, and the others all 0; when it is necessary to distinguish the order n of the circulant we shall write ${}_nC_s$.

THEOREM I. *For all values of n and s, $n > s$, ${}_n\Delta_s = {}_nC_s \cdot {}_n\Delta$.*

This appears at once if we form the product by taking ${}_nC_s$ by rows, and ${}_n\Delta$ by columns.

For example, when $n = 3$, we have

$${}_3C_2 \cdot {}_3\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = {}_3\Delta_2.$$

[1] This "exception" covers an error that is frequently made by students.

Hence, in order to determine the effect of adding the rows of ${}_n\Delta$ cyclically s in a set, it will be sufficient to evaluate ${}_nC_s$. For this purpose we shall prove the next three theorems.

THEOREM II. *If n and s have a common factor,* ${}_nC_s = 0$.

Let $n = km$ and $s = kr$, where m and r are prime to each other. In any column (as well as in any row) the elements can be grouped into m sets, k in a set. Of these there are r consecutive sets in which the elements are all 1's, and $(m - r)$ consecutive sets in which they are all zeros. Now any series of m rows, whose numbers are $i, i + k, i + 2k, \cdots, i + (m - 1)k$, where $i \ngtr k$, includes in every column one element, and only one, from each of these sets, that is, r 1's and $m - r$ zeros. Hence, if we replace any row of such a series by their sum, we shall get a new row in which each element is r. Since in this way we can make k rows alike, the determinant must be zero.

THEOREM III. $\dfrac{{}_nC_{n-s}}{n-s} = \dfrac{{}_nC_s}{s}$.

In ${}_nC_s$ take the sum of all the rows for a new first row, and take out the factor s. Then subtract the first row from each of the others, and change the signs in all these $n - 1$ rows. Then permute the rows cyclically until the last s rows become the first s rows. These operations will change the sign of the determinant $n - 1 + s(n - 1) = (n - 1)(s + 1)$ times. We now have

$$ {}_nC_s = (-1)^{(n-1)(s+1)} \cdot s\Delta_1, $$

where Δ_1 is precisely the same determinant as that obtained from ${}_nC_{n-s}$ by adding to the row whose number is $s + 1$ all the other rows, and taking out the factor $n - s$. Therefore we can write

$$ \frac{{}_nC_{n-s}}{n-s} = (-1)^{(n-1)(s+1)} \cdot \frac{{}_nC_s}{s}. $$

Now the exponent $(n - 1)(s + 1)$ is an even number except when n and s are both even. But, by Theorem II, when n and s are both even the circulants are both zero. Hence we can drop the sign factor and write the formula as in the statement of the theorem.

THEOREM IV. *When* $s < \frac{1}{2}n$, ${}_nC_s = {}_{n-s}C_s$.

If we add to the last $s - 1$ rows, respectively, the rows obtained by subtracting the first row from the next $s - 1$ rows, we shall have a determinant in which all the elements below the principal diagonal in the first s columns are zeros. Hence the first s columns and the first s rows may be dropped, and the circulant ${}_nC_s$ is thus reduced to the circulant ${}_{n-s}C_s$.

By repeating this process, if necessary, we can get

$$ {}_nC_s = {}_{n-s}C_s = {}_{n-2s}C_s = \cdots = {}_{s+r}C_s, $$

where r is the remainder when n is divided by s.

THEOREM V.[1] *When n and s are prime to each other*, ${}_nC_s = s$.

By Theorems IV and III we can reduce the subscripts n and s alternately until we have 1 for s, and a circulant equal to 1. When by Theorem III we reduce the second subscript, say from s to s', we introduce the factor s/s'. The next factor so introduced has s' in the numerator, and so each numerator cancels the preceding denominator until we come to the last factor, which has 1 for denominator. Thus we have finally only the factor s, and ${}_nC_s = s$.

THEOREM VI. ${}_n\Delta_s = 0$ *or* $s \cdot {}_n\Delta$, *according as n and s have or do not have a common factor.*

This follows at once from Theorems I, II, and V.

COROLLARY. *If m rows of a determinant* Δ *be added cyclically, s in a set, the determinant thus obtained will be equal to* 0, *or* $s\Delta$, *according as m and s have or do not have a common factor.*

Other interesting results can be found in a similar manner by taking other circulants.

The formula $\Delta_s = s\Delta$ furnishes a simple solution for the following geometrical exercises:

Let K represent the area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and K' the area of the triangle whose vertices are the middle points of its sides.

If we take the determinant

$$ {}_3\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} $$

which is equal to $2K$, ${}_3\Delta_2$ with its elements all divided by 2 will be the determinant form of $2K'$. Therefore, since ${}_3\Delta_2 = 2 \cdot {}_3\Delta$, we have $K = 4K'$.

Similarly, if V represents the volume of a tetrahedron expressed in terms of the coördinates of its vertices, and V' the volume of the tetrahedron whose vertices are the centroids of its faces, we have $V = 27V'$, by using the formula for ${}_4\Delta_3$. Furthermore, the fact that ${}_4\Delta_2 = 0$ shows that the middle points of four consecutive edges, say P_1P_2, P_2P_3, P_3P_4, P_4P_1, lie in a plane. But if we add the first three rows of ${}_4\Delta$ in sets of two and then divide these rows by 2, we get $V = 4V_1$ (Corollary under Theorem VI), where V_1 represents the volume of the tetrahedron formed by joining one vertex to the middle points of the edges of the opposite face.

The determinant ${}_n\Delta_2$ is one in which each element is a binomial. Hence it can be expressed as the sum of 2^n determinants in which each element is a monomial. That one in which all the columns are made up from the first terms of these binomials is equal to Δ, and that one in which all the columns are from second terms is equal to $\pm\,\Delta$, according as n is odd or even. But ${}_n\Delta_2 = 2\Delta$, or 0, according as n is odd or even. Therefore the sum of all the other determinants must vanish. We wish now to determine in what manner they vanish.

[1] Muir says (*Theorie of Determinants*, vol. 2, London, 1911, p. 403.) this theorem was proved by Catalan in 1846, but Muir himself does not give the proof.

To study this phase of the subject we will multiply Δ by the general circulant whose first row is $k_1\ k_2\ k_3\ \cdots\ k_n$, and in the product determinant we will write the terms in all the polynomial elements in the order of the subscripts of the k's. Then in the first row the subscripts of the letters which come from the respective columns of Δ will be in the same order, but the subscripts of these letters in the other rows will be those obtained from the first row by a cyclic permutation.

For example, when $n = 3$, we have

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ k_3 & k_1 & k_2 \\ k_2 & k_3 & k_1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} k_1a_1 + k_2a_2 + k_3a_3, & k_1b_1 + k_2b_2 + k_3b_3, & k_1c_1 + k_2c_2 + k_3c_3 \\ k_1a_2 + k_2a_3 + k_3a_1, & k_1b_2 + k_2b_3 + k_3b_1, & k_1c_2 + k_2c_3 + k_3c_1 \\ k_1a_3 + k_2a_1 + k_3a_2, & k_1b_3 + k_2b_1 + k_3b_2, & k_1c_3 + k_2c_1 + k_3c_2 \end{vmatrix}.$$

This product determinant can be expressed as the sum of n^n determinants with monomial elements. In all of these determinants the subscripts of the k's, and also those of the other letters in the first rows, form the n^n permutations that can be made with the first n consecutive integers, allowing all possible repetitions, and assuming that these other letters are always written in the same alphabetic order as in Δ. Since in each case all the elements in any one column are multiplied by the same k, these k's may be factored out leaving a determinant in which the columns are always in the order $a\ b\ c\ \cdots$, while the subscripts in any column are in direct cyclic order.[1]

Those determinants, the subscripts of whose first rows include all the permutations that can be made from a given combination of subscripts, are multiplied by a product of k's having the same combination of subscripts. Their sum may therefore be taken as the coefficient of this product of k's. To represent this sum let us write the subscripts of the first row of one of these determinants in a parenthesis, preceded by the number of permutations that can be made with the numbers within the parenthesis. The sum of the numbers within the parenthesis is the same as the sum of the numbers formed by multiplying together the subscript of each k and its exponent. This number I shall call the *weight* of the symbol, or of the determinants, and also the *weight* of the term in the k's. With this notation the above product determinant can be written

$$(111)k_1^3 + (222)k_2^3 + (333)k_3^3 + 3(112)k_1^2k_2 + 3(223)k_2^2k_3 + 3(331)k_3^2k_1$$
$$+ 3(122)k_1k_2^2 + 3(233)k_2k_3^2 + 3(311)k_3k_1^2 + 6(123)k_1k_2k_3.$$

If, now, in any one of these symbols we change n to 1, and add 1 to each of the other numbers, we get another symbol, equal to the first when n is odd, and differing only in sign when n is even. For by this operation we merely permute cyclically the rows in each of the determinants represented by the symbol, that is, we carry the top row to the bottom. By repeating this process we can always

[1] In the circulant, however, the subscripts in any column are in reverse cyclic order.

get n equal symbols if n is odd and prime. If n is composite there may be fewer than n symbols in the group. For example, there are only two symbols in the group 90(113355), 90(224466). If n is even the symbols in any group will be alternately positive and negative. Hence the expansion can be collected into groups, the number of terms in each group being n or some factor of n, and each group can be written as a polynomial in the k's alone, multiplied by the sum of determinants represented by one of these symbols, that is, by one of these symbols itself.

When the above expansion, for example, is thus written it becomes

$$(111)(k_1^3 + k_2^3 + k_3^3) + 3(112)(k_1^2k_2 + k_2^2k_3 + k_3^2k_1) + 3(122)(k_1k_2^2 + k_2k_3^2 + k_3k_1^2) + 6(123)k_1k_2k_3.$$

If, however, we expand the circulant itself, collect its terms into groups, and multiply each group by Δ, we get another form of expansion for identically the same product, since the k's are arbitrary. Hence the coefficients of the corresponding groups in these two expansions must be equal, and thus we get the value of any sum of determinants represented by one of our symbols. Such a sum must, therefore, be equal to a multiple of Δ, or to zero, according as the corresponding group of terms is present in, or absent from, the expansion of the circulant.

In the case of the above example, $n = 3$, this second method of expansion gives $(k_1^3 + k_2^3 + k_3^3)\Delta - 3k_1k_2k_3\Delta$. Therefore we have $(111) = \Delta$, [(111) is of course Δ as given.]

$$3(112) = 3(122) = 0, \qquad \text{and} \qquad 6(123) = -3\Delta.$$

THEOREM VII. *The sum of the determinants represented by one of these symbols is equal to zero when the weight is not a multiple of n.*

To prove this it will be sufficient to show that the weight of every term in the expansion of a circulant of order n is a multiple of n.[1]

If any minor, whose principal diagonal coincides with that of the circulant, be expanded in terms of the elements of its first column and their cofactors, it can readily be seen that the weights of its terms are either 1 or $n + 1$ greater than the weights of the terms in such a minor of the next lower order. Then, beginning with k_1 in the lower right-hand corner, and ending with the circulant itself, we find that the weight of every term is indeed a multiple of n.

COROLLARY. *When n is even the sum of the determinants denoted by $n!(123 \cdots n)$ is equal to zero.*

THEOREM VIII. *When the weight of one of these symbols is a multiple of n, the sum of the determinants it represents is of the form $\pm\lambda n\Delta$, where λ is some integer, including zero, when n is prime; when n is composite λ is either an integer, including zero, a fraction whose denominator is a divisor of n, or the sum of two or more such numbers. In the special cases*

$$(11\cdots1) = \pm(22\cdots2) = (33\cdots3) = \cdots = \pm(nn\cdots n) = \Delta, \lambda = 1/n.$$

[1] This was proved by R. Baltzer about 1870. (Muir, vol. 3, p. 376.)

To prove this it will be sufficient to show that the coefficients in the expansion of the circulant are numbers like λn as defined above.

The value of these coefficients is determined by the fact that the elements in the principal diagonal, and also in any diagonal parallel to it, are all alike. Hence if we start from each element in any term and take the next element in the same cyclic order in the same diagonal, we shall obtain the same term. If n is prime, we can repeat this operation n times before reaching the term from which we started. That is, there will be n terms in such a cycle. But if $n = pq$, say, such a cycle may consist of n terms, or of only p terms, or of only q terms. Moreover, all the terms in any cycle have the same sign, for at each step the column and row numbers of each element are both increased by 1, except that in passing from the last row to the first row, and from the last column to the first column, two of these numbers are changed from n to 1. The first of these operations makes no change in the number of inversions. The second makes an even number of such changes. Therefore the sign is not changed. The coefficients of the terms $k_1^n \cdots k_n^n$, given by the principal diagonal and those parallel to it, are evidently either 1, or -1.

The same terms in different cycles may not have the same sign, and so may wholly or in part cancel one another.

COROLLARY I. *When n is odd the sum of the determinants denoted by $n!(123\cdots n)$ is equal to $\lambda n\Delta$, where λ is an integer.* For when there is no letter repeated in a term of the circulant there are always n terms in each cycle.

When n is 3, 5, 7, 9, I have found these values to be, respectively,

$$(0 - 1)3\Delta = -3\Delta, \qquad (1 - 2)5\Delta = -5\Delta, \qquad (2 - 17)7\Delta = -105\Delta,$$
$$(113 - 98)9\Delta = 135\Delta,$$

where the first number in the parenthesis is the number of positive cycles, and the second is the number of negative cycles, in which the term $k_1k_2k_3 \cdots k_n$ occurs in the expansion of the circulant.

COROLLARY II. *If all possible cyclic permutations be made of the elements in each column of a given determinant, and the n^n determinants thus formed be collected into groups such that each determinant in the group shall have the same weight, and the same subscripts in the first row, these groups will be the same as those represented by the above symbols. Hence the sum of any group will be given by Theorem VIII, or Theorem VII, according as its weight is, or is not, divisible by n.*

Groups that can be obtained from each other by making the same cyclic permutations of whole rows in each determinant of the group are equal, or differ only in sign, according as n is odd or even.

THEOREM IX. *When n is odd the sum of the $(n - 1)!$ determinants of the sum denoted by $n!(123 \cdots n)$ which have one column in common is equal to $\lambda\Delta$, where λ is an integer.*

Take the $(n - 1)!$ determinants in which the first columns are the same as the first column of Δ. Permute cyclically the rows in each determinant of this group and we have a second group of determinants all having the same first column and

each equal to the corresponding determinant of the first group, n being odd. Repeating this operation we get n equal groups of this kind, all together comprising the $n!$ determinants of the sum $n!(123 \cdots n)$. As the entire sum is equal to $\lambda n\Delta$, the sum of the determinants in any one group is equal to $\lambda\Delta$.

We could start with any other column and proceed in the same way.

Thus when $n = 3$ the six determinants of $6(123)$ can be written

$$\begin{vmatrix} a_1 & b_2 & c_3 \\ a_2 & b_3 & c_1 \\ a_3 & b_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_2 & b_3 & c_1 \\ a_3 & b_1 & c_2 \\ a_1 & b_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_3 & b_1 & c_2 \\ a_1 & b_2 & c_3 \\ a_2 & b_3 & c_1 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_3 & c_2 \\ a_2 & b_1 & c_3 \\ a_3 & b_2 & c_1 \end{vmatrix} = \begin{vmatrix} a_2 & b_1 & c_3 \\ a_3 & b_2 & c_1 \\ a_1 & b_3 & c_2 \end{vmatrix} = \begin{vmatrix} a_3 & b_2 & c_1 \\ a_1 & b_3 & c_2 \\ a_2 & b_1 & c_3 \end{vmatrix}.$$

Since the total sum of the six determinants is -3Δ, the sum of the determinants in each column must be $-\Delta$.

When n is even we can form n groups in the same way and the sum of the determinants in one group will be numerically equal to the sum in any other group, but the groups in which this sum is positive will cancel the groups in which this sum is negative and the total sum will be zero. Therefore we cannot find the sum of those in any one group in this way.

When $n = 4$, for example, I have found the sum of the six determinants, in each of which the first column is the first column of Δ, to be

$$-2\begin{vmatrix} a_3 & b_1 & c_1 & d_1 \\ a_4 & b_2 & c_2 & d_2 \\ a_1 & b_3 & c_3 & d_3 \\ a_2 & b_4 & c_4 & d_4 \end{vmatrix},$$

which is not a multiple of Δ.

SOME PROPERTIES OF A SKEWSQUARE.[1]

By W. H. ECHOLS, University of Virginia.

A plane quadrilateral in which two diagonals are perpendicular and equal to each other has a number of rather interesting properties. Such a figure for convenience of reference in the present paper will be called a *skewsquare*, although I am given to understand it has at some time been called a pseudo-square.[2] The writer has seen no reference to it in print and it is possible that some or all of the properties enunciated below are not new. A number of the properties admit of easy demonstration by elementary geometry and also as easy exercises in interpreting complex number relations. Furthermore the figures involved in the geometrical constructions furnish excellent exercises in mechanical drawing inasmuch as the constructions check themselves as to accuracy in various ways.

[1] Presented at the Cambridge meeting of the Association, December, 1922.

[2] For example *Mathesis*, 1894, p. 268.—EDITORS.

The writer was led to the consideration of this type of quadrilateral in trying to place the complex number

$$w_{ij} \equiv \frac{1}{z_j - z_i} \int_{z_i}^{z_j} f(z) dz,$$

with reference to $f(z)$ in such a manner as to locate ζ that $f(\zeta) = w_{ij}$, wherein $f(z)$ is an analytic function. To any pair of points in the z-plane corresponds a point in the w-plane, the mean value of $f(z)$.

To the quadrilateral points z_1, z_2, z_3, z_4 correspond the quadrilateral points $w_{12}, w_{23}, w_{34}, w_{41}$, and in this figure it can be easily shown (by the properties of the integral) that the triangles $w_{12}w_{23}w_{31}$ and $w_{13}w_{34}w_{41}$ are, respectively, similar to $z_1z_2z_3$ and $z_3z_4z_1$. Also triangles $w_{24}w_{32}w_{34}$ and $w_{12}w_{24}w_{41}$ are, respectively, similar to $z_2z_3z_4$ and $z_1z_2z_4$. In particular, if the four z-points are the corners of a square, then the corresponding four w-points are the corners of a skewsquare. In like manner, with n z-points taken around a polygon can be constructed a skew-polygon of n w-points, and in addition the w-points corresponding to the diagonals of the z-polygon give a figure with more complicated but interesting relations.

The following are some properties of skewsquares:

(1) In a skewsquare the two squares constructed on two opposite sides as diagonals have a common vertex. The two points thus determined will be called the *foci* of the skewsquare.

(2) Conversely, if the squares constructed on two opposite sides of a quadrilateral as diagonals have a common vertex, then will the squares on the other two sides as diagonals also have a common vertex, and the figure is a skewsquare. Otherwise, if any two isosceles right triangles have the vertex common, their bases are opposite sides of a skewsquare.

(3) The foci and the mid-points of the diagonals of a skewsquare are the corners of a square (called the focal square). The center of this square will be called the *center* of the skewsquare. The mid-points of the diagonals of a skewsquare will be called the *conjugate foci* of the skewsquare.

(4) The foci lie on the bisectors of the angles between the diagonals of the skewsquare, respectively.

(5) The center of a skewsquare is the centroid of its corners, and also the centroid of the corners of the four squares having for diagonals the sides of the skewsquare.

(6) The segment joining a focus to the midpoint of a side of a skewsquare is perpendicular to and equal to half a side.

(7) The sum of the squares of the opposite sides of a skewsquare are equal.

(8) The four triangles (not right angled) whose bases are the sides of a skewsquare and whose vertices are the foci are equal in area.

(9) Segments joining any vertex of a skewsquare to the two foci make equal angles with the sides at that vertex.

(10) The foci of a skewsquare are the foci of a conic tangent to the sides of the

skewsquare. The points of contact are points in which the bisectors of the angles between the diagonals cut the sides. A point of contact divides the side in the ratio of the adjacent segments of the diagonals determined by their intersection.

(11) The midpoints of the sides of a skewsquare form a square (called the midsquare), its side is equal to half the diagonal of the skewsquare. The circle circumscribing the midsquare (called the midcircle) is the auxiliary circle on the transverse axis of the tangent conic. The square of the transverse diameter of the conic is equal to half the square of the diagonal of the skewsquare.

(12) The area of each of the four triangles in (8) is equal to the square of the semi-conjugate diameter of the tangent conic. The foci are both inside, on, or outside the boundary of the skewsquare, the tangent conic is respectively an ellipse, a line-segment, or a hyperbola. Also the product of two opposite sides of a skewsquare and the cosine of the angle between them is equal to the square of the conjugate diameter of the conic.

(13) The sides of all skewsquares, whose diagonals are perpendicular diameters of two fixed equal circles, are tangent to a fixed conic. The power of the center with respect to either circle is equal to the sum of the squares of the semiaxes of the conic.

Otherwise, the locus of the vertices of skewsquares circumscribing a given central conic is two equal circles whose centers are the other corners of a square on the segment joining the foci as diagonal, the radius of the circles is equal to the side of a square inscribed in the auxiliary circle of the conic.

(14) When one side of a skewsquare is fixed and the diagonals become infinite in given directions the conic becomes a parabola tangent to two right-angled sides at the ends of the latus rectum.

(15) The four corners of the four squares having as diagonals the sides of a skewsquare (and which are not foci) are the corners of a second skewsquare, called the conjugate skewsquare (the first being called the primitive skewsquare). The square of the diagonal of the conjugate is twice that of the primitive skewsquare. The vertices of the conjugate lie on the bisectors of the angles between the diagonals of the primitive, the foci and the conjugate foci of the primitive are respectively the conjugate foci and the foci of the conjugate skewsquare.

Otherwise, the construction of the primitive from the conjugate skewsquare follows. The vertices of four squares whose diagonals are the four segments joining each vertex of a skewsquare to the midpoint of the diagonal not containing that vertex determine only four points which are the corners of another skewsquare (called the primitive skewsquare). Its vertices lie on the bisectors of the angles between the diagonals of the conjugate, the foci and the conjugate foci of the conjugate skewsquare are respectively the conjugate foci and the foci of the primitive.

Thus, associated with any given skewsquare there is a conjugate and a primitive skewsquare which are confocal, their diagonals are collinear and their midpoints coincide, the diagonal of the first being twice that of the second.

Associated with any skewsquare there is a series of derived skewsquares, alternate members are confocal and their tangent conics are members of a confocal system.

(16) Segments joining the center to the vertices of a primitive skewsquare are perpendicular to and equal to half the corresponding sides of its conjugate skewsquare.

(17) The diagonals of a skewsquare are parallel and equal to the diagonals of the midsquare of the conjugate skewsquare.

(18) Lines through the midpoints of the sides of a skewsquare parallel to the transverse axis of the tangent conic pass through the corners of its primitive skewsquare. The segments between these midpoints and the corners of the primitive skewsquare are equal to each other and to half the distance between the foci of this conic.

(19) The center of a skewsquare is the radical center of the four circles whose centers are the corners of the skewsquare, the radius of each circle is equal to the segment joining its center to either of the two adjacent corners of the primitive skewsquare.

(20) If Z is any point in the plane of a skewsquare $ABCD$ and ZA, ZB, ZC, ZD be rotated about Z, in the same direction, through one, two, three, four right angles, respectively, to ZA', ZB', ZC', ZD', then will the last four segments be in equilibrium.

(21) The four points which divide the four sides of a skewsquare in the same given ratio $(m : n)$ are the corners of a skewsquare (m and n may be real or complex numbers).

(22) If $ABCD$ and $A'B'C'D'$ are any two skewsquares, then the four points dividing the segments AA', BB', CC', DD' in given ratio ($m : n$ real or complex) are the corners of a skewsquare.

(23) If the corners of any one of a system of confocal skewsquares are the roots of a polynomial, the roots of its second derivative and the foci are the corners of a lozenge composed of two equilateral triangles.

(24) If s_1, s_2, s_3, s_4 are the lengths of the segments into which the intersection of diagonals of a skewsquare divides those diagonals, then the two foci and the intersection of diagonals of the skewsquare (whose corners are z_1, z_2, z_3, z_4) are the roots of the derivative of the function

$$(z - z_1)^{s_1}(z - z_2)^{s_2}(z - z_3)^{s_3}(z - z_4)^{s_4}.$$

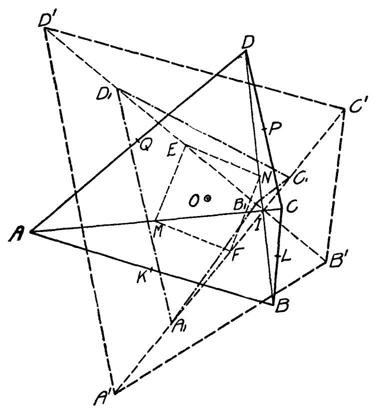
(25) If r_1, r_2, r_3, r_4 are the segments joining any point Z to the corners of a skewsquare, and s_1, s_2, s_3, s_4 are defined as in (24), then forces acting at Z whose directions pass through the corners and whose magnitudes are proportional to

$$\frac{s_1}{r_1} : \frac{s_2}{r_2} : \frac{s_3}{r_3} : \frac{s_4}{r_4}$$

are in equilibrium when Z is at either focus of the skewsquare.

The following abbreviated proofs are offered to establish the properties

enunciated above. In the accompanying diagram, $ABCD$ is a skewsquare, the diagonals intersect at I, the foci are E and F, M and N are the midpoints of diagonals and K, L, P, Q the midpoints of sides. The figure is drawn for a convex skewsquare, it may however be either reëntrant or two opposite sides may cut internally. $A'B'C'D'$ is the conjugate, and $A_1B_1C_1D_1$ the primitive of $ABCD$.



(1) The diagonal AC can be brought to BD by a rotation through a right angle, first A to B and C to D by rotation about E, second A to D and C to B by rotation about F.

(2) In the two right isosceles triangles AEB, CED the rotation AEC about E through a right angle brings A to B and C to D and AC to BD at right angles. There is a second center of rotation F accomplishing the same result, as in (1).

(3) The triangles AEC and BED are congruent, EM turns through a right angle to EN, in like manner FM turns through a right angle to FN. Hence $EMFN$ is a square.

(4) The altitudes of the congruent triangles AEC, BED are equal, hence E (and in like manner F) is equidistant from the diagonals.

(5) The mid-points of the sides of a skewsquare are the corners of a square called the mid-square. Its sides are obviously parallel and equal to half the diagonals of the skewsquare. The centroid of the corners of the skewsquare is obviously the center of the mid-square, and the centroid of the foci is O the mid-point of MN which is also the centroid of the corners of the skewsquare. The centroid of the corners of the four squares on the sides of the skewsquare as diagonals is the centroid of the corners of the mid-square.

(6) This is obvious when the focus is the vertex of the right isosceles triangle having a side as the base. In any other case such as EQ, since QL and EF bisect each other at O, then QE is parallel and equal to FL which is perpendicular and equal to half BC.

(7) $AB^2 + CD^2 = AI^2 + BI^2 + CI^2 + DI^2 = CB^2 + AD^2$.

(8) The triangles AED and BEC have equal area, for AE is equal and perpendicular to EB, ED is equal and perpendicular to EC. The included angles are therefore equal or supplementary.

(9) $\angle\, DAB$ is half a right angle plus either $\angle\, EAD$ or $\angle\, FAB$ (if E and F are outside use *minus*).

(10) The property in (9) is a well-known fundamental property of a conic having foci E, F and tangent to the sides of the skewsquare. The circumcircle of the square $AEBA'$ passes through I, therefore IA' makes with IA half a right angle and contains F, by (4). FA' cuts AB in T such that

$$\angle\, KTA' = \angle\, ETK = \angle\, FTB.$$

Since the focal radii to the point of contact make equal angles with the tangent, T is the point of contact. Since IT bisects AIB, T divides AB in ratio $AI : BI$.

(11) The sum (difference) of the focal radii to the point of contact is equal to FA'. Also PF is parallel and equal to $EK = KA'$, (6), each being perpendicular and equal to $\frac{1}{2}AB$. Therefore FA' is parallel and equal to KP. Hence the midcircle is the auxiliary circle of the tangent conic. Also $KP^2 = 2KQ^2 = \frac{1}{2}DB^2$.

(12) The product of the perpendicular from F on AB and $EK = KB$ is equal to the square on the semi-minor axis, and to the area of AFB which is equal to

$$\tfrac{1}{2}AF \cdot FB \sin AFB = \tfrac{1}{4}AD \cdot BC \sin AFB.$$

$\angle AFB + \angle LFQ = 2\pi - \frac{1}{2}\pi = \frac{3}{2}\pi$, $\angle LFQ = \pi - \angle (AD, BC)$. Therefore $\angle AFB = \angle (AD, BC) + \frac{1}{2}\pi$. Therefore $\sin AFB = \cos \angle (AD, BC)$. This establishes the statement. Obviously the eccentricity of the conic is equal to the ratio of the radius of the focal circle to that of the midcircle. In virtue of the triangles in (8) the foci E and F must be inside, on, or outside the boundary of the skewsquare, the areas of the triangles being counted positive for inside foci and negative for outside foci.

(13) The midpoints M and N are fixed and therefore the foci are also. The diagonal being of constant length, the transverse axis is fixed in length and position, and therefore the conic which the sides touch is fixed. The power of O with respect to the circle on AC as diameter is equal to

$$AM^2 - OM^2 = QP^2 - OM^2 = 2OP^2 - OM^2 = OP^2 + (OP^2 - OE^2).$$

We note by holding the midsquare and the focal circle fixed we derive the forms of skewsquares touching congruent concentric conics. If only the midsquare be fixed the skewsquares touch concentric conics having equal transverse axes. When the foci are on the boundary the skewsquare degenerates into an isosceles right triangle, one focus is the midpoint of the hypothenuse and the other is at the vertex of the right angle and the conic degenerates into the segment joining them. When two opposite sides of the skewsquare cut internally the conic is an hyperbola, since the foci lie on opposite sides of a tangent.

(14) If AB is fixed and the diagonals extend indefinitely intersecting in I, the other two sides whose ends are A and B become parallel to the diagonals and cut at right angles, in say Z. The single focus is E and the bisector IE makes equal angles, $\frac{1}{4}\pi$, with the right-angled sides, and ZE makes this same angle with these sides. This establishes the statement from the known properties of the parabola.

(15) In the figure, the circle on AB as diameter passes through A' and I, therefore AIA' is half the right angle AKA' and A' is on bisector IF, and so for each corner. By (16) FP is parallel and equal to EK, hence KF is parallel and equal to EP or $\frac{1}{2}EC'$. Therefore from similar triangles $A'KF$ and $A'EC'$, we have $A'F = FC'$. Also $A'F$ is parallel and equal to KP.

$$A'C'^2 = 4KP^2 = 8KQ^2 = 2BD^2.$$

In like manner $B'D'^2 = 2AC^2$, and E is the midpoint of $B'D'$. Hence M, N are the foci of $A'B'C'D'$.

(16) A_1O is parallel and equal to $\frac{1}{2}A'E$, therefore A_1O is perpendicular and equal to $\frac{1}{2}AB$.

(17) A_1C_1 is parallel and equal to $\frac{1}{2}KP$, by (18).

(18) $A'F = BN\sqrt{2}$ and $BN = A_1F\sqrt{2}$. Therefore $A'F = 2A_1F$. $A'A_1 = = A_1F$, $A'K = KE$. Therefore A_1K is parallel and equal to $\frac{1}{2}FE$, and so for the other points.

(19) $AA_1 = AD_1$, $BA_1 = BB_1$. The circles, centers A and B having radii AA_1, BB_1, respectively, have their radical axis passing through $A_1 \perp AB$ and therefore passing through O, by (16).

(20) The characteristic property defining a quadrilateral $ABCD$ whose corners are the complex numbers z_1, z_2, z_3, z_4, as a skewsquare is

$$z_3 - z_1 = i(z_2 - z_4),$$

which can be written

$$z_1 + \omega z_2 + \omega^2 z_3 + \omega^3 z_4, \tag{1}$$

where ω is the principal fourth root of unity, i. Any four numbers which satisfy this equation are the corners of a skewsquare, this then is the equation of a skewsquare and it establishes at once the truth of (20) on multiplying (1) through by i.

(21) If the corners of a skewsquare satisfy equation (1) in (20), then the same points satisfy the equation

$$z_2 + \omega z_3 + \omega^2 z_4 + \omega^3 z_1 = 0. \tag{2}$$

On multiplying (1) by $m/(m + n)$ and (2) by $n/(m + n)$ and adding, the resulting equation proves the statement in (21). If m, n are real numbers, division of a segment in ratio $m : n$ is conventional and familiar. When m, n are complex numbers then the point w "dividing the segment" z_1 to z_2 in ratio $m : n$ is determined, as before, from $w = (mz_1 + nz_2)/(m + n)$, or what is the same thing

$$\frac{w - z_2}{z_1 - w} = \frac{m}{n}.$$

The point w is therefore the vertex of a triangle constructed on segment z_1 to z_2 as base similar to the triangle whose sides are m and n. Therefore, if similar triangles be similarly constructed (all outwards or all inwards) on the sides of a skewsquare as bases, then the vertices are the corners of a skewsquare.

(22) Let equation (1) in (20) be any skewsquare and $z_1' + \omega z_2' + \omega^2 z_3' + \omega^3 z_4' = 0$ be any other skewsquare. Then on multiplying the first equation by $m/(m + n)$ and the second by $n/(m + n)$ and adding, the statement is verified.

(23) Take the origin at the center. Let ζ and η represent the midpoint of and half of a diagonal respectively. Then the corners are the roots of

$$(z - \zeta - \eta)(z - \zeta + \eta)(z + \zeta - i\eta)(z + \zeta - i\eta) = 0,$$

or

$$(z^2 - \zeta^2)^2 - 4\zeta\eta^2 z - \eta^4 = 0.$$

The roots of the second derivative are $z = \pm \zeta/\sqrt{3}$.

(24) Let $f(z)$ be the function. Take the logarithm and differentiate. Then

$$f'(z) = f(z)\left(\frac{s_1}{z - z_1} + \frac{s_2}{z - z_2} + \frac{s_3}{z - z_3} + \frac{s_4}{z - z_4}\right).$$

The roots of $f'(z)$, not common to $f(z)$, are gotten by the vanishing of the parenthesis. Also

$$s_1 + s_3 = s_2 + s_4 = d,$$

the length of the diagonal. Also

$$\frac{s_1 z_3 + s_3 z_1}{s_1 + s_3} = \frac{s_2 z_4 + s_4 z_2}{s_2 + s_4} = I,$$

the intersection of the diagonals. Clearing the equation of fractions and dividing by d, there results $(z - I)[(z - z_1)(z - z_3) + (z - z_2)(z - z_4)] = 0$. The quadratic in the square bracket is $z^2 - \frac{1}{2}(z_1 + z_2 + z_3 + z_4)z + \frac{1}{2}(z_1 z_3 + z_2 z_4) = 0$. Transfer the origin to the centroid (center). Then $z_1 = \zeta + \eta$, $z_3 = \zeta - \eta$, $z_2 = -\zeta - i\eta$, $z_4 = -\zeta + i\eta$, $\zeta \equiv OM$, $\eta = MA$. Then $z_1 z_3 + z_2 z_4 = 2\eta^2$. The equation then becomes $(z - I)(z + i\zeta)(z - i\zeta) = 0$, as required.

(25) In the parenthesis in (24) put $z - z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, etc. Equating to zero the real and imaginary components shows that the sum of the components along two right-angled axes vanishes, and so also must their resultant, and the segments are in equilibrium, when Z is at either focus.

THE MINIMAL PROPERTIES OF THE ISOGONAL CENTERS OF A TRIANGLE.

By H. M. LUFKIN, Dunkirk, N. Y.

An isogonal center [1] of a triangle may be defined as a point such that if lines are drawn *through* this point to the vertices of the triangle, the angular space around this point is divided into six equal angles of 60° each. There are two types of isogonal centers, one of which we shall call an I_+ point and the other an I_- point; the distinction being as follows: Let P_1, P_2, and P_3 be the vertices of the triangle taken in counter-clockwise order and draw the lines l_1, l_2, and l_3 through the isogonal center to the respective vertices; then if, as we go around the isogonal center in counter-clockwise order, we meet these lines in the order l_1, l_3, l_2, the center is an I_+ point; if in the order l_1, l_2, l_3, it is an I_- point.

The isogonal centers of a triangle may be constructed by drawing equilateral triangles on the sides of the given triangle and drawing the circumcircles of these

[1] Neuberg, *Mémoires Couronnés, L'Académie Royale de Belgique*, vol. 44, 1891, p. 12ff. M'Cay, *Mathesis*, vol. 7, 1887, p. 211ff.

equilateral triangles. These circumcircles will intersect in a point which is an isogonal center of the triangle. If the equilateral triangles are drawn outwardly, the point will be an I_+ point; if inwardly, an I_- point. From this method of construction the following facts may readily be seen:

Every triangle has a unique I_+ and a unique I_- point; except the equilateral triangle which has a unique I_+ point, but whose I_- point is anywhere on the circumcircle of the triangle.

Where one angle of the triangle is 120°, it is convenient to define the I_+ point as the vertex of this angle; and where *only one* angle of the triangle is 60°, it is convenient to define the I_- point as the vertex of this angle.

Where no angle of the triangle is greater than or equal to 120°, the I_+ point is inside the triangle, and the *half* lines from this point to the vertices divide the angular space about the point into angles of 120°, 120°, and 120°; where one angle of the triangle is greater than 120°, the I_+ point is outside the triangle and the half lines from the point to the vertices of the triangle divide the angular space about the point into angles of 60°, 60°, and 240°.

The I_- point is always outside the triangle, except in the 60° case noted above, and the half lines from this point to the vertices make angles of 60°, 60° and 240°.

A problem which has been discussed by various writers[1] is that of determining a point P_0 such that the sum of the distances ρ_1, ρ_2, ρ_3 from this point to the vertices of a triangle P_1, P_2, P_3, respectively, shall be a minimum. It has been shown that if no angle of the triangle is greater than 120°, the point P_0 is the I_+ point; and when one angle is greater than 120°, the point P_0 is at the vertex of the obtuse angle.

In this paper a further investigation of this problem will be made to determine whether the I_+ and I_- points possess any further minimum properties when we consider the functions $F_{++-} = \rho_1 + \rho_2 - \rho_3$, $F_{+-+} = \rho_1 - \rho_2 + \rho_3$, and $F_{-++} = -\rho_1 + \rho_2 + \rho_3$.[2] The functions where all the signs are negative or where two are negative need not be considered since it is evident that they have no absolute minimum.

Since the I_+ and I_- points have peculiar properties for triangles having angles of 60° and 120°, we shall divide all triangles into the following classes:

Class	I	II	II′	III	IV	V
Number of angles						
< 60°	1	1		2	2	2
= 60°		1	3			
> 60° and < 120°	2	1		1		
= 120°					1	
> 120°						1

[1] Cavalieri, *Exercitationes Geometrica*, 1647, pp. 504–519. D. Jackson, this MONTHLY, *1917*, 42–44. R. A. Johnson, this MONTHLY, *1917*, 243–244. Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, pp. 130–131. De la Vallée Poussin, *Cours d'Analyse*, vol. 1, third edition, Louvain, 1914, pp. 165–166.

[2] The cases of several of these functions are mentioned by Jacob Steiner, *Gesammelte Werke*, vol. 2, p. 729, but his results are erroneous.

The functions F_{++-}, F_{+-+}, and F_{-++} are continuous all over the plane, and $\partial F/\partial x$ and $\partial F/\partial y$ exist everywhere except at the vertices of the triangle. If we enclose a portion of the plane around the triangle, then there is for each of these functions at least one point within this region or on its boundary where this function takes on its minimum value for the region. It is evident that if we enclose the triangle with a large boundary, say a circle with its center at the circumcenter of the triangle and a radius a hundred times the radius of the circumcircle, that as we move in from the boundary the value of the function is less than on or outside the boundary; so that the minimum does not occur on the boundary, and must therefore occur at some point within the region. Since this is the case, the minimum occurs at a point where the first derivatives are zero, or where they do not exist.

Let the vertices of the triangle be $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$. The function F_{-++} is then

$$F(x, y) = -\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} + \sqrt{(x - x_3)^2 + (y - y_3)^2}$$

and denoting the three distances by ρ_1, ρ_2, ρ_3, the first partial derivatives are

$$\frac{\partial F}{\partial x} = -\frac{x - x_1}{\rho_1} + \frac{x - x_2}{\rho_2} + \frac{x - x_3}{\rho_3}, \qquad \frac{\partial F}{\partial y} = -\frac{y - y_1}{\rho_1} + \frac{y - y_2}{\rho_2} + \frac{y - y_3}{\rho_3}.$$

From the point (x, y) we shall draw a half-line in the direction of the positive end of the X-axis, and denote the angles which the half-lines to the three vertices make with this half-line, by φ_1, φ_2, and φ_3, respectively. We may then write $\partial F/\partial x = -\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3$, $\partial F/\partial y = -\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3$. From the equations

$$-\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3 = 0 \qquad \text{and} \qquad -\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3 = 0,$$ [1]

we obtain

$$\varphi_3 - \varphi_2 = \pm 120°, \ \pm 240°, \tag{1}$$
$$\varphi_2 - \varphi_1 = \pm 60°, \ \pm 300°, \tag{2}$$
$$\varphi_1 - \varphi_3 = \pm 60°, \ \pm 300°. \tag{3}$$

There are sixteen possible combinations of these values obtainable from (1) and (2), but of these, only the following six combinations satisfy the conditions of (3):

	I	II	III	IV	V	VI
$\varphi_3 - \varphi_2 =$	240°	240°	120°	−240°	−240°	−120°
$\varphi_2 - \varphi_1 =$	−300°	60°	−60°	300°	− 60°	60°

For each of these six combinations it is seen that the half-line from (x, y) to the vertex P_1 bisects an angle of 120° made by the half-lines to the other two vertices.

[1] Cf. Steiner, *op. cit.*

We may express this by saying that the point (x, y) must satisfy condition C_1. A similar argument holds for the function F_{+-+}, and the first derivatives can exist and vanish only at a point which would fulfill condition C_2, namely, that the half-line from the point (x, y) to the vertex P_2 must bisect an angle of 120° made by the half-lines to the other two vertices. A similar condition C_3 must be fulfilled for the vanishing of the derivatives of the function F_{++-}.[1]

We have then shown that for each of the functions F_{++-}, F_{+-+}, and F_{-++} the absolute minimum exists and that it occurs at an I_- point or at a vertex of the triangle, except for triangles of Class V where it occurs at an I_+ point, an I_- point, or at a vertex of the triangle.

It will be seen from the table classifying triangles that in all classes except II and II′ there is one odd angle differing from the other two. From an examination of the various classes of triangles it is seen that the half-line from the I_- point which bisects the angle formed by the other two half-lines is the one to the vertex of this odd angle. If we specify that θ_3 shall be this odd angle; then for triangles of Classes I, III, IV, and V, there is no point which fulfills conditions C_1 or C_2, and hence for the functions F_{-++} and F_{+-+} the only points at which an absolute minimum can occur are the vertices of the triangle. For triangles of Classes I, III, and IV, condition C_3 is fulfilled at just one point, namely, the I_- point; and therefore the absolute minimum for the function F_{++-} can occur at the I_- point or at one of the vertices of the triangle. For triangles of Class V, condition C_3 is fulfilled at two points, namely, the I_+ point and the I_- point; and therefore the absolute minimum for the function F_{++-} can occur at either of the isogonal centers or at one of the vertices of the triangle. For triangles of Class II, since, strictly speaking, none of the conditions C_1, C_2, or C_3 are fulfilled anywhere in the plane, the absolute minimum for all of the functions must occur at one of the vertices of the triangle. For triangles of Class II′, any point on the arc of the circumcircle from P_2 to P_3 fulfills condition C_1; and therefore the absolute minimum for the function F_{-++} can occur at points of this arc or at the vertex P_1. Similarly, the absolute minimum for the function F_{+-+} occurs at points of the arc of the circumcircle from P_1 to P_3 or at the vertex P_2, and for the function F_{++-} at points of the arc from P_1 to P_2 or at the vertex P_3.

In cases where the absolute minimum must occur at one of the vertices, this vertex can be determined from an examination of the relative lengths of the sides of the triangle. In cases where the absolute minimum can occur at an I point or one of the vertices, it remains only to investigate the relative values of the function F_{++-} at the I_- point and at the vertices of the triangle for triangles of Classes I, III, and IV; and for triangles of Class V at the I_+ point, the I_- point, and the vertices. These results are shown in the appended tables.

Since the first derivatives of the function F_{++-} vanish at the I_- point for all

[1] Steiner (*op. cit.*) has shown that the condition for the vanishing of the first derivatives of the function F_{+++} is that the half-lines from the point (x, y) to the vertices of the triangle make three angles of 120° with each other.

triangles except those of Class II, and at the I_+ point for triangles of Class V, the question arises as to whether these points yield a relative maximum or minimum where they do not give the absolute minimum. An examination of the discriminant of the second derivatives shows, however, that these points are saddle-points.

For triangles of Class II, the first derivatives do not exist at the I_- point, which is the vertex of the 60° angle. In this case, the surface represented by the equation $z = F_{++-}(x, y)$ has a conical point corresponding to the I_- point. If we pass a plane through this conical point parallel to the Z-plane, the surface lies entirely above this plane, so that the I_- point is an absolute minimum as we have previously stated; but as we go out from this point in one and only one direction the surface starts tangent to the plane and then rises above it. In the case of the surface represented by the equation $z = F_{-++}(x, y)$ there is a conical point corresponding to the I_- point, and if we pass a plane through this conical point parallel to the Z-plane, the surface rises above this plane as we move away from the conical point in every direction except one; but as we move in this direction the surface starts tangent to the plane and then drops below it, giving a peculiar sort of saddle-point.

Summary.

Character of the Isogonal Centers.

Angles of triangle.	I_+.	I_-.
$\theta_3 < 60° < \theta_2 \leqq \theta_1 < 120°$	Minimum F_{+++}	Minimum F_{++-}
$\theta_3 < 60° = \theta_2 < \theta_1 < 120°$	Minimum F_{+++}	Minimum F_{++-}
$\theta_1 = \theta_2 = \theta_3 = 60°$	Minimum F_{+++}	*See next table.*
$\theta_1 \leqq \theta_2 < 60° < \theta_3 < 120°$	Minimum F_{+++}	Saddle-point F_{++-}
$\theta_1 \leqq \theta_2 < 60°, \theta_3 = 120°$	Minimum F_{+++}	Saddle-point F_{++-}
$\theta_1 \leqq \theta_2 < 60°, \theta_3 > 120°$	Saddle-point F_{++-}	Saddle-point F_{++-}

Positions of the Absolute Minimum for the Various Functions and Classes of Triangles.

Angles of triangle.	F_{+++}.	F_{++-}.	F_{+-+}.	F_{-++}.
$\theta_3 < 60° < \theta_2 \leqq \theta_1 < 120°$	I_+	I_-	P_3	P_3
$\theta_3 < 60° = \theta_2 < \theta_1 < 120°$	I_+	$I_- = (P_2)$	P_3	P_3
$\theta_1 = \theta_2 = \theta_3 = 60°$	I_+	Arc of circumcircle from P_1 to P_2	Arc of circumcircle from P_1 to P_3	Arc of circumcircle from P_2 to P_3
$\theta_1 \leqq \theta_2 < 60° < \theta_3 < 120°$	I_+	P_1	P_1	P_2
$\theta_1 \leqq \theta_2 < 60°, \theta_3 = 120°$	I_+	P_1	P_1	P_2
$\theta_1 \leqq \theta_2 < 60°, \theta_3 > 120°$	P_3	P_1	P_1	P_2

THE MATHEMATICAL PUZZLE AS A STIMULUS TO INVESTIGATION.[1]

By WALTER B. CARVER, Cornell University.

The word investigation may be interpreted broadly enough to include all search for, and discovery of, truth which is new to the discoverer, regardless of whether it is or is not new to others. In this sense, a good high school student may be an investigator. Every teacher of mathematics has some scholars who have sufficient intellectual curiosity to ask themselves questions, and sufficient mathematical technique to discover the answers to some of these questions. All that is needed is something to arouse their interest and get them started. But the teacher, too, must be an investigator if he is to retain his enthusiasm and any freshness in his point of view. And making all due allowance for long teaching hours and heavy burdens of administrative work, the thing that primarily prevents most teachers from doing work of investigation is a certain inertia and lack of interest. As a pleasant and effective stimulus for student and teacher alike, the writer recommends the mathematical puzzle. If there is any merit in this recommendation, the best argument in its favor will be the presentation of a few examples.

From an excellent collection of mathematical puzzles by H. E. Dudeney[2] I quote the following:

Five ladies, accompanied by their daughters, bought cloth at the same shop. Each of the ten paid as many farthings per foot as she bought feet, and each mother spent 8s. $5\frac{1}{4}$d, more than her daughter. Mrs. Robinson spent 6s. more than Mrs. Evans, who spent about a quarter as much as Mrs. Jones. Mrs. Smith spent most of all. Mrs. Brown bought 21 yards more than Bessie, one of the girls. Annie bought 16 yards more than Mary and spent £3 8d. more than Emily. The Christian name of the other girl was Ada. Now, what was her surname?

Of course the point of the problem lies in the fact that the total number of farthings spent by each of the ten persons is a perfect square; and that we have the relation

$$x^2 - y^2 = 405,$$

where x^2 and y^2 represent respectively the number of farthings spent by any one of the mothers and her daughter. Since the farthing is the smallest English coin, we may assume that the values of x and y are to be integers, *i.e.*, that this is a Diophantine equation. We can find its solutions by expressing the right-hand member as the product of two factors, and equating these factors to $x + y$ and $x - y$ respectively. We thus find just five pairs of values,

$$x = 21,\ 27,\ 43,\ 69,\ 203,$$
$$y = \ \ 6,\ 18,\ 38,\ 66,\ 202,$$

[1] Read at the meeting of the Association, University of Rochester, Sept. 6, 1922.

[2] *Amusements in Mathematics*, London, 1917. The puzzle here quoted is number 140 in this collection. Dudeney conducts a puzzle page in the *Strand Magazine*. His first book *Canterbury Puzzles* is now in its second edition.

and the numerous statements in the puzzle enable us to identify these different solutions with the different mothers and daughters.

But any one with even a little mathematical curiosity will surely go further and ask himself questions. How many solutions can be found for the Diophantine equation

$$x^2 - y^2 = n,$$

where n is any given integer? Can the same methods be applied to the equation

$$x^2 - 5xy + 6y^2 = n;$$

or to any equation of the form

$$ax^2 + bxy + cy^2 = n,$$

where a, b, c, and n are any positive or negative integers? A whole field of mathematics is thus opened up, in which one may either investigate for himself or read the results of the investigations of others—or both.

The problem of finding right triangles whose sides are integers is a very old one; and it is well known that *all* positive integral solutions of the Diophantine equation

$$x^2 + y^2 = z^2$$

can be obtained from

$$x = (r^2 - s^2)t, \qquad y = 2rst, \qquad z = (r^2 + s^2)t,$$

r, s, and t being arbitrary positive integers. But less familiar is the problem of finding triangles whose sides are integers and one of whose angles is 120° or 60°, leading to the Diophantine equations

$$x^2 + xy + y^2 = z^2 \qquad \text{and} \qquad x^2 - xy + y^2 = z^2.$$

It is not difficult to show that

$$x = (r^2 - s^2)t, \qquad y = (2rs - r^2)t, \qquad z = (r^2 - rs + s^2)t$$

furnishes all solutions for the 120° case. The equation for the 60° case is rather more difficult. But one may see quite readily from a figure that if x, y, z are integral sides of a 120° triangle, then x, $x + y$, z and $x + y$, y, z are sides of two different 60° triangles; and that *all* solutions of the 60° triangle may be obtained in this way from the solutions of the 120° case. The smallest solutions are 5, 3, 7 for the 120° case, and 5, 8, 7 and 8, 3, 7 for the two corresponding 60° triangles.

A similar problem is that of finding a right circular cylinder and the frustum of a right circular cone of equal altitude and volume, the radii of the cylinder and of the upper and lower bases of the frustum being integers. If z is the radius of the cylinder, and x and y those of the bases of the frustum, we have

$$x^2 + xy + y^2 = 3z^2.$$

The simplest non-trivial solution of this Diophantine equation is 2, 11, 7. One can look for further particular solutions, or for some form that will give an

infinite number of solutions; or, most desirable of all, some form or forms which will give *all* solutions.

Consider another example.[1] Thirteen mice, one of which is white, are arranged in a circle. A cat, beginning with the white mouse as number one, counts around this circle up to a certain number n, and eats this nth mouse. Beginning with the next mouse beyond the victim as number one, and counting in the same direction, the cat again eats the nth mouse; and then repeats the process once more. What is the smallest value for n in order that the third mouse eaten may be the white mouse?

It is not difficult to see that we will have a value for n (not necessarily the smallest value) if we can find integral values for n, a, b, x, y, t, satisfying the following conditions:

$$\begin{aligned} n - 13x &= a, & n &> 0, \\ n - (13 - a) - 12y &= b, & 0 &< a < 14, \\ n - (12 - b) - 11t &= 1, & 0 &< b < 13. \end{aligned}$$

Eliminating a and b from the three equations, we have

$$3n = 13x + 12y + 11t + 26, \quad \text{or} \quad n = 4x + 4y + 4t + 9 + \frac{x - t - 1}{3}.$$

Putting

$$\frac{x - t - 1}{3} = z, \qquad \text{or} \qquad t = x - 3z - 1,$$

we have

$$n = 8x + 4y - 11z + 5, \; a = -5x + 4y - 11z + 5, \; b = 3x - 4y - 22z - 3,$$

and we thus obtain integral values satisfying the three equations by giving any arbitrary integral values to x, y, and z. It only remains to choose these values of x, y, and z in such a way as to satisfy the inequality conditions, namely

$$\begin{aligned} 8x + 4y - 11z + 5 &> 0, \\ 0 < -5x + 4y - 11z + 5 &< 14, \\ 0 < 3x - 4y - 22z - 3 &< 13. \end{aligned}$$

Regarding x, y, and z as the rectangular coördinates of a point in space, one sees that the points satisfying these five inequalities lie within a portion of space below the plane $8x + 4y - 11z + 5 = 0$ and bounded by a prismatic surface whose cross-sections are parallelograms. Considering now only the lattice of points whose coördinates are integers, we wish to find the point of this lattice lying inside the prismatic surface and at the least distance below the plane $8x + 4y - 11z + 5 = 0$. One easily deduces that z must be less than 1. For $z = 0$, the inequalities in x and y can not be satisfied with integers; but for $z = -1$ we have a number of solutions. In particular, $x = 7$, $y = 7$, $z = -1$ gives $n = 100$, the smallest value for n. Some other values for n are 104, 112, 116, 124, 128, 136, 140, 144, 148, 152, etc. If one finds thus all values of n up to

[1] This is the third part of Dudeney's puzzle number 232.

1716 (= 11 × 12 × 13), all further values can be obtained by adding multiples of 1716 to these.

For a final example, I once more quote Dudeney.[1]

Eight men had been dining not wisely but too well at a certain London restaurant. They were the last to leave, but not one man was in a condition to identify his own hat. Now, considering that they took their hats at random, what are the chances that every man took a hat that did not belong to him?[2]

Let K_n represent the number of permutations of n letters such that no letter goes into itself. One sees readily that $K_1 = 0$, $K_2 = 1$, $K_3 = 2$. If one defines, for convenience, $K_0 = 1$, the formula

$$K_n = n! - (C_n^1 K_{n-1} + C_n^2 K_{n-2} + \cdots + C_n^{n-1} K_1 + C_n^n K_0)$$

follows from the fact that the successive terms in the parentheses give respectively the number of permutations sending just one letter, just two letters, . . . just $(n - 1)$ letters, all n letters, into themselves. By means of this formula we readily deduce

$$K_4 = 9, \quad K_5 = 44, \quad K_6 = 265, \quad K_7 = 1854, \quad K_8 = 14833.$$

Hence the chance that each of eight men should get the wrong hat is 14833/8! Two recursion formulas, simpler than the one above but rather more difficult to establish, are

$$K_n = (n - 1)(K_{n-1} + K_{n-2}) \qquad \text{and} \qquad K_n = nK_{n-1} + (-1)^n.$$

The following interesting relations were pointed out to the writer by Professor W. A. Hurwitz:

$$K_n = \int_0^\infty (t - 1)^n e^{-t} dt,$$

$$\frac{e^x}{1 - x} = K_0 + K_1 x + \frac{K_2}{2!} x^2 + \frac{K_3}{3!} x^3 + \cdots + \frac{K_n}{n!} x^n + \cdots,$$

$$K_n = n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\right).$$

This last relation shows that as n increases, the chance that each of n men should get the wrong hat approaches $1/e$.

These examples are perhaps sufficient to indicate the rather serious sort of mathematics into which one may be led by following up apparently trivial or foolish puzzles.

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[1] Number 267 in his Amusements.

[2] This problem is discussed by Lucas in his *Théorie des Nombres*, chapter XIII; and references are there given to other treatments by Euler and Neuberg.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

QUESTIONS.

The following questions are proposed by Professor Harris Hancock, the first being suggested by the replies to Question **36**, 1 (*1922*, 255–7).

48. In what quadratic realms of rationality and for what values of p (which is a prime integer) is the function $\dfrac{x^p - 1}{x - 1}$ factorable?

For example,

$$\frac{x^5 - 1}{x - 1} = \left(x^2 + \frac{1 + \sqrt{5}}{2}x + 1\right)\left(x^2 + \frac{1 - \sqrt{5}}{2}x + 1\right)$$

is evidently factorable in the realm $R(\sqrt{5})$; while $\dfrac{x^7 - 1}{x - 1}$ has the factors $(x^3 - ax^2 + bx - 1) \times (x^3 - bx^2 + ax - 1)$, where $a = (-1 + \sqrt{-7})/2$, $b = (-1 - \sqrt{-7})/2$, and is therefore factorable in the realm $R(\sqrt{-7})$.

49. What is the most general value of the constant T for which the differential equation

$$\frac{d^2y}{dx^2} + \frac{ay + T}{(by^2 + cy + dx^2 + ex + f)^2} = 0$$

admits solution in the form of elliptic functions, a, b, c, d, e, and f being constants?

A special case is given by Euler, *Institutiones Calculi Integralis*, p. 153.

DISCUSSIONS.

Professor R. M. Mathews quotes a construction for a certain type of circular cubic, and discusses an exceptional curve for which the construction fails.

Mr. Philip Fitch makes the suggestion that less emphasis be placed on the problem of orthogonal trajectories, and that the more general problem be studied in which two families of curves cut one another, not at right angles, but in such a way that the slopes at points of intersection are related according to any given law. Teachers of differential equations are probably glad enough so to vary a standard problem, provided that a sufficient supply of interesting examples is available to illustrate what they are doing.

We are from time to time reminded of some inconsistency or inexactitude of expression that pervades some special branch of mathematics; but we probably agree that the general aim of mathematical writers is to exhibit extreme precision and carefulness in the use of language. The note contributed by Professor A. A. Bennett shows how easily the writer of an elementary text fails to reach this ideal. No doubt a number of the flaws to which allusion is made are the result, not so much of inexact thinking as of a desire to use the simplest possible language. Certainly this would account for the frequent use by some authors of "greater than" where the meaning is "greater than or equal to" or "numerically greater than"; though the simplicity so attained is never anything but illusory. Professor Bennett does not select his authors for the multitude of their errors. Had he chosen one of the real offenders, he might still be asking "Why?" However, let us not overlook the fact that the bad usages are fast

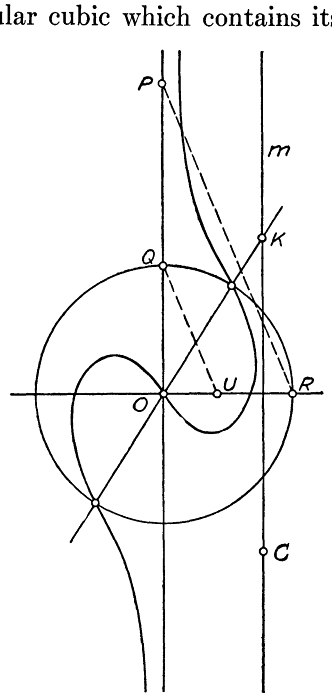
becoming obsolete, and that there is a strong movement today, of which the present paper is a sign, towards exactitude in mathematical phraseology.

I. An Exceptional Circular Cubic.

By R. M. Mathews, Wesleyan University.

It is well known that the locus of the intersection of any member of a pencil of coaxial circles with its diameter through a fixed point F is a circular cubic. The point F lies on the curve and is the *singular focus*, that is, the intersection of the tangents at the circular points at infinity. The line of centers of the circles is called the median of the curve. These curves have been studied synthetically by Schroeter, Durège and Lagrange.[1]

The converse theorem may be stated: any circular cubic which contains its singular focus F may be considered as the locus of the intersection of any member of a coaxial pencil of circles with its diameter through F. This theorem has been given by the authors named and by Loria and Hilton.[2] The object of this note is to examine a special cubic for which the diameter becomes indeterminate, and to arrive at a definite construction for this case.



The equation

$$(x^2 + y^2)x - 2gx - 2fy = 0 \qquad (1)$$

defines a proper circular cubic which is symmetric with respect to the origin O. The asymptotes are the y axis and the isotropic lines at O; so O is at once singular focus and principal point—intersection of the curve with the real asymptote. The circles of the pencil are concentric at O and so form a coaxial pencil bitangent at the circular points at infinity. The theorem is true but the simple correlation of circle and diameter is lost.

The correlation may be effected as follows. The equation (1) may be replaced by the system parametric in r

$$x^2 + y^2 = r^2, \quad (2) \qquad\qquad y = \frac{r^2 - 2g}{2f}x. \quad (3)$$

Plot the point $C = (2f, -2g)$ and through it draw m parallel to the y axis. On Ox take $OU = +1$. A circle of radius r cuts Ox at R and Oy at Q. Draw RP

[1] H. Schroeter, *Mathematische Annalen*, vol. 5, 1872, pp. 50–82; H. Durège, *ibid.*; A. Lagrange, *Nouvelles Annales de Mathématiques*, series 3, vol. 19, 1900, pp. 66–74.

[2] G. Loria, *Spezielle algebraische und transzendente ebene Kurven*, vol. 1, Leipzig, 1910, p. 35. H. Hilton, *Plane algebraic curves*, Oxford, 1920, p. 226, Ex. 28.

parallel to UQ to cut Oy at P. On m take $CK = OP$ and in the same direction. Then OK is the required line.

II. On Trajectories in General.

By Philip Fitch, Denver, Colo.

The treatment of trajectories in the better known works on differential equations is confined principally to those involving a given angle and more particularly to orthogonal trajectories. There are, however, numerous others of a different type, and having a practical value, that receive no mention.

Because of the common property of most of them, namely—that they always cut all the curves of the original family—the term "transversals" might be more appropriate.

If we let $f(x, y, \alpha) = 0$ represent the equation of a certain family of curves, we may have $\phi(x, y, dy/dx) = 0$ represent the differential equation of the same family. And in the expression $\phi(x, y, dy/dx)$ we may substitute a function of dy/dx, say $\psi(dy/dx)$, and obtain a differential equation of a new family of curves that will have some relation to the original family. Now the nature of the trajectory or transversal depends wholly upon the $\psi(dy/dx)$ and its relation and usefulness are determined by this function.

For example, take the case of bodies or groups of bodies moving in paths that are curves belonging to the same family, it might be necessary to obtain the locus of points where these bodies would possess a common property. This property could be equal momenta, equal kinetic energies, velocities in the same direction, and others, all of which may be represented as functions of dy/dx.

The value of these problems from a practical as well as an educational and purely mathematical standpoint cannot be over-emphasized.

Among the problems of this type that have been found useful is that of determining the transversals for velocity in the same direction of a series of streams of water shot out horizontally from the same level under different pressures or heads. Since the paths of these streams are parabolas with vertices and axes in common, we find the transversals to be a family of straight lines, concurrent at the point of efflux of the streams.

Another illustration is the determination of the transversals for equal kinetic energies for the streams mentioned above. In this case the transversals are a certain family of transcendental curves.

III. Why?

By Albert A. Bennett, University of Texas.

Attention has been called in this Monthly[1] to the loose attempts made to define "complex," "imaginary," "pure imaginary," as applied to numbers, in most American text-books in elementary algebra. There are certain other wide-

[1] E. S. Allen, "Definitions of imaginary and complex numbers" (*1922*, 301–303).

spread practices to which attention might also be called. For the sake of being specific, the excellent and widely-used text by Rietz and Crathorne will be mentioned, and page references will be to the "Revised Edition," 1919, of their *College Algebra.* The use of references of this sort will condense the discussion, but the writer realizes that this book may be more than usually free from lapses of the form cited, if these be indeed lapses.

P. 30, 8th line. "The square root of a negative number." One is at a loss to know whether even for a positive number, the authors would have one speak of "*the* square root," since the question of signs remains somewhat ambiguous, but surely no convention has been adopted which will justify the use of the definite article in the case of square roots of negative numbers. Why not agree that $\sqrt{n}$ is defined for non-negative real numbers *only*, and that the two square roots of $-n$, where n is positive, will be denoted by $\sqrt{n}\,i$ and $-\sqrt{n}\,i$, respectively?

P. 48, Exercise 4. Why should the illustrative example, at the bottom of page 46, be made an exercise for the student, two pages later? Is this merely to determine whether the student has read the text, or is it an accident due to the authors' not having read the exercise?

P. 58, Exercise 24. Why should a problem asking for positive integers be recorded as having an answer equal to zero?

P. 67, II. Why should the authors remark "since $r \neq 0$," when this was not given in the hypothesis, and since in the first application of the theorem, namely in III at the foot of the page, the case $r = 0$ is required?

P. 67, II. Why state the theorem, "a quadratic equation has two roots arithmetically equal but opposite in sign when and only when the term in x vanishes," as though a quadratic equation could not be in y or z instead of x? Why speak of the "known term," for a quadratic in x, meaning the term independent of x, when in the first application this "known term" contains an unknown, k, which is to be determined subject to certain conditions? Many books use the phrase "constant term" in an equation in which the unknown is at least constant, or "unknown" for a function in which the x although a variable is assuredly "known."

P. 71, Exercises. In plotting the graphs of quadratic functions, why are the exercises given in the form of quadratic functions *equated to zero?* Being given equations only, why cannot the student multiply by an arbitrary non-vanishing constant at will, and so obtain other graphs? If this is desired, why is it not explained?

P. 86, 1st line. Why are inequalities defined at this late stage after use has been made of them, P. 3, line 8; P. 13, line 1; P. 23, line 7; P. 25, next to bottom line; P. 68, 7th line from bottom; etc.?

P. 90, 4th line from bottom. Why must the "and so on" be made to reappear in each proof by mathematical induction, instead of being included in the notion of positive integers, where it belongs? Why not conduct the discussion somewhat as follows?—From the character of positive integers, we shall assume (and it may be proved) that if a theorem is true of a positive integer there must be a

first positive integer for which the theorem holds. Suppose if possible that the given proposition which we have established in the particular case $n = 1$ should fail for some positive integer. Then it must fail to be valid for a certain first one which we shall call $r + 1$. Here $r + 1$ is a positive integer at least as great as 2. Hence r is a positive integer, and one for which the proposition is valid. The argument is then carried out to show that the truth for r implies the truth for $r + 1$. Our hypothesis that the proposition fails to be valid is therefore seen to lead to an inconsistency and, unless other logical errors have been introduced, the proposition is therefore proved to be valid for all positive integers.

In this connection why not distinguish boldly between psychological induction which gains plausibility by the mere multiplicity of valid cases, and logical induction? To establish the theorem directly for $n = 2$, and $n = 3$, after the case $n = 1$ is proved, merely delays the argument. If the student finds it worth while even to attempt to prove the theorem, the stage at which psychological induction has value has been already passed, and the case $n = 1$ affords an ample start. If the proposition has a meaning for $n = 0$, this case is usually even simpler to demonstrate, and although 0 is not a positive integer, it suffices to start the induction.

Perhaps geometry is too elusive to permit of rigorous elementary text-books, but why must algebra books abound in opportunities for queries of this sort?

RECENT PUBLICATIONS.

REVIEWS.

Space—Time—Matter. By HERMANN WEYL. Translated from the German by Henry L. Brose. New York, E. P. Dutton and Co., 1922. 330 + xi pages. Price $7.50.

Professor Weyl's book was first published in 1918 and immediately achieved a remarkable success, the fourth edition being published in 1920. It is this latest edition which Mr. Brose, who has already successfully translated Einstein's own popular account of the theory of relativity, presents to the English-speaking scientific public. There is no indication of any collaboration with the author and indeed it is not clear whether the translation was "authorized" or not. This may account for many faults, some of which are indicated in detail below. It is claimed that "great care has been taken to render the mathematical text as perfect as possible." The number of misprints is, however, lamentably great and their occurrence makes the book very difficult reading for those to whom the translation can be of any possible use. The printing involved is obviously difficult, but the price asked for the book should certainly warrant better success. In a symbolism where the position of a label (above or below a symbol) is all important it is fatal for the printer to exhibit a certain carelessness in this respect.

Professor Weyl's book has been favorably reviewed by many competent

authorities.[1] It is by no means adapted for a cursory reading but demands leisurely study. And it will be of inestimable value to those who "think it out" for themselves. For the mathematician especially, the first chapter with its account of affine geometry will be very interesting. This subject is receiving considerable attention just now and Weyl deserves the credit of focussing this attention.[2] An account of Riemann's concept of a metrical space and of Weyl's own extension is given in Chapter 2. The contrast between the "action at a distance" character of Euclidean space and the "action through a medium" character of Riemann space is emphasized. The final chapters of the book deal with Einstein's idea of the relativity of space and time. In the discussion of the fundamental problem of the single statical gravitating center (p. 252) it is not clear what is meant by "the distance $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ from the centre." Some appeal from Riemann to the ideas of Pythagoras is surely being made here. But it was on reading § 35 on "The metrical structure of the world as the origin of electromagnetic phenomena" that we were finally convinced that the author belongs to that school of teachers which deprecates the lucid, orderly lecture on the ground that it is too easy and does not exercise sufficiently the mind of the reader. When we were dejected by our failure to understand a particular passage the ill-natured remark of Voltaire was recalled; to the effect that when two Germans are discussing a subject in such a manner that neither understands the other, they are talking metaphysics; but when they rise to such heights that neither understands himself, they are talking higher metaphysics. We may well be consoled, however, by the fact that the great German mathematicians—Hilbert, for example—are always clear and understandable. In making these remarks it will be clear that our strictures are meant not for what Weyl has to say but for his manner of saying it. He is, next to Einstein, the most serious contributor to the theory of relativity.

We add a list of errata and remarks for the non-expert reader of the book.

P. 21, equation (4) e^k should read e_k. It is unfortunate that upper (contravariant) labels were not used throughout for the coördinates and their differentials instead of the lower (covariant) labels. If this were done, the elegant property of the symbolism by which a summation (dummy) label occurs once above and once below in a term would be preserved. Equation (4) would read $e'^i = \Sigma \alpha_k{}^i e^k$.

P. 22, line 3 from top; $\overrightarrow{A'B'} = a' - b'$ should read $\overrightarrow{A'B'} = a'$.

P. 23, lines 4 and 2 from bottom; "quantum" should read "quantity."

P. 36, formula (23)'; for $\bar{\zeta}^l$ read $\bar{\zeta}_l$.

P. 52, line 13 from top; for $\vec{\xi}^r$ (at end of line) read $\vec{\xi}_r$.

P. 60, formula 41; for $\dfrac{\partial u^i}{\partial x_k}$ read $\dfrac{\partial u_i}{\partial x_k}$.

P. 87, at bottom; the remark that "the equation

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2$$

[1] See *Bulletin of the American Mathematical Society*, vol. 28, 1922, p. 215, for a review by G. D. Birkhoff, and *Nature*, vol. 109, 1922, p. 634, for a review by A. S. Eddington.

[2] An interesting paper on the same subject is that of W. Blaschke in *Mathematische Zeitschrift*, vol. 12, 1922, p. 262.

will hold more exactly the smaller du_1 and du_2 are taken" will grate on the ears of those accustomed to the Leibniz treatment of differentials.

P. 90, line 4 from top; for "total" read "exact."

P. 96, The remark, line 3 from top, is somewhat misleading. Riemann's definition of curvature is not an extension of Gauss' but a different definition altogether. One cannot be too careful in using the word "curvature" in these days when non-mathematicians talk so glibly about the curvature of space.

P. 96, at middle; for $(\Sigma x_i dx)^2$ read $(\Sigma x_i dx_i)^2$.

P. 107, line 8 from bottom; One must pillory the remark that "an infinitesimal element of surface is only a part (or more correctly, the limiting value of the part) of an arbitrarily small but finitely extended surface."

P. 108, line 2 from bottom; for ∂ read δ.

P. 112, line 5 from top; for $w_{ik}f_i$ read $w^{ik}f_i$. The justification for the formula immediately following is not clear.

P. 115, line 2 from top: The sneer in the words "This will relieve the minds of those who disapprove of operations with differentials" is not justifiable after the remark on page 107 (see above).

P. 136, line 12 from top; for e read ϵ.

P. 141, last line; for $d\gamma^i$ read $d\gamma_k{}^i$.

P. 161, line 12 from bottom; for "statistical" read "statical."

P. 187, line 10 from bottom; Voltaire would undoubtedly dignify by the term *high metaphysics* the statement "Suppose we have two twin brothers who take leave from one another at a world point A and suppose one remains at home . . . whilst the other sets out on voyages. . . . When the wanderer returns home in later years he will appear appreciably younger than the one who stayed at home."

P. 196, line 14 from top; for "Eichwald" read "Eichenwald."

P. 202, lines 4 and 5 from top; for $\frac{t}{dt}$ read $\frac{d}{dt}$.

P. 211, line 4 from bottom; for "deducted" read "deduced."

P. 231, in right hand side of equation of line 2; for u_k read u^k.

P. 258, lines 5 and 7; replace the minus by equality signs.

P. 262, last line; for u^i read u_i.

In conclusion we may explain why we have directed attention to the faults rather than to the merits of this translation. The widespread interest in the theory of relativity is a rare phenomenon. Not often is such an interest in the methods and results of mathematical analysis displayed by men cultured in other departments of learning. To those who believe in the importance of these methods for the progress of civilization it must seem particularly unfortunate if this curiosity is stifled and distrust aroused by difficulties due to the mode of presentation of the theory; those inherent to it are, we all know, sufficient.

FRANCIS D. MURNAGHAN.

Cours Complet de Mathématiques Spéciales, Volume 3, Mécanique. By J. HAAG. Paris, Gauthier-Villars et Cie. 1922. 8vo. viii + 188 pages.

This brief volume, although on the whole rather elementary, has enough individuality to make it interesting reading. In the opening chapters, which deal with kinematics, the motion is frequently defined by giving the relationship between displacement and time, rather than by giving the differential equation which defines such a relationship. This has certain advantages, since the differential equation naturally arises in dynamics and its solution can be taken up at that time. Considering the general scope of the book, it is surprising to encounter a Fourier series in connection with the discussion of vibratory motion.

The section on dynamics deals mainly with the dynamics of a particle. There is a chapter of fourteen pages entitled "Notions on the Dynamics of Systems" and a final chapter on "The Units of Mechanics." As there are no numerical exercises, there is no objection to this arrangement. The last two chapters deal with statics. Rolling friction, sliding friction, pivot friction and journal friction are discussed and a table of coefficients of friction for a number of different substances is also given.

Anyone using the book with a class would probably supplement it with a number of exercises.

PETER FIELD.

Leçons sur le Problème de Pfaff. By EDOUARD GOURSAT. Paris, J. Hermann. 1922. Price 30 francs.

In this book Goursat has produced what is probably the most scholarly work on the subject. The treatment is thoroughly up to date and exact references to original sources are more abundant that is usual in such treatises. In order to understand the book one must have freshly in mind the fundamental theorems relative to partial differential equations. Goursat makes frequent reference to his own work, *Leçons sur l'integration des équations aux dérivées partielles du premier ordre*, and seems to regard the present book as a sort of continuation of it.

The first two chapters deal with the "Problem of Pfaff" properly so called, particular emphasis being put on the properties of bilinear covariants. In the next two chapters a study is made of symbolic differential forms and of their application to the "Problem of Pfaff." In the fifth chapter there follows a study of integral invariants. The last three chapters are devoted to the most recent progress in the subject, particularly to that due to Cartan to whose work frequent reference is made.

The typography, etc., are good although a few misprints were noticed. The lack of binding and the uncut pages common in French books as they come from the press always annoy an American reader.

TOMLINSON FORT.

Elementary Calculus. By F. S. WOODS and F. H. BAILEY. Boston, Ginn and Company, 1922. 8 + 318 pages. Price $3.00.

As the preface states, this book is intended for first-year college students. In the light of this fact, it is not surprising that the treatment of the subject is occasionally simplified at the expense of rigor; for example, Taylor's and Maclaurin's series are discussed without mention of convergence or the remainder term. The explanations are, with few exceptions, very clear and easy to understand, and the abundance of problems which has come to be associated with the names of Woods and Bailey will add much to the usefulness of the book. In view of the fact that the book is primarily intended for engineering students, the introduction of the derivative through the concepts of velocity and accelera-

tion is a welcome departure from the geometrical presentation in terms of slope and curvature, which in this text is taken up later.

The following criticisms of a specific character may be made:

In the definition of a limit on page 2 the authors state that the difference between the variable and its limit "becomes and remains less than any small quantity which may be assigned," evidently meaning "any small positive quantity."

In connection with integration it might well have been proved that any two indefinite integrals of a function differ by a constant.

On page 5 appears the logical error of dividing $\frac{1}{2}$ mile by $\frac{1}{60}$ hour.

The discussion of pressure on page 68 calls attention to the confusion which appears in many books between the two meanings which are attached to the word "pressure," namely, *total force* and *force per unit area*.

In the proof, on page 120, that

$$\lim_{h \to 0} \left(\frac{\sin h}{h} \right) = 1$$

it will probably not be clear to the student that $BD + DB' > BAB'$.

The definitions of the inverse trigonometric functions, on page 130, as multiple-valued functions lead to much unnecessary complication in the formulæ for their derivatives.

The use of the term "linear acceleration," on page 135, to denote the tangential component, d^2s/dt^2, of the acceleration in plane curvilinear motion conflicts with the customary usage and, in particular, with the common expression of Newton's second law of motion, namely, $f = ma$, where f, m, and a are the measures of force, mass, and acceleration respectively, in terms of suitably chosen units.

It is difficult to understand the reason for introducing, as is done on p. 181, the notation $(df/dx)_y$ for $\partial f/\partial x$, and other analogous notations, and making practically no use of them later.

On the whole, many of the subjects treated seem a bit beyond the comprehension of the average freshman, but the book would probably be very satisfactory with sufficiently mature students.

HORACE L. OLSON.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

THE NEWTONIAN CLUB OF THE UNIVERSITY OF ALABAMA, University, Ala.

Seventeen students, with the coöperation of Professor Fort, Professor Lewis, and Mr. Dahlene, organized the Newtonian Club on September 13, 1921, to "*add* to the interest in mathematics, *subtract* from its apprehensive terror, *multiply* its benefits, and *divide* its snares." They elected as president, F. L. Davis, student-assistant in mathematics; vice-president, Reese Malette; secretary-treasurer, Louise Sandifer. Membership is limited to twenty including both active and

honorary members. Active members are elected by a two-thirds vote from students who have shown proficiency in analytical geometry and differential calculus; honorary members are the faculty members. The club meets on the first and third Tuesday nights in each month, generally at the family or fraternity homes of its members. In 1921–1922 the lives of Newton, Leibnitz, Napier, Einstein, and women mathematicians were studied; also such topics as the invention of the calculus and of logarithms, and the Einstein Theory. At one meeting a concert given in Philadelphia was heard by radio. For the year 1922–1923 the following officers were elected: President, Ramsey Reed '23; secretary-treasurer, Jesse Williams '23; program committee, the officers and Professor Fort.

September 12, 1922: Organization and election of new members.

October 10: "A type of algebraic correspondence" by Professor I. A. Lewis.

November 14: "Reasons for studying mathematics" by Professor Tomlinson Fort; "Circles connected with a triangle" by Harold Friedman '23.

December 12: "Determinants" by Esther Frank '25, John Steiner '23, and S. Rubrai '23.

(Reported by Miss Sandifer and Professor Fort.)

The Mathematics Club of Iowa State College of Agriculture and Mechanic Arts, Ames, Ia.

The Mathematics Club of Iowa State College was organized in October 1920 with a membership of fifteen including junior, senior, and graduate students majoring in mathematics. The purpose of the club is to stimulate interest in mathematics outside the usual text books and to promote sociability between students and faculty. The following officers were elected for the year 1920–1921: President, Ruth Dewey, Gr.; secretary-treasurer, Lillian Willson 22; faculty adviser, Professor Marian Daniells.

The program for the first year included papers on "Mathematical fallacies," "Mathematics of life insurance," "Mathematicians past and present" and a discussion of the Newton-Leibniz controversy. The final meeting of the year was a dinner at which were served non-isotropic ellipsoids, stuffed paraboloids, permutations and combinations and similar dainties. The program which followed the dinner included the play "The Flatlanders" and vaudeville by the Mathematics Department quartet. The first meeting of the year 1921–1922 was a picnic. At the October meeting the following officers were elected: President, Mary Battell '22; secretary-treasurer, Gilbert Witmer,—who was succeeded in the winter quarter by Alberta Wolfe, Gr.; faculty adviser, Professor Daniells.

November 2, 1921: "Why study mathematics" by Professor E. R. Smith.

December 7: "Mathematics and chemistry" by Professor E. I. Fulmer, of the Department of Chemistry.

January 4, 1922: "History of the development of logarithms" by Thelma Tollefson '23; "Life of Edward Wright" by O. A. Bock '23.

February 1: "Teaching of high-school mathematics" by Professor Gertrude Heer.

March 1: "The game of Nim" by A. L. Young, Gr.

April 5: "The planimeter and integration by mechanical means" by Mary Battell '22; "The abacus and other forms of counting machines" by Alberta Wolfe, Gr.

May 3: "Mathematics and music" by Florence Catlin, Gr.

In May the faculty of the department entertained at dinner the members and prospective members of the club. The dinner was followed by a program of music and toasts on the following subjects: "The integrating factor," "Essential singularities," "Harmonic progressions," "Absolute values," "A plane talk," and "Reductio ad absurdum" consisting of episodes from *Alice in Wonderland* and *Through the Looking Glass.*

(Reported by Professor Daniells.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3011. Proposed by E. T. BELL, University of Washington.

In a certain paper it is stated that "it is easy to prove that, if $p > 0$ is an integer, the relation $a_1 \sin \frac{\pi}{2p} + a_2 \sin \frac{2\pi}{2p} + \cdots + a_{p-1} \sin \frac{(p-1)\pi}{2p} + a_p = 0$ necessitates $a_1 = a_2 = \cdots = a_{p-1} = a_p = 0$, the a's being integers." Prove it.

3012. Proposed by R. A. JOHNSON, St. Paul, Minn.

It is well known that, of all polygons with given sides, that one which can be inscribed in a circle has maximum area. Show how to construct a polygon with its vertices concyclic and with its sides in order equal to the sides of a given polygon.

3013. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that $\lim\limits_{n \to \infty} \sum\limits_{r=0}^{r=m-1} \frac{4m}{4m^2 + (2r+1)^2} = \frac{\pi}{4}$.

3014. Proposed by the late T. M. BLAKSLEE, Ames, Iowa.

If 1, p, d are the radius and sides of the regular inscribed pentagon and decagon, and if a triangle be formed from these three lengths, determine the angle opposite p without the use of trigonometric tables.

3015. Proposed by J. G. COFFIN, New York City.

A heavy particle at the end of a weightless, flexible, inextensible string is set in motion. The string is attached to the side of a cylindrical post of radius 7 and the initial motion is such that, when the string is tangent to the post, the particle is moving as a conical pendulum. What is the curve made by the string on the post? What is the curve in space described by the particle? What time is required to wind up completely? As an interesting variation assume that the string is extensible.

3016. Proposed by R. B. ROBBINS, University of Michigan.

Solve the following recurrence relations, expressing the solution in terms of initial values U_1 and M_1 and the variable n which is always a positive integer:

$$2M_n = 2K_nU_n = M_{n-1} + U_{n-1}, \qquad K_n = M_{n-1} + U_{n-1} - M_{n-1}U_{n-1}.$$

SOLUTIONS.

2932 [1921, 467]. Proposed by R. C. ARCHIBALD, Brown University.

De Ville gave in 1629 the following construction for an approximation to the side of a regular polygon of n sides inscribed in a given circle: On a diameter AB construct an equilateral triangle ABC; divide AB into n equal parts, join C to D, the first point of division of AB, and let CD produced intersect the circumference in E; then the chord of the arc AF, the double of AE, will be approximately equal to the required side of the polygon. Construct a table of values for $n = 5$ to $n = 20$ exhibiting the extent, in minutes and seconds, to which the above construction

is in error in connection with the central angle subtended by the constructed side of the polygon. [Somewhat similar tables have been made for constructions by Bosse and Bernard.]

Solution by H. S. Uhler, Yale University.

Take a system of rectangular axes in such a position as to make the coördinates of A, B, C respectively $(a, 0)$, $(-a, 0)$, and $(0, -a\sqrt{3})$. Then $\overline{DA} = 2a/n$ and the coördinates of D are $(a - 2a/n, 0)$.

The equation of the line CDE is

$$y + a\sqrt{3} = \frac{\sqrt{3}}{1 - \frac{2}{n}}\,x.$$

Making the last equation simultaneous with that of the circle

$$x^2 + y^2 = a^2$$

the ratio of the ordinate to the abscissa of E is found at once to be

$$\tan\theta_n = \frac{\sqrt{3}\left[\sqrt{3 - 2\left(1 - \frac{2}{n}\right)^2} - 1\right]}{2\left(1 - \frac{2}{n}\right)},$$

where $\theta_n = \angle AOE =$ one half of the central angle subtended by the chord $\overline{AF}$.

Using Barlow's tables for the roots, and Bauschinger and Peters' eight place logarithmic tables I calculated the data tabulated below. All the numbers in the middle columns were verified by independent computation involving the corresponding formula for $\sin\theta_n$.

n	$\theta_n - 360°/n$			=	$(error)''$
5	0°	42′	41″.8		2561.8
6	0	56	48.7		3408.7
7	1	5	53.7		3953.7
8	1	11	25.8		4285.8
9	1	14	33.4		4473.4
10	1	16	3.8		4563.8
11	1	16	29.1		4589.1
12	1	16	10.7		4570.7
13	1	15	23.1		4523.1
14	1	14	16.3		4456.3
15	1	12	57.1		4377.1
16	1	11	30.2		4290.2
17	1	9	59.0		4199.0
18	1	8	25.7		4105.7
19	1	6	52.1		4012.1
20	1	5	19.1		3919.1

Also solved by W. C. Eells.

2933 [1921, 467].

Dudeney's[1] Problem, 1902: With ruler and compasses only, divide an equilateral triangle into four rectilinear pieces which may be put together so as to form a square.

Solution by H. C. Bradley, Massachusetts Institute of Technology.

Let ABC be the given triangle, D and E the middle points of the sides AB and BC. Construct S, the side of a square equal in area to the given triangle. This may be found by ruler and compasses since it is the geometric mean between AD and the perpendicular from C on AB. Describe a circle with center D and radius S cutting AC in F; draw the straight line DF and take

[1] Dudeney gave without proof in *Canterbury Puzzles* a solution which he presented to the Royal Society. Editor.

G on FA so that FG is one half CA. If FG is not greater than FA, or what is the same thing, if F falls on the segment CM where M is the middle point of CA, the construction is possible. Drop perpendiculars from E and G upon FD with feet at J and H, respectively. The figure will then be divided into four pieces, $DBEJ$, $HGAD_1$, CFJ_1E_1, $G_1H_1F_1$, where $D_1 \equiv D$, $J_1 \equiv J$, etc. The first three may be reassembled by putting D_1A and DB, EB and E_1C together. Then FC and AG will lie in a straight line, since the angles at A, B and C form 180°. Then the fourth piece may be placed with F_1G_1 in coincidence with FCG, since $FC + AG = FG$ by construction. The resulting figure is a rectangle for the adjacent angles at E, E_1 are supplementary; similarly for the angles at F, F_1, D, D_1 and G, G_1. Also the angles at H, H_1, J_1, J are right. It may be shown that $JD + D_1H = S$ and this suffices to show that the rectangle is a square.

Since no use has been made of the fact that the triangle is equilateral, the above process of dissection applies to any triangle in which certain conditions, seen from the above, are satisfied.

If a triangle is such that its altitude is equal to its base and neither of the two base angles exceeds 90°, the four pieces may be cut out by two straight cuts only. Let AC be the base, the altitude $BD = AC$, and E, F the middle points of AB and BC, respectively. Project E and F upon AC in J and K, respectively; then $EJKF$ is a square and it may be easily shown that the two cuts along the diagonals EK and JF give the required four pieces.

NOTE BY OTTO DUNKEL, Washington University.

Proof that the rectangle is a square. Since in an equilateral triangle $CD > AD$, the geometric mean S must satisfy the inequality $CD > S > AD$. Now $DM = AD$ and hence M lies within the circle of radius S and center D and C lies without. Hence F lies within the segment CM and G within MA.

Project C, A, B upon FD in the points C', A', B'. Then $C'F + HA' = FH$ from the construction for G; $C'J = JB'$; and $A'D = DB'$. Subtracting the first equation from the second there results $FJ - HA' = JB' - FH$ or $FJ + FH = JB' + HA' = JD + DB' + HD - A'D = JD + HD$. On the other hand we have $(FJ + FH) + (JD + HD) = 2S$ and, since it has been shown that the two parentheses are equal, it follows that $(FJ + FH) = (JD + HD) = S$.

Also solved by ARTHUR PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Miss CONSTANCE WIENER, formerly a graduate student at the University of Chicago, has been appointed instructor of mathematics at Smith College.

At Wesleyan University, Middletown, Conn., Mr. H. E. ARNOLD and Mr. G. W. BAIN have been appointed instructors of mathematics. Professor B. H. CAMP has been granted leave of absence for the year 1923–1924.

Dr. V. E. POUND, of Toronto University, has been appointed instructor of mathematics at the University of Buffalo.

At Union College, Mr. W. L. WARNER has resigned his instructorship of mathematics, and Mr. A. P. J. BOUDREAU has been appointed instructor.

At Lafayette College, Mr. R. J. W. TEMPLIN has resigned his instructorship, and Mr. J. A. BENNER, of Pennsylvania State College, has been appointed instructor of mathematics.

Mr. P. R. YODER, of the University of Kansas, has been appointed professor of mathematics and physics at Blue Ridge College, New Windsor, Md.

Miss HARRIET L. HASKINS, a graduate of Vassar College, has been appointed to teach mathematics and science at Eastern College, Manassas, Va.

Mrs. ANNETTE HIGHSMITH, of Peabody College, has been appointed instructor of mathematics at Meredith College for the year 1922–1923.

At Hampden-Sidney College, Professor J. S. MILLER has resigned to accept a position at Emory and Henry College. Mr. MACON REED, of Columbia University, has been appointed head of the department of mathematics, and Mr. C. D. LAWS, of the University of Georgia, an instructor.

At Greenville (S. C.) Woman's College, Miss ISABEL HARRIS, of Columbia University, served as professor for the academic year 1921–1922. Miss ROSE B. WOOD, of Barnard College, is professor of mathematics at present, and Miss MARY L. GAMBRELL has been instructor since the fall of 1921.

At the University of South Carolina, Mr. M. A. HILL and Mr. C. C. EDWARDS have been appointed instructors of mathematics.

At the Georgia School of Technology, Professor FLOYD FIELD has been elected Dean of Men. Mr. R. M. MUNDORFF has been appointed assistant professor and Mr. H. K. FULMER and Mr. E. R. C. MILES have been appointed instructors of mathematics. Mr. W. H. BOERCKEL resigned his instructorship in June, 1922, to accept a position in the high schools of Philadelphia.

Mr. R. G. DEMAREE, of the University of Chicago, has been appointed head of the department of mathematics at Kentucky Wesleyan College, Winchester, Ky.

At Maryville College, Maryville, Tenn., Professor J. A. HYDEN has been granted a year's leave of absence and is studying at Vanderbilt University. Mr. PHIL SHEFFEY, of Maryville College, is taking his place for the year.

At Mississippi College, Clinton, Miss., Mr. J. R. HITT has been promoted to a full professorship, and Mr. C. A. LOVELL has been appointed instructor of mathematics.

At St. Ignatius College, Cleveland, Ohio, Professor E. J. O'LEARY has been transferred to St. Louis University, and Mr. H. P. HECKEN has been appointed professor of mathematics.

Miss EMMA L. KONANTZ, associate professor of mathematics at Ohio Wesleyan University, has accepted a permanent appointment at Peking University, effective the second semester of this year. Professor Konantz, while on leave of absence 1919–1921, taught in Peking University, and collaborated with Professor CHEN in his work on "History of Chinese Mathematics." (See *1922*, 276.)

Dr. M. A. NORDGAARD, of Columbia University, has been appointed professor of mathematics and head of the department at Antioch College, Yellow Springs, Ohio.

Mr. WAYNE DANCER, formerly a teaching fellow, has been appointed instructor of mathematics at Toledo University.

At the University of Notre Dame, Mr. DANIEL HULL has been promoted to an assistant professorship, and Mr. W. L. SHILTS, of Notre Dame, has been appointed instructor of mathematics.

At Hanover College, Hanover, Ind., Professor C. A. REAGAN has resigned to accept a position as field agent of Missouri Valley College, Marshall, Mo., and Miss LENA R. COLE, of the University of Missouri, has been appointed to fill the vacancy.

Mr. A. O. BOATMAN, recently a graduate student at De Pauw University and the University of Indiana, has been appointed professor of physics at Carthage College, Carthage, Ill. His duties include assisting in the department of mathematics.

Mr. BURGOYNE GRIFFING, of the University of Kansas, has been added to the staff of the mathematics department of Des Moines University.

At Beloit College, Beloit, Wis., Professor W. A. HAMILTON has been appointed chairman of the Administrative Committee, which is in charge of the college during the interim while a new president is being sought to replace President M. A. BRANNON, recently appointed Chancellor of the University of Montana. As this work requires all of Professor HAMILTON's time, Professor H. H. CONWELL has been appointed acting head of the department of mathematics.

Mr. J. M. JACOBY has been appointed instructor of mathematics at St. Mary's College, St. Mary's, Kansas.

Mr. DON BROUSE, of Purdue University, has been appointed assistant professor of mathematics at Baker University, Baldwin, Kansas.

At Kansas State Agricultural College, Mr. W. H. ROWE, Mr. W. C. JANES, and Miss THIRZA A. MOSSMAN have been appointed instructors of mathematics.

Miss LAURA DUERNER, of the University of Oregon, has been appointed instructor of mathematics at North Dakota Agricultural College.

Mr. C. R. HILLARD, a graduate of Franklin and Marshall College, has been appointed head of the mathematics department in the College of the Ozarks, Clarksville, Ark.

Dean W. A. BARTLETT, of Pomona (Cal.) Junior College, has been elected president of the mathematics section of the Southern California Teachers' Association.

Professor C. G. DARWIN, who has been lecturing this year at the California Institute of Technology, has been appointed to the newly instituted Tait chair of Natural Philosophy at the University of Edinburgh.

At the University of Georgia, Professor D. F. BARROW presented a paper "On square roots of integers" to the Georgia Academy of Science at the annual meeting in December. Professor R. P. STEPHENS was elected president of the Academy.

The Minister of Education of France has approved a regulation admitting as candidates for the degree of Doctor of Science, students possessing certain foreign degrees, regarded as equivalent or superior to the "diplôme de licence." Amongst these are candidates from those American universities belonging to the Association of American Universities who possess a master's or doctor's degree, or a certificate stating that they have completed at least two years of graduate study leading to the Ph.D. Such candidates need not present the diplôme de licence.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Summer Meeting of the Association, Vassar College, September 5–6, 1923

Eighth Annual Meeting, University of Cincinnati, December, 1923

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

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IOWA, Cornell College, Mount Vernon, April 27–28

KANSAS, Topeka, January 20

KENTUCKY, University of Kentucky, Lexington, April

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OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April

SOUTHEASTERN, Agnes Scott College, Decatur, March 10

TEXAS, Houston, December 1–2

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE CARUS MATHEMATICAL MONOGRAPHS.

A REPORT OF PROGRESS BY H. E. SLAUGHT.

In the Autumn of 1921, Mrs. Mary Hegeler Carus made a notable gift to the Mathematical Association of America in the form of an annual contribution of $1,200 for five years, beginning in January, 1922, for the purpose of enabling the Association to prepare and publish a series of mathematical monographs. The full text of the deed of gift and acceptance by the Trustees of the Association is printed in this MONTHLY for October, 1921, pages 353, 360.

The purpose in the mind of Mrs. Carus, as indicated both by word of mouth and in her written communications, is to popularize mathematical intelligence by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in the field of mathematics. These presentations are to be set forth in booklet form in a manner comprehensible not only to teachers and students specializing in mathematics, but to scientific workers in other lines. It is furthermore desired to reach that still wider circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, are quite willing and eager to extend that acquaintance indefinitely along informational lines, provided it can be done without prolonged and painful study of the mathematical treatises which abound in extreme rigor and endless detail.

The Trustees immediately appointed a committee consisting of Professors W. D. Cairns, Oswald Veblen and H. E. Slaught, "to nominate to the Trustees an editorial board on monographs and to formulate a statement of powers of this board." This committee, having no instructions other than those just quoted, found itself confronted with a very serious responsibility, for it recognized that upon the wisdom of its action depended in large measure the success or failure of this enterprise. Many questions at once arose: Should the editorial board contain a large number or a small number of members? Should they be widely separated geographically or concentrated near one center? Should the members represent all the leading divisions of mathematics,—analysis, geometry, number theory, etc.? Should this board be empowered to act in an editorial capacity only or should its members be chosen as well for their fitness to prepare some of the monographs themselves? These and numerous other questions required most careful consideration and led to much consultation and extended correspondence.

After prolonged study of these questions, the nominating committee reached the following conclusions:

(1) The editorial board should be limited initially to three members and should be so chosen that they can actually get together for personal conference with little expense and loss of time. This need has been demonstrated in the case of other committees, but seems to be imperative in this case since an entirely new project has to be worked out, one which could hardly be done by correspondence or widely separated meetings.

(2) This central body should have full and final authority in selecting authors, passing upon manuscripts and all other matters pertaining to the preparation and publication of these monographs, subject only to restraint or veto of the Board of Trustees. They should, however, be authorized to call into service any members of the Association, especially in viséing manuscripts, reading proof, or supplying any advice or assistance that may be needed, assuming that this is a work of public welfare in mathematics and that any member of the Association should consider it his patriotic duty to serve in whatever capacity this editorial board may call upon him.

(3) One member of this editorial board should be chosen for his qualifications as a business manager, since many questions of a business character are bound to arise at every turn. The other two should be chosen not only for their editorial qualifications, but also for their ability and fitness actually to produce the first two monographs and this they should be instructed to do.

The reasons for this latter recommendation are self-evident upon reflection. A somewhat careful examination of the numerous existing series of mathematical monographs in Europe and the few in this country shows that none of them are designed to fulfill completely, and most of them not even approximately, the purpose indicated for the Carus Monographs, and hence this editorial board has a new and exceedingly difficult task to perform. For this reason the two author-editors should be particularly well qualified to collaborate in working out the desired form of presentation and in setting the standard for future numbers of the series.

(4) It goes without saying that these two men must have unquestioned scientific standing in the mathematical world and particularly throughout this country and that they must have had successful editorial experience in the mathematical field, together with a good appreciation of literary qualities, especially as to clear and forceful use of English. Moreover, these men must have firmness and independence sufficient to reject any manuscript from whatever source if it does not conform to the standards set up for these monographs; they must have the tact and good sense which will enable them to deal with authors in such a way as to bring out in these publications the very best that is possible through suggestions and criticism. They must have a keen appreciation of the aim and scope of these proposed monographs as intended by Mrs. Carus and an enthusiastic determination to see these objectives realized to the fullest possible extent. And finally they should be willing to enter upon a five-year program, since it is hardly possible that the success of this undertaking can be demonstrated in a shorter time, and surely one of the most important factors is continuity of purpose and control.

(5) Taking into full consideration all of the foregoing specifications, the nominating committee considered itself exceedingly fortunate in being able to secure the acceptance of PROFESSORS DAVID RAYMOND CURTISS of Northwestern University and GILBERT AMES BLISS of the University of Chicago as the author-editor members of the editorial board for the Carus Monographs. The committee also insisted by a two-thirds vote that PROFESSOR H. E. SLAUGHT be selected as the business manager of the editorial board.

The foregoing recommendations were presented in oral outline to the Board of Trustees of the Association and approved at the annual meeting in Cambridge in December, 1922. The present elaboration of these recommendations may be considered a part of the nominating committee's report and a copy of the same will be filed with the minutes of the Cambridge meeting.

The Editorial Board has held numerous meetings and has agreed upon certain general policies and procedure, as well as upon many details, some of which are herewith given.

(1) The choice of subjects for the early monographs was a question much in the minds of all who gave the matter any serious thought. This choice was at once limited, as was intended, to topics in analysis by the selection of Professors Curtiss and Bliss as the author-editors of monographs one and two. After careful consideration, it was agreed that Professor Curtiss would write on "Functions of a Complex Variable" and Professor Bliss on "Calculus of Variations," and that they would collaborate in the closest possible manner on all questions of scope and form of presentation.

(2) It is agreed that for the present, and until pretty definite standards have been established, no general call will be made for authors to present completed manuscripts for consideration and acceptance, which procedure might easily swamp the editors at a time when all their energy is needed in the actual determination and working out of these standards. On the other hand, the editors are carefully considering the selection of other authors with whom they believe effective collaboration can be established most surely and directly, and whose selection will insure a proper variety and range of topics.

(3) The editors do wish most emphatically to appeal for general and widespread coöperation on the following lines:

(*a*) They desire lists of topics suggested for monographs, arranged in the order of precedence preferred.

(*b*) They desire names of persons who are believed to be especially competent to prepare such monographs together with reasons in detail for such belief.

(*c*) Any person interested in the preparation of a monograph is earnestly requested to communicate with the editorial board, stating the topic proposed to be treated, giving an outline of the proposed treatment, and giving a sample chapter all in typewritten form. Such procedure will enable prospective authors and the editors to become acquainted on a practical working basis with the least possible delay and with no chance for misunderstanding or embarrassment.

It is agreed to be desirable in general that a monograph should be limited to about 125 to 150 pages, size $3\frac{1}{2}$ by $5\frac{3}{4}$ inches, printed with nine point leaded type, that there should be a minimum of display formulas and equations, and that the expository form of presentation should be used as much as possible. The books are intended primarily for the private reader rather than for the class room.

(4) For the benefit of prospective authors, the editors wish to suggest that it will be desirable in the preface of each monograph to state explicitly the maxi-

mum mathematical attainment on the part of the reader pre-supposed by the author and that the author should adhere religiously to this agreement at every point. For example, the editors have agreed that in the first two monographs this maximum requirement shall be an introductory course in the Calculus. In the case of other subjects, the maximum may doubtless be made lower and should be whenever feasible; while in all cases it will be the purpose to keep the prerequisites within the range of the widest possible circle of readers, certainly including all teachers of mathematics in high schools and colleges and all students and general readers who are acquainted with the usual undergraduate courses in mathematics.

Naturally, such monographs cannot contain rigorous proofs of all theorems presented, but they can give a readable exposition of the significance of such theorems with definite references where rigorous proofs may be found, and they can show how these theorems are used to build up in logical order the underlying doctrine of each subject, how this doctrine dovetails with that of other subjects, and how the whole fabric of mathematical truth, as related to this subject, finds useful and powerful applications in scientific affairs. Such monographs intended primarily for very moderately informed readers can be produced successfully only by the highest skill of the best informed writers.

(5) It is agreed, with the consent of Mrs. Carus, that a certain part of the fund shall be used to remunerate authors for clerical service, typewriting, etc., needed in connection with the preparation of monographs. It is further agreed that, in lieu of royalty payments to authors from the sale of the monographs, a certain lump sum shall be paid as an honorarium to an author out of the sales income from the first thousand copies and that this amount may be augmented by additional payments to be determined by the rate of sale realized beyond the first thousand.

It is confidently believed that at least one thousand copies of each monograph will be sold to the members of the Association to whom the price will be adjusted on the cost basis. The distribution to the general public will be through business channels such as the Open Court Publishing Company of Chicago, in whose general catalog of publications, which has a very wide circulation both in this country and in Europe, announcement has already been made of this projected series under the auspices of the Mathematical Association of America.

Finally, it should be said that the range of such a series of monographs is as wide as the field of mathematics itself. There may be fifty, or one hundred, or even two hundred topics in pure and applied mathematics, including historical and biographical subjects, which are capable of successful treatment in this series. At any rate, since the generous donor proposes to endow this fund, if the plan proves to be feasible and practicable after five years of preliminary trial, the only cause for eventual cessation will be either lack of appropriate topics or scarcity of qualified authors.

There is no intention of limiting the work of preparing these monographs to American authors. The editors will look with great pleasure for suggestions from

or concerning authors in any part of the world who may be interested in this project, the only condition being that an accepted manuscript shall be prepared and printed in English.

ON THE SUBJECT MATTER OF A COURSE IN MATHEMATICAL STATISTICS.[1]

By H. L. RIETZ, University of Iowa.

1. Introduction. It was suggested by Professor Huntington as Chairman of our Program Committee that this paper give, among other things, a brief report on such a partial survey of existing courses in mathematical statistics as I could make in the short time available to obtain the necessary information. My first step in attempting such a partial survey consisted in an examination of the recent catalogue announcements of a large number of colleges and universities. These announcements show that many departments of mathematics offer at least a beginning course in statistics, and that a relatively small number of departments—seven among the institutions whose catalogues I examined—offer also an advanced course in mathematical statistics. My next step consisted in writing letters of inquiry to a number of teachers of the courses in statistics. This correspondence disclosed the fact that the courses differ so widely in their prerequisites, organization, and content as to make it difficult to present a precise report without going into details. But neglecting details, we find at one extreme certain experiments in the teaching of freshmen courses in statistics. At the other extreme, we find advanced courses in mathematical statistics taken primarily by graduate students of mathematics. Although I have been unable, in general, to form more than a rough judgment as to the content of the courses as given in a particular department, it appears that the courses differ so much as to be widely scattered over the interval between the two extreme types of courses just mentioned. Moreover, it is but natural to expect that these relatively new courses based on material selected from a large and rapidly growing field would show much variation both as to prerequisites and subject matter.

As the subject matter of a course in mathematical statistics is determined to a considerable extent by the influences which have led to the development of such courses and by the primary aims of the course, let me attempt to describe briefly these influences and what I conceive to be the chief aims of such a course. The influences which have led to the development of the courses have come from at least two sources. A wholesome influence has come from the revival of interest in the pure theory of probability. This revival of interest is expressed both in recent papers on probability theory and by the appearance of books on probability by Poincaré, Borel, Bachelier, Czuber, Fisher, and Keynes. The main influence, however, in the development of courses in mathematical statistics has come from the demand for better statistical methods in handling the data of the rapidly

[1] Presented before the Association at the Symposium held in Cambridge December 29, 1922.

growing statistical sciences. The dominant factor is thus to be found on the side of the applications.

To illustrate the operation of the influence of practical statistics on the development of mathematical statistics, consider the situation of Francis Galton when he was working on the problem of likeness of parents and offspring with respect to human stature. His intuition for statistical inference was well developed. It was no doubt fairly obvious to him from a rough survey of his data on family history that there is neither the kind of perfect dependence between the heights of father and son that is given by a simple usable mathematical function nor the independence ordinarily assumed in probability theory. Galton thought that the analysis of his data presented an important mathematical problem. His suggestion of the problem to J. Hamilton Dickson, Karl Pearson and others initiated a development of correlation theory which is now being applied to a remarkably wide range of data.

The influences which have led to the development of courses in mathematical statistics are closely connected with the chief aims in the teaching of the subject; and I shall next attempt to make a statement of what seem to me to be the leading aims in the teaching of mathematical statistics. It seems to me that one aim is to develop in the minds of the students the meanings of what I shall call appropriate norms or patterns for the elucidation of the mass phenomena to which statistical data apply. The development of such patterns in the mind seems to me to be as fundamentally necessary for the description of statistical data as is the development of the appropriate elementary geometric forms or patterns in the mind of the student for the description of the shapes and sizes of objects in the space of experience. In statistical analysis, the patterns are represented by binomial distributions, probability curves, Bernoulli, Poisson and Lexis series, tetrachroic functions, regression curves, correlation surfaces, and so on.

The second important aim in the teaching of mathematical statistics is to develop the principles and theorems for the solution of practical problems involving the determination of significant differences and relations among statistical variables. The solution of these real problems consists largely in giving a satisfactory description of a collection of items. In his recent book, Keynes says that the first function of the theory of statistics is purely descriptive. While this statement seems to be correct, it is also correct to say that appropriate mathematico-statistical methods go much further than an ordinary non-mathematical description. The mathematical analysis makes possible inferences which could in all probability not have been otherwise made and it should offer safeguards against incorrect inferences and pitfalls. In this use of mathematical statistics there are, however, some dangers. The greatest danger seems to me to be the failure of some persons applying formulas to recognize the limitations of the formulas they are using. Thus, the formality of the method of measuring correlation by a coefficient is likely to be carried too far. For instance, let me cite the particular case where a few years ago I gave some lectures on correlation

before some instructors in agriculture. Presently I found one of them so taken up with correlating things in his farm survey work that I told the dean of the college that we must stop this man "correlating."

With the above mentioned aims in mind, suppose we attempt to determine the subject matter of a course. We do not get far until we are confronted with the problem of selecting the subject matter adapted to the mathematical preparations of the rather heterogeneous group of students from whom the demand for the course comes. Thus, we have to consider the subject of prerequisites. While the demand for the course comes from many classes of students with respect to mathematical preparation, it has seemed to me to be good educational policy and fair economy to recognize especially two classes of students with respect to qualifications to enter upon the study of statistics. The one group consists of those who have had training in college mathematics at least through a thorough first course in calculus and have the degree of maturity of superior seniors or graduate students. The other group consists of the rather mature students, say juniors, seniors, and graduate students from departments such as economics, biology, and psychology. These students are often dealing with statistical data but are lacking entirely or almost entirely in training in college mathematics. The two classes of students thus recognized would include no freshmen or sophomores. It is not unlikely that there are situations in which it is also desirable to give statistics in the freshman or sophomore year, but ordinarily I fear such a policy is likely to result in side-tracking a good many students from the regular mathematical courses which they should be encouraged to take.

Since it is obvious that a much better course can be given to students with good mathematical training than to those without such training, the very natural reaction of the mathematics teacher would be to dispose of the problem by specifying a thorough first course in calculus as a prerequisite for any course in mathematical statistics. Let me say that I wish this were the practical solution of the whole problem, and perhaps it will in time be the solution, but we are confronted with a condition which calls for consideration. Under actual conditions, this view of requiring calculus as a minimum prerequisite for any course in statistics is not so practicable nor perhaps so generous in meeting a real situation as it may seem to be at first reaction. The actual situation which has confronted me and perhaps most teachers of mathematical statistics is a demand from seniors and graduate students of economics, biology, psychology and other departments interested in some statistical project for which they feel the need of better statistical methods. It is hardly necessary to say that such a group of students is usually represented by suitable attorneys in the form of professors in these various departments, who urge that their students should have the course.

When the interested student has had at least a year of college mathematics, it has been my experience that he can often be convinced that it is to his best interest to take a course in calculus before entering a course in statistics, but when he has had no college mathematics, he usually feels that it is impracticable

for him to take all the preliminary mathematics up to and through a course in calculus. Moreover, it must be granted that a substantial amount can be taught about statistical methods without requiring calculus as a prerequisite. To meet the situation, certain departments are giving two kinds of courses in statistics. They are giving a sort of self-subsisting (*selbständig*) course for students who are interested in the analysis of data, but have had little or no college mathematics, and they are giving another course requiring a substantial amount of mathematics as a prerequisite. In what follows, I shall for convenience call the former an elementary course and the latter an advanced course.

To illustrate from my experience, let me say that I have for some time been giving two courses of three hours each throughout the academic year. The elementary course requires no college mathematics as a prerequisite, but at least junior standing in college. It has been taken by superior juniors and seniors and graduate students from departments other than mathematics. The course may be described as a combination of statistics and of necessary mathematical preliminaries adapted to rather mature students interested in the analysis of data. The elementary and advanced courses alternate from year to year. In the present year, I am giving the advanced course to eleven students of whom ten are graduates and one is a senior. Except for one graduate student of education who has had very little mathematics beyond a first course in calculus, each student has had several courses in mathematics beyond the sophomore calculus.

Last year slightly over half the students of the elementary course were juniors and seniors of the department of economics, and the remainder of the group was made up of graduate students from economics, biology, and psychology. I mention these classes of students in order to give a sort of picture of the kind of students interested in these courses.

The elementary course to which I have referred is conducted by having two lectures or class periods a week, and three hours in the statistics laboratory provided with computing machines. Such a combination of class room and laboratory work I regard as a distinct success in handing applications to real statistical problems.

Although it is not my intention to mark out a course to the extent of giving a syllabus, it has seemed to me that our discussion would probably be stimulated by giving a pretty definite notion of the subject matter of the two types of courses which I carry in my mind, with special reference to places for emphasis in the teaching of certain topics, and to certain points which may provoke discussion; for I understand it is my duty to be provocative on this occasion even at some risk of not being able to defend my position in all cases. It seems convenient to group my remarks and comments around certain groups of topics somewhat like chapter headings. Perhaps I should say that these groups of topics are not selected for purposes of logical organization of material, nor in all cases to indicate an order of presentation, but largely for convenience in shortening my comments —a subject in which I assume we are all interested.

2. Elementary course. Apart from a brief historical introduction, and a

few lectures on the chief sources of statistics, the course begins with an elementary treatment of frequency distributions, averages and measures of dispersion. Even at the risk of being tedious, let me explain to a slight extent the procedure in starting this course.

The work is begun by giving each student a sheet of data, say the monthly rainfalls of Iowa City for the past twenty-five years, and asking him to write out any facts of interest which he can draw from the data by ordinary common sense observation. After defining frequency distribution our next step is to require each student to exhibit the rainfall data in the form of a frequency distribution, and to make another frequency distribution by throwing coins or dice, say you have them carry out the experiment of throwing sets of seven coins 128 times making a frequency distribution of the number of heads per throw. Carry these concrete problems both from actual statistics and from games of chance through the process of constructing frequency polygons and freehand frequency curves.

The elementary treatment of different kinds of averages seems to me to be pretty well standardized as is shown by an examination of a number of textbooks. It may, however, be worth while to emphasize the teaching of short methods of computation and appropriate forms for computation. With these students who are inexperienced in numerical computation, special consideration should be given to such questions as the number of significant figures to be retained, to the relative errors in products and quotients, and to other questions of accuracy in numerical calculations. Another place for emphasis is in the selection of suitable concrete problems to illustrate the adaptation of an average to a particular purpose.

Since much of the discussion of the use of different types of averages has taken place in the preparation of index numbers, it seems desirable to give some simple applications to index numbers although the methods of preparation of index numbers should constitute a later chapter where we should lead up to the "best average" for index numbers as given recently by Irving Fisher, and as supported in a careful analysis by Allyn A. Young.

Let us consider next diagrammatic and graphical representation. I have sometimes thought that too much time is likely to be spent in making attractive pictures about obvious comparisons and relations. The Committee on Mathematical Requirements has very properly recommended that some graphical representation of statistics be taught in high schools. However, by a careful selection of material, I feel sure that my students have found a brief treatment of this topic of value. For practice exercises, I have found it useful to go through W. C. Brinton's *Graphical Methods of Presenting Facts* (New York, 1914), and make class exercises to illustrate practically every important point on graphical representation discussed in this book. It requires but short time to do these exercises and I think the students have found this work profitable. Among the large variety of detailed schemes, there is, of course, included the plotting of statistical variables on coördinate paper. A statistical curve with time as the abscissas supplies a very simple concrete basis for interpolation. This graphical

work leads naturally into the graphical representation of a mathematical function. Indeed the next chapter deals with the graphical representation of mathematical functions and the subsequent two groups of topics may be fairly well described as pure mathematical preliminaries to further progress in the course.

It is just as desirable in mathematical statistics as in other fields of mathematics that the meaning of mathematical function be clear, and that the pictures of changes in the function which correspond to changes in the assigned values of the variable be vivid. The tables of corresponding values in the case of the mathematical function are so analogous to certain statistical tables that we have a good setting for the development of the notion of a mathematical function. This graphical representation early in the course should be carried at least as far as it is carried in those of our college algebras which emphasize the subject, and I should carry the work somewhat further by developing the equation of a straight line to satisfy given conditions, and by finding the parabola $y = a + bx + cx^2$ through three points.

Let us consider next interpolation and graduation. Interpolation is so much used in statistical analysis that its treatment should be carried as far as the method can be appreciated by these students with no training in college mathematics. With the equation of the straight line through two points given it seems well to make graphical ideas the basis of interpolation by proportional parts. In my experience, it has been found practicable to introduce higher differences in this course, and to teach something of the meaning and use of both Newton's and Lagrange's interpolation formulas. Some of the simpler methods of graduation of certain data so as to remove minor irregularities may be taught very naturally immediately following interpolation.

Let us turn next to permutations, combinations, the binomial theorem, the elements of probability, logarithms, logarithmic and exponential functions. These topics hardly require any comment, except possibly the very general one to the effect that the more thoroughly the elements of these topics are studied early in the course, the more is likely to be accomplished later with practical problems of statistics. One special comment may be worth while to say that emphasis should be placed on statistical probability, on experiments with urn schemata, and on the uses of logarithmic and semi-logarithmic paper.

The study of the elements of probability leads us naturally to the binomial distribution. It seems practicable to me to follow Yule's chapter on this topic pretty closely in demonstrating that the standard deviation of the binomial distribution $(p + q)^s$ of the frequencies is $\sqrt{spq}$ and of the corresponding relative frequencies is $\sqrt{pq/s}$. It is important that the students experiment in making distributions by the use of urn schemata, and compare their results with the *a priori* most probable values.

The binomial distribution forms a natural introduction to the normal probability curve. It is surely of first rate importance that the student appreciate not only the value of the normal curve in statistics, but also the limitations on its applications in the description of frequency distributions. For the purposes

of this class, the curve may well be obtained as a limiting case of a symmetrical binomial distribution. It should be held prominently before the students that the normal probability function thus obtained represents a very special law of probability, and sufficient practice problems should be given in fitting the curve to actual distributions to make sure that the student makes proper application of both the tables of the probability integral and of the ordinates of the curve in calculating the theoretical frequencies given by the normal curve. The failure to fit reasonably well most of the given distributions forms the basis of a natural introduction to generalized frequency curves, but it seems impracticable to attempt more than a graphic treatment of generalized frequency curves in this elementary course.

Following the study of the normal probability curve, we may well give a sort of informal introduction to sampling theory. For example, it seems appropriate to introduce at this point in the course a suitable selection of material from such a chapter as that of D. C. Jones on probability and sampling which deals with the meaning of probable error, and with the derivation of the probable error in a simple relative frequency and in an arithmetic mean.

Let us consider next the Lexis theory of dispersion of relative frequencies. My class last year responded with enthusiasm to the classification of series according to the Lexis theory. Each member of the class carried out several experiments on urn schemata to give a concrete picture of the Bernoulli, Poisson, and Lexis types of dispersion. This theory of Lexis may be described as a method of breaking up statistical distributions into selected sets of sub-series, with a view to analyzing the stability of the relative frequencies among the sub-series. It is only the part of common sense to assume that the breaking of a statistical series into sub-series for examination would facilitate analysis, and form an important safeguard against erroneous conclusions from averages based on the aggregate only. The three types of urn schemata in the Lexis theory constitute a useful set of norms for setting up analogies with actual statistical series.

Let us turn next to the correlation theory. The treatment of correlation theory should be made one of the main topics of the course. The preparation of convenient forms for numerical calculation of the correlation coefficient should be stressed; for the inaccuracy of the beginner in the calculation of correlation coefficients is usually appalling. It is of first rate importance to emphasize the meaning and limitations of the correlation coefficient both from a common sense and from a precise technical standpoint. Bring out the fact by illustrations that the use of a summary formula like the correlation coefficient does not relieve us entirely of the necessity of close investigation of the data in subgroups. Make much of the mean-square-error of estimate—that is, of the standard deviation of arrays of the correlation table. Misunderstandings and extravagant interpretations of the accuracy of prediction by the use of a correlation coefficient arise from lack of appreciation of this point. For example, I feel sure that certain college administrators, who do not know correlation theory, fail to realize with what little security you can predict the achievement of an individual student

from an intelligence test because there is found to be a correlation of .6 between the results of such a test and actual achievement in college work. As you well know, the facts are that with $r = .6$ the average standard deviation of an array under the most ideal conditions is .8 of the standard deviation of the whole group, and it would obviously be a grave injustice to an individual to judge his achievement from such a test. On the other hand, this correlation of .6 based on large numbers gives, under the same ideal condition of linear regression, a close prediction as to the average achievement of those who make an assigned grade in the test.

The chief way that has occurred to me to guard against the dangers of extravagant interpretations of correlation coefficients is to give a diversity of applications, and have the student carry several problems through to completion and formulate conclusions, subject to limitations on the particular problem in hand. The correlation ratio should be as fully and carefully treated as the correlation coefficient, and the form for its calculation may well accompany that for finding the correlation coefficient as is shown in a paper by Crathorne in the *Journal of the American Statistical Association* for September, 1922.

To show what correlation means in relation to urn schemata, I have found it a valuable exercise for these students to experiment with urn schemata with certain limitations on independence in the probability sense, and investigate the correlation from data thus obtained.

With the group of students from psychology, it seems important to teach Pearson's treatment of correlation in ranks, and to point out the meaning of Spearman's "footrule"; for students of psychology are almost sure to ask whether Spearman's footrule is not good enough for measuring correlation when only small numbers are available.

At least the meaning of partial and multiple correlation should be presented, and I should carry the class as far in this topic as their mathematical preparation permits. You are perhaps aware that educational psychologists are at present making considerable use of partial correlation methods, and in this connection it is surely important to emphasize the meaning and limitations of the methods. It is practicable and desirable to carry through some simple applications of the correlation of deviations from a line of general trend, thus giving an introduction to correlation in time series.

As perhaps the final topic in the course, it is important to return to a consideration of random sampling. The idea of random sampling has been developed to a slight extent in an earlier chapter, but it seems that more should be done with it. The students in this course are almost sure to bring up questions which make it desirable to discuss the meaning of Pearson's criterion of goodness of fit and to apply it to some of the problems we have had earlier in the course where we passed judgment on goodness of fit mainly by a common-sense process. It needs hardly be said here that the limitations of the method should be especially stressed with this class of students.

In bringing to a close my comments on the elementary course, let me direct special attention to the fact that in this course theory is taught in so far as it is

practicable to do so in carrying out the aims of the course, but much is made of concrete problems and experiments in statistical probability. In closing my remarks on this course let me say that by the elimination of the chapters on mathematical preliminaries, the course seems well adapted to mature students who have had freshmen mathematics. Such students may well take the course with a reduction in credit on account of such duplication of work as is likely to occur.

3. Advanced course. Turning now to a brief consideration of the advanced course, there is hardly a subject in the elementary course which cannot be greatly enriched when the students taking the course have adequate mathematical preparation to be introduced gracefully to the theory of statistics. In my comments on this course, I shall follow to some extent the order of topics of the elementary course, and interpolate additional topics.

Our treatment of averages in the elementary course is one to which we might conceivably care to make but slight extension, but even here the introduction of the average of an infinite number of elements, the identity of centroid and arithmetic mean, of the radius of gyration and standard deviation, set up associations of real interest to the student.

It was suggested that the treatment of binomial distributions in the elementary course might follow Yule rather closely. In the advanced course, it is fairly clear that a treatment similar to that in E. Czuber's *Wahrscheinlichkeitsrechnung* (Leipzig, 1910–1914) or to that in A. Fisher's *Mathematical Theory of Probabilities* (New York, 1922) is to be preferred. Closely associated with the binomial distribution is the theorem of Bernoulli. My experience indicates that this central theorem may well be developed in practically the form in which it is stated in the German and French encyclopedias of mathematics, although this statement includes much in addition to the original statement of Bernoulli. With respect to the consideration of the well known controversy about the application of the principle of Bayes [1] in the derivation of the converse theorem, it seems best to me to do little more than direct attention to the controversy at this point, and to defer its careful consideration until near the end of the course.

A substantial chapter on interpolation by the use of higher differences should be among the early subjects in this course and this chapter may well be followed by one on the methods of graduation of statistical data. The principles of the methods of least squares and of moments should likewise be taught early in the course with some simple applications to curve fitting. Certain quadrature formulas should be developed for the application of the method of moments of areas. In particular, Sheppard's corrections of raw moments of frequencies should be derived.

The treatment of the normal probability curve in the elementary course is an almost purely experimental treatment. In the advanced course, such experiments on fitting the curve to data are important, but the various derivations of

[1] Compare this MONTHLY, *1920*, 347–354: "The average reading vocabulary; an application of Bayes's theorem."—EDITORS.

this law of probability, such as those of Gauss, Hagen, Morgan-Crofton, Thompson and Tait, and others, may be profitably studied. Particular emphasis should be placed on the assumptions on which each derivation rests, and on the question of our knowledge or lack of knowledge as to whether the assumptions are valid for the real problems of statistics. After this study of the normal probability function, it seems appropriate to take up the Poisson-exponential—the Bortkewitsch "law of small numbers." This law applies to cases in which the probability of occurrence is very small. Applications should be given to fit the Poisson-exponential to distributions such as the number of deaths per month or year from a minor cause, or to distributions such as that of the Rutherford and Geiger experiment (*Philosophical Magazine*, series 6, volume 20, 1910, p. 698) on the frequency of the number of α-particles radiated from a disc per one eighth second.

It is interesting to experiment with this law in comparison with the Gaussian law of probability in assigning deviations above and below a certain most probable value. The criticisms of Miss Lucy Whitaker of the Bortkewitsch representation should receive attention, although it seems to me her main criticism is invalid.

In this advanced course, there should be included a thorough treatment of generalized frequency curves. With both the Pearson system of generalized frequency curves and what I shall call the Charlier system available, it is a question of difficulty for some of us to decide upon the most appropriate treatment of the subject. It seems to me that the derivations of both systems should be studied in this course, with emphasis first on the character of the assumptions from which they are developed. Then the student should make a variety of applications of both systems to actual distributions as experiments in the facility with which each system applies. My students have shown a good deal of interest in using both systems, but I am not ready to announce a general conclusion from these experiments. I will say that there remains no doubt in my mind that the Charlier system is born in a higher region of mathematical thought than the Pearson system, but the final test consists in trying these systems on many classes of data to see how well they work. The Charlier system has not had extensive use in this country largely because the auxiliary tables were not easily accessible. Fortunately, Fisher in his 1922 edition of *Mathematical Theory of Probabilities* has given such tables to four places, and more extensive tables are given by Glover in his *Tables of Applied Mathematics*, 1923. Among frequency curves deserving special consideration are also those arising by the transformation of the independent variable of well-known frequency curves.

The Lexis theory of dispersion should be studied thoroughly in this advanced course. This study should include the generalization by Coolidge in the *Bulletin of the American Mathematical Society*, volume 27, pp. 439–442, 1921, in which he generalizes the averages which fall under the Lexis theory, and makes the probabilities defined by Lexis a special case.

In correlation theory, we have one of our chief opportunities to extend the work in the elementary course. This should involve the development of the

theories of non-linear regression, and of partial and multiple correlation. It should include the methods of Persons and of Crum for the study of the correlation of time series. It is also of interest to study the correlation surfaces in three and higher dimensions and the curves and surfaces of equal distribution. Another set of configurations of interest which may well be included are the curves and surfaces with the density of distribution which exists under independence.

In the treatment of random sampling we are able to do much in this advanced course where we could do very little but discuss meanings in the other course. After developing the theory of probable errors of various frequency constants, it seems especially important to study the work of Student, Soper and Fisher in *Biometrika,* volumes 6, 9 and 10, on the distribution of means, standard deviations, and correlation coefficients from small samples. The development of the Tchebycheff's inequality should be included because the distribution in this case can be a more or less arbitrary continuous function. As a final section on sampling theory, Pearson's criterion of goodness of fit of theory and observation (*Philosophical Magazine,* 1900) should be developed.

The applications of harmonic analysis to statistical data including the periodogram method should be treated in this course by carrying through the numerical work on some actual data.

Near the end of the course, it is well to deal with the present status of such controversial questions as the validity of the principle of Bayes on inverse probabilities. This question is again of special interest at present because Professor E. T. Whittaker gave in the *Transactions of the Faculty of Actuaries,* volume VIII, 1920, p. 163, a very clear argument to show the invalidity of Chrystal's well-known criticism of the inverse theory, and Karl Pearson gave also in *Biometrika,* volume XII, 1920, p. 11, the derivation of a sort of generalized result on inverse probability which does not make the much criticized assumption of equal distribution of the unknown probability.

If the time were available for the introduction of additional topics, it seems appropriate to include the mathematical theory of the flow of populations, and the mathematical theory of risk, including Landre's theory of maximum risk. Possibly a chapter earlier in the course should be given to the analysis of mortality statistics, but it has been my assumption that this important topic may be appropriately regarded as a special one to be included in a course in actuarial theory, or in a special course in vital statistics. It is desirable, however, to draw upon population and mortality statistics for applications. The analysis of mortality statistics naturally suggests the relation of courses in actuarial theory to courses in the general theory of statistics. Actuarial theory is, of course, concerned with a special field. Although I am much interested in the teaching of actuarial theory, it seems doubtful whether it would be economical for more than a small number of institutions to offer courses in this special field. On the other hand, it seems to me that a general course in mathematical statistics touches such a diversity of interests and applications that it would be justifiable for all large institutions to develop courses in the subject.

In conclusion, let me say that the courses which I have tried to describe are changing from year to year with new developments and are in the very nature of the case defective in many respects. It is my hope that the discussion of this paper will supply some of these defects, and point the way to progress. As to the outlook for further progress in the movement to develop courses in statistics, let me say first that in the eighteen years in which I have given such courses, there has been a healthy growth in interest and also a great improvement in the subject matter of the courses, and I see no indications that the movement will come at once to a standstill. In fact, I look forward to great improvement in these courses in the future, because they are concerned with the development of those ideal patterns which form the very basis of precise statistical description, and because there is every reason to expect an increasing demand for better statistical analysis along with the ever-increasing tendency to enter upon scientific projects which involve the accumulation of statistical data.

HISTORICAL-MATHEMATICAL PARIS.[1]

By DAVID EUGENE SMITH, Columbia University.

III. La Rive Droite.

Few of the older mathematicians had abiding places on the right bank of the Seine, most of them, as already seen, preferring the Quartier Latin. Guillaume Budé,[2] the prime mover in the founding of the Collège de France, however, died at No. 203 of Rue Saint-Martin, as an inscription shows. This is a long street running parallel to the Boulevard de Strasbourg from near the Conservatoire des Arts et Métiers to the Seine. Budé was one of the leading scholars of his day and wrote a work *De asse et partibus ejus libri V* (1516) which was very well known.

Even the Louvre, which we ordinarily look upon as the ancient abode of royalty and the modern home of the graphic and plastic arts, has a certain connection with mathematics. In the years immediately preceding the Revolution it was the seat of the academies and the lodging place of certain savants. Even under the first empire it served in the latter capacity, as witness a letter from Legendre in which he begs for quarters for himself in the older part of the palace. Lagrange[3] lived at No. 124, Rue du Faubourg-Saint-Honoré, a street which, after crossing the Rue Royale, bears the shorter name of Rue Saint-Honoré. A little to the north, near the Gare Saint-Lazare, the Hungarian pianist, Franz Liszt, lived at No. 63, Rue de Provence, and took an interest in the arithmetical prodigy, Henri Mondeux.[4] Among my autographs is an invitation to a musical séance which Liszt gave "au bénéfice du jeune Pâtre mathématicien" on April 30, 1841, with a page of Mondeux's writing on the back.

[1] This is the conclusion of an article which begins on page 107 of the March–April number.

[2] Born at Paris in 1467; died there in 1540.

[3] Joseph-Louis, Comte Lagrange, born at Turin in 1736; died at Paris in 1813. He was probably the greatest all-round mathematician of his time.

[4] He died in 1861.

In the vicinity of the ancient Temple, at No. 41 of Rue de Bretagne, was the Marché des Enfants-Rouges (opened in 1628), which at one time belonged to the Cassinis.[1] It took its name from a hospital nearby. César-François Cassini de Thury lived for a time at No. 8, Rue Simon-le-Franc, an ancient street known to have existed in 1200, and taking its name from Simon Franque, a wealthy merchant of the time. It is a little south of the Rue Michel-le-Comte which is mentioned later. The Rue de Bretagne crosses the Rue des Archives, a modern thoroughfare which includes the ancient Rue du Grand-Chantier on which, at No. 5 (old numbering), Jean Picard [2] lived.

Returning for a moment to the Ile de la Cité, facing the Palais de Justice is Notre Dame, near the north tower of which stood, until 1748, the small church of Saint-Jean-le-Rond [3] on the steps of which Mme. de Tencin's[4] child had been left on a winter's night and had been saved to become the great d'Alembert. The church has long since disappeared, and Mme. de Tencin has long since been dust under the pavement of Saint-Eustache, across the street from Les Halles Centrales. North of the site of Saint-Jean-le-Rond is the Rue des Ursins on which (at No. 9) stood, seven centuries ago, the cell of Abailard [5] who may possibly be ranked among the mathematicians for his study of Euclid in connection with his lectures on logic. He is buried with Héloïse by his side, in Père Lachaise. Two minutes' walk from here, across the little bridge to the Ile Saint-Louis, is the Rue Budé which takes its name from the mathematician, already mentioned, who prevailed upon Francis I to found the Collège de France. Returning to the right bank, a short distance south of Les Halles Centrales is the short Rue des Déchargeurs,[6] off which runs the Rue du Plat-d'Étain in which, at No. 1, was a cabaret in d'Alembert's time which he is said to have frequented. He was also a welcome guest at the salon of Mme. Geoffin, "le Ministre de la Société," as she was called, who lived not far away at No. 374, Rue Saint-Honoré. This street ends near Les Halles, and not far to the northeast lies the little Rue Michel-le-Comte where d'Alembert spent his early years in the home of a glazier. There is an old saying, "Ça fait la rue Michel," meaning "Ça fait le compte," a pun on "Michel-le-Comte." The Rue Michel-le-Comte terminates in the Rue du Temple which formerly, south of this point, was called the Rue Saint-Aroye. The Rue de Braque runs eastward from near this corner, and a letter was written

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[1] One of the most remarkable families of mathematicians and astronomers. The leading members were Giovanni Domenico (1625–1712), who was known as Jean Dominique after going to Paris (1669); Jacques, his son (1677–1756); César-François de Thury (1714–1784), the son of Jacques; and Jacques Dominique de Thury (1742–1845), the son of César-François. They were all astronomers royal of France.

[2] 1620–1682, a pupil of Gassendi's, and known chiefly for his work in geodesy and astronomy.

[3] Like the ancient Saint-Denis-du-Pas, it stood within the close (cloister, cloître) of Notre Dame.

[4] The father, General Destouches, died in 1726.

[5] Or Abélard.

[6] It dates from the early 14th century and takes its name from the stevedores who lived there.

[7] Alexis Claude Clairaut, born at Paris in 1713; died at Paris in 1765, well known for his work in higher plane curves, dynamics, and astronomy. He became a member of the Académie des sciences at the age of 18.

by Clairaut[7] from "rue St Aroye vis á vis la rue de braque" on November 24, 1764. He died, very likely there, the following May.

To the east of Les Halles there runs the Boulevard Sébastopol, and a little to the north is the Square des Arts et Métiers, with the Conservatoire facing upon it. In this the mathematician and physicist will find a storehouse of material, including a model of Foucault's pendulum and a collection of calculating machines dating from those of Pascal to the period just preceding the recent great development of these nerve- and labor-saving devices.

Across the street to the south lies the Église Saint-Nicolas-des-Champs, the Temple de l'Hymen of the Revolution. Here are buried the Guillaume Budé already mentioned, and also Pierre Gassendi,[1] who was professor of mathematics at the Collège Royal (1645) and author of various works, chiefly on astronomy. The south side of the Conservatoire faces upon the modern (1851) Rue Réaumur. In the part to the west of the Boulevard de Sébastopol was the Rue Thévenot where, at No. 23 (old numbering), there lived, in 1808, Charles Bossut,[2] well known for his history of mathematics[3] and for his various textbooks and his monographs on geometry.

In the same general vicinity, on the Rue de Rivoli at the Square Saint-Jacques, there stands the Tour Saint-Jacques, dating from 1521. It is all that remains of the old gothic church of Saint-Jacques-la-Boucherie, built upon the site of a chapel of the 8th century, called the Capella in Carnificeria (we might, in America, say the church in the stockyards) and dedicated to Saint James (Saint Jacques). The church was taken down in the year 5 of the Republic (1796), and in 1856 the tower was put in its present condition, with a statue of Pascal between the arches. It was on this tower that Pascal made his experiments with a barometer before making them at Puy-de-Dôme (1648). The story that these were made in the tower of the church of Saint-Jacques-du-Haut-Pas, in the Rue Saint-Jacques, is unfounded. This latter church has some mathematical interest, however, for it is here that the geometer Lahire[4] is buried.

A little east of the Tour Saint-Jacques and north of the Rue de Rivoli is the Rue Sainte-Croix-de-la-Bretonnerie, formerly (1228) the Champs-des-Bretons. At No. 16 is the hôtel of Lalande, whom Montucla, in one of his letters, sarcastically called "the divine Lalande," and who edited the second edition of his Histoire des Mathématiques (1799–1802).

A short walk to the north along the Rue du Temple brings one to the Rue de Rambuteau, part of which, where now stands the Archives, was formerly called the Rue de Paradis (not to be confused with another street by the same name southwest of the Gare Saint-Lazare), and it was at No. 1 of this street that

[1] Born in 1592; died at Paris in 1655.

[2] Born near Lyons in 1730; died at Paris in 1814. A letter written by Delambre on February 9, 1808, was sent to him at this address.

[3] *Essai sur l'histoire générale des mathématiques*, Paris, 2 vols., 1802. English translation, London, 1803.

[4] Philippe de Lahire, born at Paris in 1640; died at Paris in 1718. The name often appears as La Hire.

Delambre[1] lived in 1802, and at No. 16 in 1807. Between this street and the Rue Sainte-Croix-de-la-Bretonnerie is the Rue des Guillemites, part of which was called at one time the Rue des Singes, and here the father of the unfortunate Bailly [2] had his home and here no doubt his gifted son was born.

IV. Street Names.

The custom of naming the streets after the great men of a city or of the world is not peculiar to Paris; it is seen in all continental countries, but it is more evident in Paris than in other places owing to the size of the city and its long history as an intellectual center.

Among the streets that bear the names of mathematicians there is, in the vicinity of the Boulevard Sébastopol, the Rue Borda, opened in 1816 and named in honor of Jean Charles Borda,[3] who was distinguished not only in mathematics but also in physics and navigation. In the same general region there is a street that was (from 1780 until 1884) known as the Rue de Breteuil, so called in honor of Émilie de Breteuil, Marquise du Châtelet, but now called the Rue du Général-Morin. A short distance away is the Rue Bailly, the name of which may suggest the mathematician elsewhere mentioned in this article, but which is really due to the bailliage of Saint-Martin-des-Champs to which church the property formerly belonged. Similarly, the Rue Dupin, which branches off from the Rue du Cherche-Midi near the Boulevard du Montparnasse, is not named after the mathematician, Charles, but after his brother André Marie Jean Jacques (1783–1865), the lawyer.

In the newer parts of the city little attention has been paid to the naming of streets with respect to their proximity to schools of any kind. In the northwest section, a short distance south of the Boulevard des Batignolles, is the Rue Bernoulli,[4] named after the mathematician who, to use the misprint in the Marquis de Rochegude's work, "découvrit le calcul expérimental!" [5]

South of the Arc de Triomphe de l'Étoile, in the relatively new section, are the Rue Euler,[6] the Rue Newton,[7] and the Rue Galilée,[8] all coming together on the Avenue Marceau. The Rue Keppler[9] is in the same vicinity. The Rue Galilée ends at the west in the Avenue Kléber, across which, extending from the Place Victor Hugo, is the Rue Copernic,[10] known before 1856 as the Rue des Bassins. North of the Place Victor Hugo runs the Rue Léonard de Vinci,[11]

[1] Jean-Baptiste-Joseph Delambre, born at Amiens in 1749; died at Paris in 1822. His greatest work was in geodesy and astronomy.

[2] See page 107.

[3] Born in 1733; died at Paris in 1799.

[4] Jean Bernoulli (1667–1748).

[5] For "exponential."

[6] Léonard Euler, born at Basel in 1707; died at Petrograd (Petersburg) in 1783.

[7] Sir Isaac Newton, born at Woolsthorpe in 1642; died at Kensington in 1727.

[8] Galileo Galilei, born at Pisa in 1564; died at Florence in 1642.

[9] Johann Kepler, born at Weil in Würtemberg in 1571; died at Ratisbon in 1630.

[10] Nicolaus Copernicus (Coppernicus), the astronomer (1473–1543), who also wrote on trigonometry.

[11] Leonardo da Vinci (1452–1519).

named for one who would have been a great mathematician if he had not been greater in nearly every other line that his genius touched. Just west of the Gare du Nord is the Rue Condorcet, with its little cité Condorcet at No. 27. At No. 21, Rue Chanzy, which branches off from the Boulevard Voltaire at No. 212, there lived Jean Macé (1815–1894) who, perhaps, deserves to be included in our list because of his little *L'Arithmétique du Grand-Papa* (1862). Near the Gare de Lyon and running south from the Avenue Daumesnil, is the short Rue Charles-Bossut [1] and the Rue Michel-Chasles,[2] the latter being part of the land occupied by the old prison de Mazas. The Rue Abel [3] was opened at about the same time (1901) and in the same territory. Some distance to the west, and near the Métro station of Place de Vaugirard, is the Rue Gerbert, leading from the church Saint-Lambert to the Rue de Vaugirard and bearing the name of the most learned mathematician of his day,[4]—a day in which mathematics, however, was near its low ebb.

In the Gobelin region the Boulevard Arago[5] was opened in 1859, and at the western section is the observatory, near which is the Rue Méchain.[6] Arago was director of the observatory and died there in 1853. In the court is a statue of Leverrier,[7] by Chapu. The observatory grounds are bounded on the north by Rue Cassini,[8] known in the 17th and 18th centuries as Rue des Deux-Anges, des Deux-Maillets, and Rue Maillot. Its present name dates from 1790. The Cassini house was No. 2. The boulevard north of the one which bears Arago's name is Boulevard de Port Royal. At No. 119 was the ancient Abbaye des Religieuses de Port-Royal which, in the 17th century, served as a retreat for many of the learned world, including, as is well known, Blaise Pascal.

North of the cemetery of Montparnasse are the Rue Delambre[9] and the Rue Huyghens.[10] South of the cemetery are the Rue Fermat,[11] the Rue Gassendi, and the Rue Lalande. Still farther south, and branching out from the Avenue d'Orleans, are the Rue Bézout and the Rue Sophie-Germain.

In the Quartier Latin there is the Rue Laplace,[12] so called since 1864, having been called the Rue de l'Allemanier as early as 1300. It lies north of the Panthéon.

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[1] See page 168. The street received its name in 1873.

[2] Born at Épernon in 1793; died at Paris in 1880. He is known chiefly for his *Aperçu historique*, Paris, 1837, and his work on higher geometry (1852). The street was opened in 1902.

[3] Niels Henrik Abel, born at Findöe, Norway, in 1802; died at Arendal in 1829. Known chiefly for his work on elliptic functions and higher algebra.

[4] He became Pope Sylvester II in 999 and died in 1003.

[5] Dominique François Jean Arago (1786–1853), the astronomer.

[6] Pierre-François-André Méchain, born at Laon in 1744; died at Castellon de la Plana in 1804. He was one of the chief geodesists who worked on the base of the metric system.

[7] Leverrier is buried in the nearby cemetery of Montparnasse.

[8] See this MONTHLY, *1921*, 123, 369.

[9] The street was opened in 1839.

[10] French spelling for Huygens. Christian Huygens, born at The Hague in 1629; died at The Hague in 1695. Known as one of the greatest physicists and geometers of his time.

[11] Pierre de Fermat, born at Beaumont de Lomagne, *c.* 1608; died at Castres or Toulouse in 1665. Known for his work on theory of numbers and analytic geometry.

[12] Pierre-Simon, Marquis de Laplace, born at Beaumont-en-Auge in 1749; died at Paris in 1827. He wrote not only on astronomy but on probability and the higher calculus.

A little to the east is the Rue Descartes, known in the 13th century as the Rue Bordel (Bourdeille). As stated on page 174, Descartes was finally buried in Saint-Germain-des-Prés, this being recorded on a mural slab in the first chapel on the south after passing the sacristy. On the Rue Descartes stands the École Polytechnique[1] on the site of the ancient College de Navarre,[2] a school with which have been connected some of the greatest names in the history of French mathematics during the last century and a quarter. Back of the school runs the Rue Monge, named (1864) in honor of the great geometer. It is one of the most prominent of the many streets which were cut through ancient Paris in the 19th century and adds a posthumous glory to the memory of one who rose to great prominence but was allowed by his countrymen to die in poverty. The Square Monge lies next to the École Polytechnique where he worked for several years. There is also a Place Monge, and a little further south the Rue Pestalozzi[3] opens to the west.

Parallel to Boulevard Saint-Michel (the "Boul Mich" of the student jargon), and to the east, there runs the Rue Saint-Jacques, past the Sorbonne. From this there branches off to the west the short Rue Malebranche,[4] named in honor of one who was chiefly a philosopher but nominally a mathematician.

South of the Luxembourg gardens and running from the winding Rue Notre-Dame-des-Champs to the Rue d'Assas, is the Rue Leverrier, named after the scholar who applied mathematics to the determination of the position of Neptune. If one is visiting this section he may walk along the Rue Notre-Dame-des-Champs to No. 17, and see the house where Ampère[5] lived for a time. He may also walk over to the Rue d'Assas and see, at No. 28, the site of the house[6] in which Foucault died and in which he made some of his preliminary experiments on the pendulum.

Cutting the Boulevard Arago and running through to the Avenue des Gobelins is the Rue Pascal, opened in 1827.

In the northwestern part of the city, west of the cemetery of Montmartre and ending in the Avenue de Clichy, is the Rue Clairaut, before 1867 known as the Rue Sainte-Thérèse. The next street to the south, and a much longer one, is the Rue Legendre, which is cut by the Passage Legendre, formerly (1867–1877) known as the Passage Saint-Paul. Rue Legendre is cut farther to the west by the Rue Lemercier of which the southern part bears the name of Biot,[7] opened in 1850 but receiving its present name in 1864. Still farther to the west is the

[1] Founded in 1794.

[2] Founded in 1304 by Jeanne de Navarre, wife of Philippe le Bel.

[3] Johann Heinrich Pestalozzi, born at Zürich, 1746; died at Brugg, 1827. Known for his reform in elementary teaching of arithmetic and as the teacher of Steiner.

[4] Nicolas Malebranche (1638–1715). His mathematical work was chiefly in connection with astronomy.

[5] André Marie Ampère, born at Lyons in 1775; died at Marseilles in 1836. He was professor of mathematics at the École Polytechnique as well as professor of physics at the Collège de France. He wrote numerous monographs upon the calculus, higher plane curves, and probability, as well as upon physical questions.

[6] The present building is the École libre de Jeunes Filles.

[7] Jean-Baptiste Biot (1774–1862), who applied mathematics to physics and astronomy.

Boulevard Malesherbes on which is the Lycée Carnot, known as the École Monge until 1875, and this is bounded on the north by the Rue Viète,[1] so called in honor of the greatest of the early French algebraists. At the northern end of the same boulevard is the Rue Nicolas-Chuquet,[2] named for one of the few mathematical scholars of France at the close of the 15th century. From about the middle of the boulevard there runs to the westward the Rue Ampère, with an array of modern houses. It is cut by the Avenue de Wagram, which in turn is cut by the Rue Poncelet [3] with its small Passage Poncelet. Five minutes' walk to the west will bring one to the Rue Torricelli,[4] and another five minutes leads to the Rue Vernier and the Rue Roger Bacon,[5] before 1907 known as the Rue Bacon from its former proprietor. A short walk to the south brings one to the Avenue de la Grande Armée, between the Place de l'Étoile and the Bois de Boulogne, from which, near the Porte de Neuilly, branches the Rue Denis-Poisson, which went by divers names before the present one was adopted (1907). From the Étoile there starts, just north of the Avenue de la Grande Armée, the Avenue Carnot [6] which was so named in 1880 in honor of the Carnot of the Revolution.

Not far to the east of the cemetery of Montmartre, in the maze of narrow streets of the section, is the Rue Francœur,[7] a name given to the street in 1875.

Passing over to the northwestern part of the city, there lies, a short walk to the north of the Parc des Buttes Chaumont, the Passage Barrême, named in memory of the author of a very successful textbook on arithmetic,[8] one which still makes his name synonymous with all sorts of tabular devices for ready computation. Still in the eastern section, but near the Père-Lachaise cemetery, is the Rue Ramus,[9] recalling the death of the philosopher-mathematician who perished at the massacre of Saint Bartholomew, the signal for which was given by the bells of Saint-Germain-l'Auxerrois, across the street from the Colonnade of the Louvre. Branching off from the Rue Ramus is a small Passage Ramus, and a short walk to the east brings one to the Rue Vitruvius, calling to mind one

[1] François Viète, commonly known as Vieta, born at Fontenay-le-Comte in 1540; died at Paris in 1603.

[2] A native of Paris. He wrote the *Triparty en la Science des Nombres* at Lyons in 1484, but it was not printed until the 19th century.

[3] Jean-Victor Poncelet, born at Metz in 1788; died at Paris in 1867. He is known for his great work in projective geometry.

[4] Evangelista Torricelli, born in or near Faenza in 1608; died at Florence in 1647. Known not only as a physicist but for his contributions to the study of higher plane curves.

[5] Roger Bacon (1214–1294), the foremost scientist of his time, familiar with much of the older mathematics. He had an unusual knowledge of the latter science considering the time in which he lived.

[6] Lazare-Nicolas-Marguerite Carnot, born at Nolay in 1753; died at Magdeburg in 1823. A great military leader as well as a great geometer.

[7] See this MONTHLY, *1921*, 254.

[8] François Barrême, a native of Lyons. He died at Paris in 1703. His *Arithmétique* (Paris, 1677) went through many editions.

[9] Pierre de la Ramée, better known as Peter Ramus, born at Cust in 1515; died in 1572.

of the few Romans who had any appreciation of even the lower forms of mathematics.[1]

V. School Names, Statues, Tombs.

Paris has not only honored many of her mathematicians by the names of streets, but she has impressed upon her children the achievements of her learned citizens by occasionally giving such names to her schools. Thus we have the École Sophie-Germain, a higher primary school for girls at No. 9, Rue de Jouy, the street in which Richelieu was probably born and which takes its name from an abbey of the 13th century which stood at Nos. 13, 15, and 17. The Lycée Saint-Louis, on the Boulevard Saint-Michel, was at one time (1848) called the Lycée Monge, but the name was short-lived. The Lycée Condorcet is at No. 61, Rue d'Amsterdam, in the northwest section. The Lycée Voltaire (1890) is in the Avenue de la République, which leads out from the Place de la République. Besides these there are at No. 10, Boulevard Lannes, the École Pascal, and near the Place de la Nation the École Municipale Arago.

Paris has also many busts and statues and bas-reliefs of those who have made the science, some of relatively little importance, like the *médaillon* of Francœur on the building of the Société pour l'Instruction Élémentaire at No. 6, Rue du Fouarre, the ancient Rue des Escoliers of the 12th century, where several colleges were established in that period. Part of the street is now (since 1887) the Rue Lagrange. Budé [2] better deserves a statue, and one by Bourgeois (1882) stands in the Cours d'honneur of the Collège de France. Here are also the busts of Gassendi, Ramus, Lalande, Ampère, Chasles, and many others of scientific prominence.

On the chief façade of the Sorbonne is a statue representing mathematics and in the building are statues of Archimedes, Descartes, and Pascal. The garden of the Luxembourg has a statue of Bailly by the sculptor André, but it would be quite uninteresting to attempt anything like a complete list of efforts at portraiture of this kind to be found in the city. In the line of paintings, the most interesting portrait is the well-known one of Descartes, by Frans Hals, in the long gallery of the Louvre.

If the wanderer wishes to make a pilgrimage to the last resting-places of the mathematicians of Paris, his steps will naturally turn first to Père-Lachaise. Here he will do well to proceed by the Entrée principale up the main route, passing the graves of Arago (d. 1853) on the right (with a bust by David d'Angers) and of Poinsot (d. 1859) on the left, and then branching off to the Avenue Latérale du Sud to see the resting-place of Delambre (d. 1822). He may then proceed to the chapel, turn to the right, and pass on to the Carrefour du Rond, where he will find the graves of Monge (d. 1818), Hachette (d. 1834), and Fourier (d. 1830). A little farther along the Avenue des Acacias he will find the burial places of Chasles (d. 1881) and Comte (d. 1857). A few others are buried in different

[1] His great work on architecture was written between 20 and 14 B.C. He was well trained in the engineering of the time and in the ancient science of optics.

[2] See page 166.

parts of the cemetery, but the graves of many of the older group are to be found in less prominent cemeteries or in the ancient churches. Louis Étienne Lefébure de Fourcy[1] (1787–1869), for example, is buried in the cemetery Montparnasse, as are also Poincaré (d. 1912) and Joseph Bertrand (d. 1900). Pascal is buried in Saint-Etienne-du-Mont and Descartes in Saint-Germain-des-Prés.[2] There is a monument to Maupertuis (d. 1759) in Saint-Roch, on the Rue Saint-Honoré, in the usual ultra-dramatic style of the times, and he was finally buried here.

Such are a few of the interesting historical spots of a city which has exerted a remarkable influence upon mathematics, pure and applied, in the last four centuries, and which is certain to have an equally remarkable influence in the centuries to come.

SOME UNSOLVED PROBLEMS IN SOLID GEOMETRY.[3]

By JULIAN LOWELL COOLIDGE, Harvard University.

One of the delightful uncertainties connected with mathematical research arises from doubt as to the life of this or that part of the subject. We have no vital statistics bearing on the longevity of different mathematical topics. No man can say at what moment any particular branch of the science may wither and die for lack of new life and growth. A process of senility seems to have set in in algebra, in the elementary theory of numbers, and in other parts of our science. One elementary branch alone seems exempt from this general law of decay, elementary geometry. For some reason this ancient branch of learning seems to bear in itself an inexhaustible fountain of youth. It is just a hundred years since Feuerbach first published his classic theorem about the nine-point circle, eleven years ago a Japanese mathematician published eleven new proofs.[4] The Brocard figures were discovered some fifty years ago, the number of published articles dealing with them quickly ran up towards a thousand.

The great majority of the best theorems of elementary geometry are, and always will be, "in plano." This is partly owing to the greater inherent simplicity of two dimensional figures, partly to the equality of all angles inscribed in the same circular arc, a property which has no analogue in space. Yet I am personally persuaded that there are a lot of simple and elegant theorems in solid geometry, waiting for discovery by skillful and painstaking geometers. Who could ask for a better theorem than that which tells us that if a sphere be inscribed in a tetrahedron, the three angles subtended at each point of contact, by the three edges coplanar with that point, are the same in every case. Yet this theorem, so far as we know, was first found in 1897, and did not come to general notice till mentioned by Franz Meyer at the International Mathematical Congress at Heidelberg in 1904. I wish that I knew the way through the wood

[1] He wrote several works on algebra and geometry.

[2] See this MONTHLY, *1921*, 62.

[3] Read before the Mathematical Association of America at Cambridge, December 29, 1922.

[4] V. Sawayama, "Nouvelles démonstrations d'un théorème relatif au cercle des neuf points," *L'Enseignement Mathématique*, vol. 13, 1911, pp. 31–49.

to the enchanted palace where numbers of such mathematical beauties are still asleep. Alas, I do not. In the present paper I must confine myself to the more prosaic task of pointing out various directions in which I believe interesting mathematical truths may lie, hoping that some geometers whose skill and patience exceed my own will go boldly ahead to discover them.

There are a large number of so-called notable points connected with the triangle. Every high-school pupil learns about the center of the circumscribed circle, where the perpendicular bisectors of the sides meet, the center of the inscribed circle where the bisectors of the angles meet, the center of gravity which lies on all the medians, and the orthocenter on the altitudes. Further study will lead him to Ceva's theorem giving the conditions for concurrence of lines through the vertices of a triangle, and if he take up the modern geometry of the triangle, he will hear about the Brocard points, the symmedian points, Miquel's point, Tarry's point "et hoc genus omne." How many of these points have their counterparts in the geometry of the tetrahedron?

The first steps in the search for such points are encouraging. The perpendicular bisectors of the edges of a tetrahedron meet in the center of the circumscribed sphere, the bisectors of the dihedral angles meet in the center of the inscribed sphere, the lines to the centers of gravity of the faces meet in the center of gravity of the tetrahedron, it all looks very hopeful. But when we examine whether the altitudes be concurrent or not, the trouble begins. There is a very widespread ignorance on this simple matter. I once asked the combined mathematical department of a large New England college, and not one of them had the ghost of an idea. As a matter of fact these altitudes are not, usually, concurrent. If at the orthocenter of each face we erect a perpendicular to that face, it is easy to see that each of these perpendiculars intersects three altitudes and is parallel to the fourth. We have two sets of four lines, all lines of one set intersecting all of the other in finite or infinite points, yet neither set are all parallel to one plane. This shows that they are generators of the same hyperboloid. It is true that the center of this hyperboloid makes a pretty good notable point,[1] but the first natural reaction is to feel discouraged from searching further. Yet such pessimism would, perhaps, be immature. Here is a ground for hope:

Suppose that we connect a point in the plane of a triangle, not on the circumscribed circle, with the three vertices, and reflect the three connecting lines in the bisectors of the angles; the reflected lines will also be concurrent. The two points of concurrence, which are said to be "isogonally conjugate," have a reciprocal relation, they are the two foci of a conic tangent to the lines of the sides of the triangle, and if we take a system of homogeneous trilinear coördinates, where a point is represented by numbers proportional to its distances from three non-concurrent lines, the coördinates of one point are proportional to the reciprocals of the coördinates of the other. Lastly if we drop perpendiculars from the two points on the three lines, the feet of these are six points on one circle, called the "pedal circle" of the two points.

[1] C. Intrigila, "Sul tetraedro," *Rendiconti R. Accademia delle Scienze di Napoli*, vol. 22, 1883, pp. 69–95.

Now these relations, contrary to what one would naturally fear, hold unaltered in three dimensions. Each point, not on a certain cubic surface, has a single isogonal conjugate with regard to a tetrahedron, and the two have the same pedal sphere, the tetrahedral coördinates of the one are the reciprocals of those of the other. These facts suggest the possibility of further progress. In the plane the center of the circumscribed circle and the orthocenter are isogonally conjugate, as are the center of gravity and the symmedian point.

PROBLEM 1. What are the geometrical properties of the points which are isogonally conjugate to such notable points of the tetrahedron as are already known?

In the plane, if a point trace a straight line, its isogonal conjugate will trace a conic through the vertices, and vice versa. When the line goes through the center of the circumscribed circle, the conic is an equilateral hyperbola whose center is on the nine-point circle. These facts, which do not correspond directly to anything in space, lead at once to the properties of the nine-point circle.

PROBLEM 2. What are the properties of the cubic surfaces and cubic space curves which correspond isogonally to planes and lines? When will these figures reduce to combinations of simpler figures?

We mentioned on a previous page that the altitudes of a tetrahedron are usually the generators of a hyperboloid, and so, not concurrent. There are, however, special cases where all go through one point. If two pairs of opposite edges have the property that their lines are mutually perpendicular in direction, the same is true of the third pair. The foot of each altitude is, in this case, the orthocenter of a face and the four altitudes are concurrent. We call such a tetrahedron an "orthogonal" one. If the faces be called 1, 2, 3, 4, and the angle between faces i and j be θ_{ij}, the condition for an orthogonal tetrahedron is immediately found to be

$$\cos\theta_{12}\cos\theta_{34} = \cos\theta_{13}\cos\theta_{24} = \cos\theta_{14}\cos\theta_{23}.$$

In the case of the orthogonal tetrahedron there are two twelve-point spheres, quite analogous to the nine-point circle.[1] We are led to:

PROBLEM 3. How are problems of isogonal conjugacy modified in the case of the orthogonal tetrahedron?

There is a popular dictum, which almost has the force of dogma, that all great inventions are essentially simple. Without going into the question of the limits of validity of this general statement, we may safely say that the most fruitful theorems in geometry are generally the simplest. The following is a case in point. Let us start with a triangle and mark a point on the line of each side. We now pass a circle through each vertex and the two marked points on lines through that vertex. Each circle and the opposite line form a cubic curve, and the three cubics have eight common points. They are, consequently, linearly dependent, and the three circles pass through a common point called the Miquel

[1] A. Transon, "Sur une question relative à la géométrie de l'espace," *Nouvelles Annales de Mathématiques*, series 2, vol. 2, 1863, pp. 138–143.

point.[1] If we invert the figure from a point either in the plane or outside of it we have a curvilinear triangle determined by three concurrent circles of a plane or sphere, with a point marked on each circle. The three circles, each through a vertex and two adjacent marked points, are concurrent. In fact we have two sets of reciprocally related concurrent circles. This relation may be immediately generalized in space if we go about it in the right manner. Let us start with a tetrahedron, and mark a point on the line of each edge, then pass a sphere through each vertex and the adjacent marked points. In each face we have the Miquel configuration, hence the given spheres are concurrent by threes in the planes of the faces. On each sphere we have the Miquel figure generalized by inversion, hence the four spheres go through a point.[2] At the same time the question may fairly be asked whether we have generalized Miquel's theorem in the most natural way. Would it not be more natural to mark a point in each plane? This leads to:

PROBLEM 4. In the plane of each face of a tetrahedron, a point is marked. What is the necessary and sufficient condition that the three spheres, each through a vertex and the marked points in the adjacent planes, should be concurrent?

This may also be restated in another form, more favorable to analytic handling:

PROBLEM 5. Given two sets of points, $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. If the four spheres $A_iB_jB_kB_l$ be concurrent, when will the spheres $B_iA_jA_kA_l$ also be concurrent?

Returning to the plane, a very good way to mark points on three given lines is by their intersections with a fourth line. We are thus led to the theorem that the circles circumscribing the four triangles formed by sets of three out of four given lines are concurrent; the point of concurrence being, in fact, the focus of the parabola which touches the lines. Let us say that this point is "associated" with the four lines. If five lines be given, each set of four will be associated with a point, and the five points lie on a circle, associated with the five lines. If six lines be given, each set of five are associated with a circle, and the six circles are concurrent in a point associated with the six lines. Keeping on in this way we find that each even system of lines will be associated with a point, and each odd set with a circle in such a way that the point associated with $2n$ lines lies on the circle associated with each sub-set of $2n - 1$, and the circle associated with $2n - 1$ contains the point associated with each sub-set of $2n - 2$. This figure is called a "Clifford Chain," after the first of some three or four independent discoverers.[3] Other such chains have been found by Grace,[4] Pesci[5] and the

[1] A. Miquel, "Sur quelques questions relatives à la théorie des courbes," *Journal de Mathématiques pures et appliquées* (Liouville), vol. 3, 1838, pp. 202–208.

[2] S. Roberts, "On certain tetrahedra specially related to four spheres meeting in a point," *Proceedings of the London Mathematical Society*, vol. 12, 1880–1881, pp. 117–120.

[3] W. K. Clifford, "Synthetic proof of Miquel's theorem," *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1870, pp. 124–141.

[4] J. H. Grace, "Circles, spheres and linear complexes," *Transactions of the Cambridge Philosophical Society*, vol. 16, 1897–1898, pp. 153–190.

[5] G. Pesci, "Dei circoli circoscritti ai triangoli etc.," *Periodico di Matematica*, vol. 5, 1890, pp. 120–127.

present author.[1] It seems self evident that there must be analogues in space to many of these chains, although, so far, only the last seems to have been found.[2] This is curious. Roberts' theorem seems to suggest natural extension. If we cut a tetrahedron by a plane, we mark a point very nicely on each edge line, and the spheres circumscribing the four tetrahedra, each having faces in three of the given planes, while its remaining face is in the secant plane, are concurrent. They are the spheres circumscribing four of the five tetrahedra determined by the five planes. But when we take the five tetrahedra and five circumscribing spheres they are not all concurrent in one point, but are concurrent by fours in five different points, one in each plane. It is not certain whether these five points are cospherical or not, I have the impression that they are not, but can not say why. At any rate there is very little known about possible extensions.

PROBLEM 6. Are there any three-dimensional analogues to the chains of Clifford, Grace and Pesci?

That master of geometrical craft, Jakob Steiner, devoted much time and attention to the following problem. Suppose that we have two circles, one surrounding the other. Can we construct a chain of circles, finite in number, each of which is tangent not only to the given circles but its two next neighbors in the chain? The problem is easily handled if we invert our two circles into concentric ones. It then appears that if we can construct one such chain of circles, we may construct an infinite number of them. Returning to the general case, let the line of centers cut the first circle in A_1B_1 and the second in A_2B_2, so that the two A's lie on one side, and the two B's on the other. Construct the two circles whose diameters are A_1B_2 and A_2B_1. If their angle of intersection be expressible in the form $2\pi m/n$ there will be a chain of n circles making m circuits.

If we generalize in space in such a way as to seek a chain of spheres tangent to three given spheres there is nothing essentially new, for it is evident that the problem of finding spheres tangent to two concentric spheres and to some third sphere goes right back to the two-dimensional case.[3] Suppose, however, we have only two given spheres, we then get:

PROBLEM 7. Can there be such a system of spheres tangent to two given spheres that each sphere is tangent to at least 3 others of the system?

Steiner's ring problem is but an example of a common type of problem to which the name of "closure" has been attached. Numbers of geometers, following Poncelet, have looked into the question of what relations must exist between two conics, if a polygon of n sides can be inscribed to the one and circumscribed to the other. This suggests

PROBLEM 8. What relation must exist between two spheres in order that

[1] J. L. Coolidge, "Some circles associated with concyclic points," *Annals of Mathematics*, series 2, vol. 12, 1910–1911, pp. 39–44. See article by F. V. Morley, "Note on the incenters of a quadrilateral," this MONTHLY, *1920*, 252–255.

[2] C. Intrigila, *loc. cit.*, pp. 78–79.

[3] K. Th. Vahlen, "Ueber Steiner'sche Kugelketten," *Zeitschrift für Mathematik und Physik*, vol. 41, 1896, pp. 153–160.

it may be possible to inscribe a tetrahedron in one which is circumscribed to the other?

It is hard to believe that this problem has never been solved, I can only affirm that I have never seen a solution, and there is no reference thereto in Simon's bibliography of elementary geometry in the nineteenth century.[1]

Steiner's contact problem leads naturally to an even more famous one of a slightly different nature. How can we put three circles inside a triangle in such a way that each shall touch two sides and the other two circles? This is known as Malfatti's problem,[2] and the literature bearing on it is depressingly extensive, one reason for its popularity being that Steiner published the following simple construction without proof: [3]

Bisect the angles of the triangle. Inscribe a circle in each triangle formed by one of the given sides and the adjacent bisectors. The bisectors are transverse common tangents to these circles, two by two. Draw the other transverse common tangents. The circles required each touch two of these last tangents and two of the sides.

The first demonstration was given thirty years later by Hart,[4] a priority which the Germans did not particularly relish.

Let us discuss the corresponding problem in three dimensions. It is evident that there are some tetrahedrons in which we can place four Malfatti spheres, for if we start with four spheres, each of which touches the other three externally, we can build a tangent tetrahedron outside. On the other hand it seems unlikely that four Malfatti spheres can be placed in every tetrahedron. There is just one triad of spheres mutually tangent, each of which touches the base of a tetrahedron and two lateral faces, inside. There are an infinite number of spheres tangent to the three lateral faces inside, but they have only one degree of freedom, so that we can hardly require one of them to touch the three spheres already found. It seems likely that two independent conditions must be imposed upon a tetrahedron if we are to find four Malfatti spheres. We find those conditions by using an identity apparently due to Frobenius.[5] Suppose that six spheres or planes are indicated by the numbers 1, 2, 3, 4, 5, 6. Let the angle between spheres i and j be indicated by θ_{ij}. Then

$$\begin{vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} & \cos\theta_{15} & \cos\theta_{16} \\ \cos\theta_{21} & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos\theta_{61} & \cdot & \cdot & \cdot & \cos\theta_{65} & 1 \end{vmatrix} = 0.$$

[1] M. Simon, *Über die Entwicklung der Elementar-Geometrie im XIX Jahrhundert*, Leipzig, 1906.

[2] *Memorie di matematica e di fisica della Societa Italiana delle Scienze Modena*, 1803. I have never been able to see this article.

[3] Steiner, "Einige geometrische Betrachtungen," *Journal für die reine und angewandte Mathematik* (Crelle), vol. 1, 1826, pp. 161–184 (see § IV, p. 178).

[4] A. S. Hart, "Geometrical investigation of Steiner's construction for Malfatti's problem," *Quarterly Journal of Mathematics*, vol. 1, 1857, pp. 219–221.

[5] G. Frobenius, "Anwendungen der Determinantentheorie auf die Geometrie des Maasses," *Journal für die reine und angewandte Mathematik* (Crelle), vol. 79, 1874, 185–247.

In the present case we may take the angle between each two spheres as π while the sphere i makes an angle φ_i with the like numbered plane and 0 with the other 3 planes. Taking the planes i and j and the four spheres, we have

$$\begin{vmatrix} 1 & \cos\theta_{ij} & \cos\varphi_i & 1 & 1 & 1 \\ \cos\theta_{ij} & 1 & 1 & \cos\varphi_j & 1 & 1 \\ \cos\varphi_i & 1 & 1 & -1 & -1 & -1 \\ 1 & \cos\varphi_j & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \end{vmatrix} = 0.$$

The development of this determinant requires care rather than skill. The resulting equation is

$$\cos\varphi_i \cos\varphi_j (\cos\varphi_i + \cos\varphi_j) + A_{ij}\cos\varphi_i\cos\varphi_j + B_{ij}(\cos\varphi_i + \cos\varphi_j) + C_{ij} = 0,$$

where A_{ij}, B_{ij}, C_{ij} are simple polynomials in $\cos\theta_{ij}$. There will be six such equations. Eliminating the four angles φ_i we must have two eliminants in the six angles θ_{ij}, which are probably independent.

PROBLEM 9. Is there any connection between these equations and the conditions for an orthogonal tetrahedron?

PROBLEM 10. When the conditions are fulfilled, how are the Malfatti spheres constructed?

LINEAR OPERATIONS AND GENERALIZED ELEMENTARY SYMMETRIC FUNCTIONS.

By ALBERT A. BENNETT, University of Texas.

Many of the simplest operations of mathematics are linear, and among these the linear homogeneous operations are the most interesting. When attention is confined to rational algebraic steps with a single variable x the most general linear homogeneous operation available is that of multiplying x by a constant. The Cauchy functional relation, $F(x + y) = F(x) + F(y)$, which features in this connection has also celebrated discontinuous solutions.

Without attempting to classify all linear operations, it may be of interest to point out five special illustrations of the general class of homogeneous linear operations. In each of these there are two sets of quantities, $[a]$ and $[F(a)]$, having elements, $a, b, c, \cdots$, and $F(a), F(b), F(c), \cdots$, respectively. In each case, $F(a + b) = F(a) + F(b)$, addition being defined for both sets. The significance of the following discussion lies in the fact that it is applicable equally to all of these distinct cases.

While it is possible to discuss the merely additive properties of linear functions, the existence of a basis, finite or infinite, the nature of the roots of a linear equation, and so forth, we shall here be interested in the application of multiplication to linear homogeneous operations. We shall assume indeed that both for the

set [a] and for the set [$F(a)$] commutative associative multiplication is possible, this multiplication being distributive with respect to addition. The five illustrations or examples of the general theory are as follows:

1. [a] denotes the set of expressions, $a = (x_1, x_2, \cdots, x_n)$, $b = (y_1, y_2, \cdots, y_n)$, $\cdots$, for a fixed n. $F(a)$ denotes $\sum_{i=1}^{n} x_i$. By the product ab will be meant the element of [a] that may be written, $(x_1y_1, x_2y_2, \cdots, x_ny_n)$. The other sums and products are defined as is usual in algebra.

2. [a] denotes the set of algebraic numbers of a given algebraic field of order, n. By $F(a)$ is meant the trace (*Spur*) of a, so that $F(a)$ is always a rational number. The sums and products are as usual in algebra.

3. [a] denotes the set of matrices which are polynomials (with scalar coefficients) in a given non-singular square matrix of order n, with distinct invariant factors. $F(a)$ is the sum of the elements in the main diagonal of a. The product ab is the usual matric product, which in this case is commutative. The sum $a + b$ is the matric sum. The sum and product for elements of [$F(a)$] is as in algebra.

4. [a] denotes the set of finite subsets, repeated elements being permitted, of a given set of objects. $F(a)$ denotes the number of elements in the subset a, any element occurring exactly n times in a, yielding the number n in the count $F(a)$. In defining the sum $a + b$ each element that appears in either term appears in the sum, and the number of times that it occurs in the sum is the sum of the number of times that it appears in the given terms. By the product ab is meant the subset, possibly the null-set, containing only elements common to both factor sets, the number of times that any given element appears being the product of the number of times that it appears in the given factors. The sum and product for [$F(a)$] are defined as in arithmetic.

5. [a] denotes the set of subsets, finite or infinite, without repeated elements, of a given set of elements. $F(a)$ is the same set as a, but with different rules of combination. The sum $a + b$ is the arithmetic sum defined in 4, and $F(a) + F(b)$ coincides with it. The product ab is as in 4, but a new definition is given to the product $F(a)F(b)$. This is in fact defined as the set of all unordered pairs of elements, one each from $F(a)$ and from $F(b)$, respectively, any pair consisting of the same element twice, once each from $F(a)$ and from $F(b)$, counting merely as this element itself.

The sets [a] and [$F(a)$] and their rules of combination having been once settled upon, it is possible to introduce a sequence of functions of more than one argument, which functions play an important part in the theory of each of the illustrations and constitute the basis of the general theory. These will be denoted by $F_1, F_2, F_3, \cdots, F_n, \cdots$ where the subscript in each case identifies the function and also denotes the number of independent arguments which enter into its definition. Thus these are respectively of the form $F_1(a)$, $F_2(a, b)$, $F_3(a, b, c)$, $\cdots$, $F_n(a_1, a_2, \cdots, a_n)$. Each $F_n(a_1, a_2, \cdots, a_n)$ will be seen to be linear and homogeneous in each of the arguments, as a consequence of the defining relations that will be given presently. Furthermore it will be shown that each integral

rational function of the F's with arguments which are themselves integral rational functions of the independent arguments a, and which is such as to be linear and homogeneous in each of these independent arguments, is uniquely expressible as a linear combination of the F's.

The functions in the sequence $F_1, F_2, \cdots, F_n, \cdots$ are themselves linearly independent. The simplest method of defining the relations among their products is perhaps to give a sequence of equations, each introducing in turn a single one of these functions. On the basis of these, other relations can be derived which must be valid in all cases and not dependent upon the special interpretation of $[a]$ and $[F(a)]$. Incidentally the theory of the elementary symmetric functions of algebra follows by a specialization in illustration 1. The definitions adopted are as follows:

$$
\begin{aligned}
&F_1(a) = F(a),\\
&F_1(a)F_1(b) = F_1(ab) + F_2(a, b),\\
&F_1(a)F_1(b)F_1(c) = F_1(abc) + F_2(ab, c) + F_2(ac, b) + F_2(bc, a) + F_3(a, b, c), \qquad (1)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&F_1(a_1)F_1(a_2)\cdots F_1(a_n) = F_1(a_1a_2\cdots a_n) + \sum F_2 + \sum F_3 + \cdots\\
&\qquad\qquad\qquad\qquad\qquad + F_n(a_1, a_2, \cdots, a_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Here the sign $\sum$ in each case covers all distinct integral partitions of the n arguments, $a_1\, a_2, \cdots, a_n$, into products to the number required by the subscript indicated. It is now seen that each F, in turn, is symmetrical in its arguments. The summation sign covers a determinate but, except in the simplest cases, large numbers of terms. The expansion of the product $F_1(a)F_1(b)F_1(c)F_1(d)F_1(e)$ will amply illustrate the formation of these sums. For this case, $\sum F_2$ contains the following fifteen terms,

$$
\begin{aligned}
&F_2(abcd, e) + F_2(bcde, a) + F_2(acde, b) + F_2(abde, c) + F_2(abce, d)\\
&\quad + F_2(abc, de) + F_2(abd, ce) + F_2(acd, be) + F_2(bcd, ae) + F_2(abe, cd)\\
&\quad + F_2(ace, bd) + F_2(bce, ad) + F_2(ade, bc) + F_2(bde, ac) + F_2(cde, ab).
\end{aligned}
$$

On the other hand $\sum F_3$ contains twenty-five terms,

$$
\begin{aligned}
&F_3(abc, d, e) + F_3(abd, c, e) + F_3(acd, b, e) + F_3(bcd, a, e) + F_3(abe, c, d)\\
&\quad + F_3(ace, b, d) + F_3(bce, a, d) + F_3(ade, b, c) + F_3(bde, a, c) + F_3(cde, a, b)\\
&\quad + F_3(ab, cd, e) + F_3(ac, bd, e) + F_3(ad, bc, e) + F_3(ab, ce, d) + F_3(ac, be, d)\\
&\quad + F_3(ae, bc, d) + F_3(ab, de, c) + F_3(ad, be, c) + F_3(ae, bd, c) + F_3(ac, de, b)\\
&\quad + F_3(ad, ce, b) + F_3(ae, cd, b) + F_3(bc, de, a) + F_3(bd, ce, a) + F_3(be, cd, a).
\end{aligned}
$$

There are but ten terms in $\sum F_4$, namely,

$$
\begin{aligned}
&F_4(ab, c, d, e) + F_4(ac, b, d, e) + F_4(ad, b, c, e) + F_4(ae, b, c, d) + F_4(bc, a, d, e)\\
&\quad + F_4(bd, a, c, e) + F_4(be, a, c, d) + F_4(cd, a, b, e) + F_4(ce, a, b, d) + F_4(de, a, b, c).
\end{aligned}
$$

The defining equations (1) exhibit the products of F's when these are of the simplest sort, as linear expressions in F's. It remains to show that under proper conditions of homogeneity alone, as already mentioned, any product of F's is itself expressible as a linear combination of F's. Owing to the linearity of the

F's in the several arguments, no further discussion is needed to extend the results from mere products to general rational functions suitably homogeneous. It suffices to consider only products of two F's, since if the theorem is true for all such products, it will follow for products of more than two factors by mere repetition and summation. Consider therefore the product $F_m(A_1, A_2, \cdots, A_m)F_n(A_1, A_2, \cdots, A_n)$, where each A is either an a, or a product of a's. We shall first cast the equations (1) into another form,

$$F_n(a_1, a_2, \cdots, a_n) = F_1(a_1)F_1(a_2) \cdots F_1(a_n) - F_1 - \sum F_2 - \sum F_3 - \cdots - \sum F_{n-1}. \tag{2}$$

Making use of the form (2) for each of the two factors F_m and F_n, we obtain a sum of terms in which the subscript m is replaced by smaller integers, and likewise n is replaced with smaller integers, together with terms involving F_1 as a repeated factor. By repeated applications, the original product is eventually replaced by a sum of terms each containing exclusively repeated factors F_1. Finally, each of these products is evaluated as a linear expression by use of formula, (1). An indication of the result to be expected in simple cases may be inferred by reference to the product, $F_3(a, b, c)F_1(d)$ which reduces after the above mentioned operations have been performed to $F_3(ad, b, c) + F_3(bd, a, c) + F_3(cd, a, b) + F_4(a, b, c, d)$.

We are now prepared to interpret $F_2(a, b)$ in each of the five illustrative cases mentioned above. In 1, $F_2(a, b)$ denotes the sum of all products x_iy_j, $i \neq j$. In 2, $F_2(a, b)$ is always a rational number. It is a symmetric function of the pair of algebraic numbers a, b, but has no commonly current name, other than $S(a)S(b) - S(ab)$, where $S(x)$ is the trace of x. In 3, $F_2(a, b)$ is a symmetric scalar function of the pair of matrices, a, b, but enjoys no special name. In 4, $F_2(a, b)$ is a number dependent upon the number of elements which are common to the subsets a and b. If a contains n elements not necessarily distinct, $F(a)$ is n. If b contains m elements, $F(b)$ is m. If no element of a is also in b, $F(ab) = 0$. If however a and b should coincide, but the elements of each should be distinct, then $m = n$, and $ab = a = b$, so that $F(ab) = n$, and $F_2(a, b) = n(n - 1)$. If a consists of a single element counted n times and b consists of the same element counted m times, $F(a)F(b) = F(ab)$, so that $F_2(a, b) = 0$. In 5, the expression, $F_2(a, b)$ consists of the set of unordered pairs of distinct elements one each from a and from b.

The theory of the functions F_m with m distinct arguments is fundamental, but in most applications functions are used in which the arguments are not independent but are all powers of a single argument. In the most familiar cases, indeed, the arguments are all equal, so that F_1, F_2, F_3, etc., are determined by a single quantity a. Equations (1) become in this important case,

$$\begin{aligned} F_1(a) &= F(a), \\ F_1^2(a) &= F_1(a^2) + F_2(a, a), \\ F_1^3(a) &= F_1(a^3) + 3F_2(a^2, a) + F_3(a, a, a), \\ F_1^4(a) &= F_1(a^4) + 4F_2(a^3, a) + 6F_2(a^2, a^2) + 4F_3(a^2, a, a) + F_4(a, a, a, a). \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

It may be noted that from the same function $F_2(a, b)$ are obtained in the fourth of the above relations two distinct functions of the single argument a. To determine the number of times that a given function of several distinct arguments will appear in the equation for a given power of $F(a)$, say the mth, we must find the number of distinct partitions of which m is capable, where the number of parts for each partition is equal to the subscript of the function in question.

Making use of the specialization explained above, we are in the position to give further interpretations in the five illustrative cases. It will be convenient to refer occasionally to the expression,

$$x_n - F_1(a)x^{n-1} + (F_2(a, a)/2)x^{n-2} - \cdots + (-1)^n(F_n(a, a, a, \cdots, a)/n!). \quad (4)$$

In 1, where $a = (x_1, x_2, x_3, \cdots, x_n)$, $F_2(a, a)$ consists of the sum of products x_ix_j, $i \neq j$, and contains each of these products twice, namely once in the form x_ix_j and once in the form x_jx_i. Thus $F_2(a, a)/2$ is the elementary symmetric function of the second order. More generally, $F_m(a, a, \cdots, a)/m!$ is the mth elementary symmetric function of x's in a. In this illustration, the expression (4) is a polynomial in x whose roots are the elements of a. For this reason the expression $F_m(a, a, \cdots, a)/m!$ may be regarded in general as a generalized elementary symmetric function.

In 2, a is an algebraic number in a field of order n. This does not require that a is itself of the nth order, but in any case there is a unique polynomial of the nth order defined having a as a root, and which is either irreducible in the domain of rational numbers or else is merely a power of an irreducible polynomial. This polynomial will be expressible in the form (4). Here $F_1(a)$ is the trace and $F_n(a, a, \cdots, a)/n!$ is the norm of a. The intermediate coefficients are likewise rational numbers featuring in the theory.

In 3, the well-known "characteristic equation" furnishes the first elements of the theory. The characteristic equation does not completely define the matrix a, if not all the roots of this equation are distinct, there being additional arithmetic invariants in this case. If the roots be distinct, however, no further invariants exist for the transformations usually considered in the algebraic or geometric applications. The characteristic equation is obtained by putting (4) equal to zero. Thus the coefficient $F(a, a, \cdots, a)/n!$ is merely the determinant of a.

In 4, $F_m(a, a, \cdots, a)/m!$ is a non-negative integer for each choice of m. The term contributed to F_m by any repeated element of a is $n!/(n - m)!$, where n is the number of times that the given element appears in a. This functional symbol is of course to be understood to denote 0, whenever m exceeds n, and to denote $n!$ whenever $m = n$. Since the binomial coefficient $n!/[m!(n - m)!]$ is an integer it follows that in each case, F_n is divisible integrally by $m!$ and vanishes if no element of a is counted as many as m times.

If in 5, a be thought of as in some sense a "linear" set, $F_2(a, a)$ will be twice a triangular set, $F_3(a, a, a)$ will be six times a tetrahedral set, and so forth, $F_n(a, a, \cdots, a)$ being $n!$ times a set which is a simplex of n dimensions. These simplexes constitute a generalization of the binomial coefficients.

Given the problem of expressing the symmetric function of more than four variables represented by $\sum x_1^2 x_2^2 x_3$, in terms of the elementary symmetric functions of these variables, a possible method of solution, in case that the number of variables is not excessive, is actually to expand the indicated summation as the first step. Since the form of the result is the same no matter what the number of variables so long merely as there are enough to give all of the indicated sums a meaning, it is clear that there is no necessity of making the expansion. The solution can be effected by use of the relations (1) and relations derived therefrom alone. The problem can therefore be put into a more general form. We can inquire into the representation of the expression $F_3(a^2, a^2, a)/2$ [1] as a sum of products of the elementary functions $F_1(a)$, $F_2(a, a)$, $F_3(a, a, a)$, etc., alone. The proof that a solution is always possible may be made in the usual manner provided that care be taken in the wording of the proof. To simplify the notation, it is usually convenient to adopt the nomenclature current in the theory of the elementary symmetric functions of algebra, and write $F_n(a, a, \cdots, a)/n!$ in the condensed form $E_n(a)$. Using this notation, we have as the result of a few steps, $F_3(a^2, a^2, a)/2 = E_2E_3 - 3E_1E_4 + 5E_5$. This result may be interpreted in connection with each of the illustrations already considered. It is of particular interest to consider 5. In this case since $a^2 = a$, it follows that $F_3(a^2, a^2, a)$ is the same as $F_3(a, a, a)$ which is itself equal to $6E_3(a)$. Thus for this particular illustration, $3E_1E_4 + 3E_3 = E_2E_3 + 5E_5$. When, in particular, the set a is finite and consists of n elements, the count of the number of elements gives a relation among the binomial coefficients, but the theorem is one applicable to any sets. To take a simpler example, from the general relation $F_2(a^2, a) = E_1E_2 - 3E_3$, we have for 5, $2E_2 = E_1E_2 - 3E_3$, or in other order, $E_1E_2 = 3E_3 + 2E_2$. Geometrically interpreted, this means that the triangular prism obtained from a given linear set, consists of two triangular interfaces and three tetrahedra, a relation familiar in solid geometry, but here valid for a general "linear" set.

THE AREA OF RULED SURFACES BY VECTORS.

By J. B. REYNOLDS, Lehigh University.

When a surface may be considered as generated by the movement of a straight line, we may think of the line as a vector and thus easily lead up to an expression for the area generated by the moving line.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be vectors from the origin of vectors to the ends of the generating vector $\mathbf{l}$, as shown in figure 1. If then $\mathbf{r}$ is the vector to any point on the surface, we have $\mathbf{r} = \mathbf{r}_1 + s\mathbf{l}$, in which s is a scalar quantity. From this

$$d\mathbf{r} = d\mathbf{r}_1 + sd\mathbf{l} + \mathbf{l}ds,$$

[1] The occasion for the factor 2 is that in $F_3(a^2, a^2, a)$, itself, the following terms for example would all appear as though distinct, $x_1^2x_2^2x_3$, $x_2^2x_1^2x_3$, $x_2^2x_3^2x_1$, $x_3^2x_2^2x_1$, $x_3^2x_1^2x_2$, $x_1^2x_3^2x_2$, whereas these are actually equal by pairs, and the notation $\Sigma x_1^2x_2^2x_3$ would be interpreted as including but one of each pair, that is, as being equal to exactly one half of $F_3(a_1^2, a^2, a)$.

the hyperbola $x^2 - y^2 = a^2 \cos^2 \frac{1}{2}\alpha$ about the y-axis from $y = -a \sin \alpha/2$ to $y = +a \sin \alpha/2$.

This method may be followed in finding a definite portion of a tangent surface of a space curve or in finding the surface generated by a given part of the normal or binormal to a curve.

For a tangent surface, since $\boldsymbol{l}$ is a portion of the line tangent to the curve defining $\boldsymbol{r}_1$, $\boldsymbol{l}$ is parallel to $d\boldsymbol{r}_1/dt$ so that $\boldsymbol{l} \times d\boldsymbol{r}_1/dt = 0$, simplifying the formula to

$$S = \int_{t_1}^{t_2}\int_0^1 \left(\boldsymbol{l} \times \frac{d\boldsymbol{l}}{dt}\right)_0 s\,dt\,ds = \tfrac{1}{2}\int_{t_1}^{t_2}\left(\boldsymbol{l} \times \frac{d\boldsymbol{l}}{dt}\right)_0 dt.$$

For example, to find the area of the surface generated by the length a of the tangent to the helix $\boldsymbol{r} = \boldsymbol{i}\cdot a \cos t + \boldsymbol{j}\cdot a \sin t + \boldsymbol{k}\cdot ct$ in one revolution, we have

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{i}a \sin t + \boldsymbol{j}a \cos t + \boldsymbol{k}c;$$

whence

$$\boldsymbol{l} = \frac{a}{\sqrt{(a^2 + c^2)}}(-\boldsymbol{i}a \sin t + \boldsymbol{j}a \cos t + \boldsymbol{k}c),$$

the absolute length being a, and, therefore,

$$\frac{d\boldsymbol{l}}{dt} = \frac{a}{\sqrt{(a^2 + c^2)}}(-\boldsymbol{i}a \cos t - \boldsymbol{j}a \sin t)$$

and

$$S = \frac{1}{2}\frac{a^3}{a^2 + c^2}\int_0^{2\pi}[-\boldsymbol{i}c \sin t - \boldsymbol{j}c \cos t + \boldsymbol{k}a]_0 dt$$

$$= \frac{1}{2}\frac{a^3}{a^2 + c^2}\int_0^{2\pi}\sqrt{(a^2 + c^2)}dt = \frac{\pi a^3}{\sqrt{(a^2 + c^2)}}.$$

In case $c = 0$ we get $S = \pi a^2$, the area of the annulus of outer radius $\sqrt{2}a$ and inner radius a.

AN ENGLISH TEXT ON MATHEMATICS WRITTEN ABOUT 1810.[1]

By ELIZABETH B. COWLEY, Vassar College.

This three-volume treatise is owned by Dr. Charles C. Godfrey of Bridgeport, Conn., who purchased the books in a second-hand store in New Haven. He says that they were part of a large library of works on mathematics collected by an Englishman and sold as part of his estate and later brought to this country. He has recently written that he has learned that these three volumes once belonged to Mr. Thomas P. Stowell of Rochester, N. Y., who was probably connected with the Alexandria Boarding School.

These three bound volumes are written by hand in ink. Many of the problems

[1] Read at the Summer meeting of the Association, Rochester, N. Y., Sept. 6, 1922.

have elaborately drawn illustrations colored in water colors; and each volume has a full page fanciful frontispiece in colors. The pages are not numbered consecutively in ink, but on the first page "No. 1" is written in pencil in a large hand; on the seventeenth page "No. 2"; on the thirty-third page "No. 3," etc. This pencil numbering is found in each volume, each number usually indicating 16 pages. The last number in vol. 1 is 38; in vol. 2 is 29; and in vol. 3 is 30.

There is no statement of date, place, or author. I have not found any printed text of which this is a copy, although some parts of it bear a resemblance to portions of the *Mathematical Institutions* of W. Leybourn, which was published in London in 1704. In volume 3 there is written in a small hand in ink in the upper right-hand corner of the first fly-leaf

W. B. Falcon
May 20th, 1811
Whitehaven.

This date and place fit in well with some stray bits of internal evidence. For example, one problem tells of Allonby-bay on the coast of Cumberland and of houses across on the Scotch shore. A rebus refers to Carlisle. The description of the "Carpenter's Rule, commonly called Coggeshall's sliding rule" agrees with the description found in Barlow's *Dictionary* (1814). The illustrated problem on the coal wagon would not be out of place in that mining district. Volume 3, which starts out with "Book keeping Moderniz'd," begins the "Waste Book" the first of January 1807, London. Volume 2 gives the "Journal of a Voyage from London to Madeira and Teneriffe in the Heroine of Workington, Thomas Falcon Commander," which begins April 12, 1807. There may or may not be any significance in these two references to London.

Volume 1 begins with "Practical Geometry." From the 42 definitions I quote two: "A right line is that which lies evenly between its extreme points." "The measure of an angle is the arc of a circle contained between the two lines which form that angle, the angular point being the centre." There are 51 problems. The first 40 of these are ordinary constructions of plane geometry. The next 10 are problems in which, when the measurements of certain sides or angles of right or oblique triangles are given, it is required to construct the triangle and measure the other parts. The 51st problem is "To make a plane scale or to construct a scale of Sines, Tangents, Chords, Semitangents, Longitudes, and Rhumbs to any radius."

This "Practical Geometry" serves as an introduction to "Plain Trigonometry," which starts in with definitions of trigonometry and of "the lines applied to a circle." For example, "the right sine of an arc is a right line drawn from one end or termination of an arc to the radius and perpendicular thereto." There are 6 cases of right-angled triangles, each of which is solved by 3 methods:—by geometrical construction, by Gunther's scale and compasses, and by logarithmic sines, tangents, etc. There are 4 precepts which the writer says "must be well impressed upon the memory of all who wish to be ready and dexterous in this most excellent art": 1. "When a side is required, begin the operation with an

angle and when an angle is sought begin with a side." 2. "As the sine of an angle : its opposite side :: the sine of any other angle : its opposite side." 3. "As either leg : tangent of the radius 45° :: the other leg : tangent of the other opposite angle." 4. "Extend the first term to the third and that extent will reach from the second to the fourth." The four cases of the oblique triangle may be solved by these and two new precepts. The first is our ordinary law of tangents, stated as a proportion. The second precept, which may be reduced to our law of cosines, is stated thus : "As the longest side : sum of the other two sides :: the difference of the said two sides : the difference of the segments of the base made by a perpendicular let fall from the vertical angle."

"Altimetry and Longimetry" deal with the applications of trigonometry. There are 18 problems, each illustrated by an elaborate and carefully drawn ink picture, colored with water colors. Half of the problems deal with the ever-popular theme of towers. Their pictures are varied—including round and square, ruined and well-preserved, some set on hills and others beside the sea or a stream. From one tower the British flag is flying. We find also a May pole, a church steeple, a number of fully rigged sailing vessels, and houses. In one case the houses on both sides of a street are drawn. The problem is the familiar one dealing with the ladder that leans first against a house on one side of the street and then is turned over to lean against one on the other side. In addition to these 18 problems collected here, there are others amongst the miscellaneous problems at the end of volume 2. These involve castles, ships, and farm houses. All these problems would repay study, but we shall mention here only the twelfth in the first collection. It is the most elaborate and it has so much local color. On the seacoast of Cumberland stand Dubmill and Bankend on opposite sides of Allonby-bay from each of which appear two houses, which for distinction we call A and B, on the Scotch shore, the distances of which from Dubmill and Bankend and from one another are required from the data subjoined. Suppose at Dubmill the line to house B bisects the angle there between Bankend and house A, then admit 80 chains measured from Dubmill on the line from house A so that a line to Bankend was perpendicular thereto, and there a signal put up or a fire made. Next at Bankend suppose the following angles observed:

1. Between the signal or fire and Dubmill 12° 30′.
2. Between Dubmill and house A 54° 30′.
3. Between houses A and B 61° 30′.

The water color illustration is elaborate, showing Dubmill with its wheel, and the other places mentioned.

The next subject taken up is the "Mensuration of Superficies." This involves areas of parallelograms, triangles, and other polygons, of circles, sectors and segments of circles, and the Pythagorean theorem. No attempt is made to prove a theorem, instead there are "rules."

"Conic sections" is devoted to definitions and the problems of constructing and finding the areas of segments of the ellipse, parabola, and hyperbola. The mensuration of solids deals with the convex surface and solidity of the parallel-

opipedon, cone, sphere, spheroid, parabola and hyperbola of revolution, parabolic spindle and the "cylindrical ring." In each case there is a rule followed by examples solved by the rule.

Artificers' work includes that of bricklayers, masons, carpenters and joiners, slaters and tilers, plasterers, painters and glaziers, etc. There is a description of the carpenters' rule with examples on timber measure, etc. The last half of the first volume is devoted to spherics, including stereographic projection, spherical trigonometry, and astronomical problems solved by the globe construction and by calculations.

Volume 2 treats of navigation, starting in with sailing:—plane, traverse, parallel, middle latitude, etc. There are other problems such as "Construction and use of maps." It is in this volume that we find the journal of the voyage of the "Heroine of Workington," which I have previously mentioned. At the end of the volume there is an odd assortment of questions which are solved by algebra and there are several rebuses. One problem and one rebus refer to "Green Row." I shall be glad to have any suggestion as to what "Green Row" is.

Volume 3 opens with "Bookkeeping Moderniz'd." There are a "Waste Book," Journal, Index, and Ledger. This volume also contains algebra, with questions involving "simple" and quadratic equations. It is concluded by a bit on converging series and some "promiscuous questions" and applications of algebra to geometry. The first part of the book is the most interesting. The "Waste Book" is dated London, the first of January 1807. It starts out "Inventory of money, goods and debts belonging to me AB, as also of the debts due by me to others." As we read we learn that "AB" is a merchant doing business with Liverpool, Bristol, Glasgow, Edinburgh, Dublin, Cork, etc., as well as with foreign ports such as New York, Boston, Philadelphia, Amsterdam, Funchall, Cadiz, Hamburg, Leghorn, Smyrna, etc. He deals in many different kinds of merchandise extending from "blue Hungary pearl ashes," broadcloth, fustians, kerseys, linen, serges, figs, flour, flax, hops, Canary wine, rum, tobacco, sugar, raisins, herring, beef, pork, and shoes. He had a house in Fleet St. and another in Charing Cross and he paid house rent and window tax. He purchased two lottery tickets and he gained a wager on the safe arrival in Smyrna of the ship "Happy Janet." He was interested in Drummond's shoe factory and in the Royal Bank of Edinburgh: he opened a store in Boston and he had part interest in several ships. On May 9 he enters the statement "George Barclay is broke." Even the list of family names of those with whom he transacted business is interesting. The last entry for Dec. 31 is "Laid out for the use of my family since the first of January last £300."

In conclusion may I suggest that the usefulness of old books like these is not confined to the person who has a special interest in the history of mathematics. Last year I showed these books to my freshmen when they were studying trigonometry. They were especially interested in the problems on heights and distances and were eager to solve them and compare their solutions with those given. Later I showed them the photostats of a fourteenth-century manuscript

from Italy upon which I was working. From glimpses of old books and manuscripts they begin to realize that mathematics is not a cold, dead subject the same to-day as it was 1000 years ago and as it will be centuries hence; but rather that it is closely interwoven with the life of any period and that there is a mutual dependence between the progress of mathematics and that of civilization.

A GEOMETRIC PARADOX.

By B. H. BROWN, Dartmouth College.

We are indebted to Professor Coolidge for the remark that "there is nothing more instructive than an apparent paradox." The following situation involves interesting theorems, the answer is not too obvious, and the student may find in the simple explanation a concept which is new to him.

One of the curious facts in elementary geometry is that a plane bitangent to a torus cuts it in two circles, a theorem due to Villarceau.[1] Hence, in addition to the meridian and parallel circles there are two other systems of circles such that each circle of one system cuts every circle of the other system, but no circle of one system cuts any other circle of the same system. Furthermore Schoelcher[2] proved the "joli théorème" that these Villarceau circles are loxodromes of the meridian and of the parallel circles. Now consider the following chain of theorems:

1. *A cone of revolution can be inverted into a torus;*
2. *The lines of curvature on a cone of revolution are the rulings and parallel circles;*
3. *The lines of curvature on a torus are the meridian and parallel circles;*
4. *Inversion carries lines of curvature into lines of curvature;*
5. *Inversion carries circles into circles (straight lines being considered circles);*
6. *Inversion preserves angles (except for sign).*

If then we invert a cone into a torus, where can we find any circles on the cone to invert into the Villarceau circles? And these circles on the cone must be loxodromes of the rulings, an exceedingly difficult thing to visualize.

Let us carry through the details of a particular transformation. The inversion with center at $(0, 0, i)$ and power -2, whose equations are

$$\begin{aligned} \bar{x} &= \frac{-2x}{x^2+y^2+(z-i)^2}, \qquad \bar{y} = \frac{-2y}{x^2+y^2+(z-i)^2}, \\ \bar{z} &= i + \frac{-2(z-i)}{x^2+y^2+(z-i)^2} = \frac{i(x^2+y^2+z^2+1)}{x^2+y^2+(z-i)^2}, \end{aligned} \tag{1}$$

will carry the (imaginary) cone

$$\bar{x}^2+\bar{y}^2+\tfrac{1}{2}\bar{z}^2 = 0 \tag{2}$$

into the real torus

$$(x^2+y^2+z^2+1)^2 = 8(x^2+y^2). \tag{3}$$

[1] Cf. *Nouvelles Annales de Mathématiques*, series 1, vol. 7, 1848, pp. 345–347. An elementary proof for a special case is given later in this paper.

[2] Cf. *Nouvelles Annales de Mathématiques*, 4th series, vol. 3, 1903, pp. 105–107.

The torus (3) is formed by the revolution about the z-axis of a circle of radius unity, the plane of the circle passing through this axis, and the center of the circle lying in the xy plane at a distance $\sqrt{2}$ from the origin. The plane $x = z$ is bitangent to the torus. If we rotate the torus about the y-axis until this plane becomes the xy plane, the curve of intersection of plane and torus may be found by transforming (3) by

$$x = \bar{x}\tfrac{1}{2}\sqrt{2} - \bar{z}\tfrac{1}{2}\sqrt{2}, \qquad y = \bar{y}, \qquad z = \bar{x}\tfrac{1}{2}\sqrt{2} + \bar{z}\tfrac{1}{2}\sqrt{2},$$

and setting $\bar{z} = 0$. This gives a factorable quartic

$$(\bar{x}^2 + \bar{y}^2 - 1 + 2\bar{y})(\bar{x}^2 + \bar{y}^2 - 1 - 2\bar{y}) = 0;$$

that is, two Villarceau circles of different systems with centers on the y-axis at $(0, 1, 0)$ and $(0, -1, 0)$ and with radii $\sqrt{2}$. These two circles cut orthogonally, and by Schoelcher's theorem and a corollary thereto, the Villarceau circles of this torus cut every line of curvature under the angle 45°, and each Villarceau circle of one system cuts every circle of the other system orthogonally.

Under the inversion (1) it is clear that a plane through the z-axis is invariant and consequently the two rulings in which it cuts the cone invert into the two meridian circles in which the plane cuts the torus. Again, spheres orthogonal to the z-axis invert, in general, into spheres orthogonal to the z-axis. Such spheres cut the cone in two parallel circles, and their transforms cut the torus in two parallel circles. We leave to the reader the explanation of the following situation. A plane $z = c$, cutting the cone in one circle, inverts into a sphere normal to the z-axis cutting the torus in two circles!

But where do the Villarceau circles come from? Let us invert the torus back into the cone and see where they go. Two co-planar Villarceau circles of different systems are given by the intersection of a plane

$$x \cos\theta + y \sin\theta - z = 0, \tag{4}$$

and the torus (3). The inverse of these two circles is given by the intersection of

$$x^2 + y^2 + \tfrac{1}{2}z^2 = 0, \qquad \text{and} \qquad x^2 + y^2 + z^2 + 1 - 2ix\cos\theta - 2iy\sin\theta = 0,$$

or, more simply, by the intersection of

$$x^2 + y^2 + \tfrac{1}{2}z^2 = 0, \qquad \text{and} \qquad x^2 + y^2 + 2ix\cos\theta + 2iy\sin\theta - 1 = 0. \tag{5}$$

Now (5) is factorable into

$$(x + iy + ie^{i\theta})(x - iy + ie^{-i\theta}) = 0; \tag{6}$$

so that the inverses of one system of Villarceau circles are given as intersections of the cone and the planes

$$x + iy = c; \qquad c \neq 0, \tag{7}$$

and of the other system by the cone and the planes

$$x - iy = c; \qquad c \neq 0, \tag{8}$$

the relation between c and θ being evident from (6).

These are the transforms of the Villarceau circles, and the following reasoning

will make clear that these are circles. The intersection of the plane at infinity and the cone (2) is a conic bitangent to the Absolute at $(1, i, 0, 0)$ and $(i, 1, 0, 0)$ and the planes (7) and (8) are tangent to the Absolute at these two points respectively. The intersection of a plane (7) or (8) with the cone (2) is a most curious conic: a parabola because it is tangent to the line at infinity in its plane, but a circle because it goes through the two (coincident) circular points in its plane.

We may now verify analytically that these parabolic circles cut each other orthogonally, and that they cut the rulings and parallel circles under an angle of 45°. In fact it is easy to show that the parabolic circles on any cone of revolution

$$x^2 + y^2 + kz^2 = 0$$

cut the parallel circles under the angle $\cos^{-1} \sqrt{k}$, and that one system of parabolic circles cuts the other system under the angle

$$\cos^{-1} (1 - 2k).$$

Recalling the theorem [1] that every Dupin cyclide can be inverted into a cone of revolution we see that from the standpoint of inversion there is a single infinity of distinct Dupin cyclides characterized by the angle between their Villarceau circles.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. AN APPROXIMATE CONSTRUCTION OF THE SIDE OF A REGULAR INSCRIBED HEPTAGON.[2]

By T. R. RUNNING, University of Michigan.

The construction of Figure 1 gives a remarkably close approximation to the side of a regular inscribed heptagon. If the segment AP be stepped around the circumference of the circle from A seven times, the terminal point will fall short of A by an arc approximately equal to 0".000432.

If the construction be carried out for a circle the size of the earth's equator and AP stepped around the circumference seven times, the terminal point will fall short of the initial point by about 0.53 inch.

[1] Mannheim gave a very neat proof of an equivalent theorem: every Dupin cyclide can be inverted into a torus. Cf. GOURSAT-HEDRICK, *A Course in Mathematical Analysis*, vol. 1, p. 525.

[2] A number of historical references for this problem were given by R. C. Archibald in this MONTHLY (*1921*, 473–9). In particular, allusion is there made to Röber's construction, the accuracy of which (the total error being half a second) astonished Hamilton and De Morgan. Professor Running's very simple construction involves an error less than one thousandth of this.

A word may be said with regard to the idea that the simplicity of such a construction depends on the number of square roots involved (see *loc. cit.*, p. 479). This, of course, is not the case. If $\cos (2\pi/7) = m/n$ to 20 decimal places, the first degree equation $n \cos (2\pi/7) = m$ suffices within these limits without the taking of any square roots. In the present construction, the steps are both few and simple, the principal labor being the division of two segments each into seven equal parts. EDITOR.

Figure 1 shows the construction. The point F, determined as indicated is joined to E by a straight line. The diameters, AE and LM, are perpendicular to each other. The point S is the intersection of FE with the line HS parallel to LM. The line SC continued intersects AK in D. CD is bisected in Q, and the line AQ drawn, intersecting OB in P. AP is the approximate side of a regular heptagon inscribed in the circle whose radius is AO.

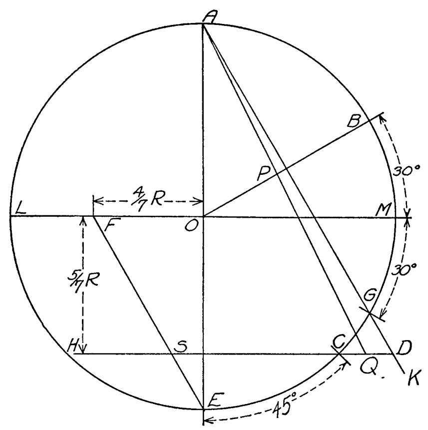

FIG. 1.

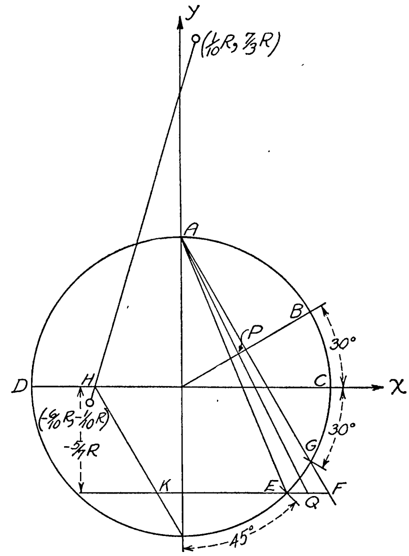

FIG. 2.

It is to be noted that HS and SC do not form the same straight line.

A modified construction which differs from this only in the addition of one more line is indicated in Figure 2.

The point H is determined as the intersection of the line joining $(\frac{1}{10}R, \frac{7}{3}R)$ and $(-\frac{6}{10}R, -\frac{1}{10}R)$ with DC. The rest of the construction is the same.

In this case the defect is only $0''.000185$ or 0.23 inch.

II. Evaluation of the Determinant $|1/(r+s-1)!|$.

By L. L. Dines, University of Saskatchewan.

I have recently had occasion to evaluate the determinant

$$D_n = \begin{vmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(n+1)!} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} \end{vmatrix}.$$

Although it belongs to the class of determinants sometimes called *orthosymmetrical,* and specially studied by Hankel,[1] the principal theorem relating to this class of determinants is not directly effective. Inasmuch as I have not found the evaluation of D_n nor of any determinant including it as a special case [2] in the books I have at hand, it may be of some interest to record it.

If the elements of each column of D_n be multiplied by the reciprocal of the last element in that column, and compensation be made by means of appropriate multipliers for the determinant, we may write

$$D_n = \frac{1}{n!(n+1)!\cdots(2n-1)!} \underset{(r,\, s\,=\,1,\,2,\,\cdots,\,n)}{|a_{rs}|},$$

where

$$a_{rs} = \frac{(n+s-1)!}{(r+s-1)!}.$$

Every element of the last row of the determinant $|a_{rs}|$ is equal to 1. Hence it may be reduced to a determinant in which all elements of the last row are zero with the exception of the last one, by subtracting the second column from the first, the third from the second, and in general the $(s+1)$th column from the sth column for $s = 1, 2, \cdots, n-1$. By this transformation, the element in the rth row and sth column becomes

$$\frac{(n+s-1)!}{(r+s-1)!} - \frac{(n+s)!}{(r+s)!} = -(n-r)\frac{(n+s-1)!}{(r+s)!} \quad (r, s = 1, 2, \cdots, n-1);$$

and by taking the factor $-(n-r)$ out of the rth row for $r = 1, 2, \cdots, n-1$, we may reduce the determinant $|a_{rs}|$ as follows:

$$\underset{(r,\, s\,=\,1,\,2,\,\cdots,\,n)}{|a_{rs}|} = (-1)^{n-1}(n-1)! \underset{(r,\, s\,=\,1,\,2,\,\cdots,\,n-1)}{|a_{rs}^{(1)}|}$$

where

$$a_{rs}^{(1)} = \frac{(n+s-1)!}{(r+s)!}.$$

Each element of the last row of $|a_{rs}^{(1)}|$ is equal to 1. Hence a transformation similar to that used on $|a_{rs}|$ will suffice to reduce its order by one. We find

$$\underset{(r,\, s\,=\,1,\,2,\,\cdots,\,n-1)}{|a_{rs}^{(1)}|} = (-1)^{n-2}(n-2)! \underset{(r,\, s\,=\,1,\,2,\,\cdots,\,n-2)}{|a_{rs}^{(2)}|}$$

where

$$a_{rs}^{(2)} = \frac{(n+s-1)!}{(r+s+1)!}.$$

Again, the elements of the last row of $|a_{rs}^{(2)}|$ are equal to 1. The process of reduction of order can be repeated successively, the jth repetition giving the

[1] See Kowalewski's *Einführung in die Determinantentheorie,* page 112.

[2] For the evaluation of determinants of a very similar character, see however Scott's *Treatise on the Theory of Determinants,* page 79.

reduction formula

$$\underset{(r,s=1,2,\cdots,n-j+1)}{|a_{rs}^{(j-1)}|} = (-1)^{n-j}(n-j)!\underset{(r,s=1,2,\cdots,n-j)}{|a_{rs}^{(j)}|}$$

where

$$a_{rs}^{(j)} = \frac{(n+s-1)!}{(r+s+j-1)!}.$$

After $n-1$ reductions we obtain a determinant of a single element

$$(-1)\underset{(r,s=1)}{|a_{rs}^{(n-1)}|} = -1.$$

Assembling the results of the successive reductions, we have [1]

$$D_n = (-1)^{\frac{n(n-1)}{2}}\frac{1!\,2!\,3!\cdots(n-1)!}{n!(n+1)!\cdots(2n-1)!}.$$

III. A Correction.

By R. M. Mathews, Wesleyan University.

Dr. B. H. Brown has called my attention to an error caused by uncritical zeal in my note in the October (1922) Monthly.[2] The following paragraph on p. 348 should be deleted:

"It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions."

The statement is true but vacuously so, for an inversion does not raise the order of a bicircular quartic; the curve already passes twice through the circular points at infinity, and a change of the pole of inversion does not change these points. In the general case of a quadratic transformation (of which inversion is a special case) successive transformations will double the order of the curve each time if we take a fundamental triangle entirely different from the preceding.

The error in question is found in the article by Cazamian as cited.

RECENT PUBLICATIONS.

REVIEWS.

Einleitung in die Mengenlehre. By A. Fraenkel. Berlin, Julius Springer, 1919. vi + 156 pp.

This book is an introduction to the researches of G. Cantor—a field which, even among professional mathematicians, has a special reputation for subtleties and abstractions. And yet, with all of its dignity of subject-matter and of style,

[1] A somewhat different treatment would be to multiply the rth row by $(n+r-1)!$, and divide the sth column by $(n-s)!$, for every r and s. The elements are then binomial coefficients, and the determinant as modified easily found to be ± 1. This same method evaluates any minor of D_n standing in the upper right-hand corner. Several references to determinants involving binomial coefficients may be found in E. Pascal, *I Determinanti*, Milan, 1897, pp. 173, 175. Editor.

[2] "Concyclic Points on an Equilateral Hyperbola and its Inverses," this Monthly (*1922*, 347–8).

Fraenkel's *Einleitung* is intended for the mathematical amateur. It is thus a contribution to the scanty field of popular literature that is concerned with new mathematical discoveries. Nearly every person with a capacity for abstract thinking can read it with understanding. The actual mathematical knowledge needed by the reader barely goes beyond high-school algebra. For example, a proof is inserted of the irrationality of $\sqrt{2}$; and more than a page of fine type is devoted to showing that there exist only a finite number of algebraic equations, with integral coefficients, having a given rank (*Höhe*)—the rank of the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ being defined as $(n - 1) + |a_0| + |a_1| + \cdots + |a_n|$. Of course, it is to be expected, since the book is written in German, that the popularity is of the German kind. And so it is: there is no soft-pedalling of abstractness or of Gründlichkeit.

The style is clear and vivid; concrete examples are copious, especially in the earlier portion. The sentences are wrought with a literary and logical swing that must have given joy to the author. There is a freedom and a warmth that may be ascribed in part, no doubt, to the way in which the book originated, "the incentive came in conversations with comrades of war (non-mathematicians), whose desolate hours I could now and then abbreviate by initiating them in the thought-paths of the theory of aggregates." The author succeeds well in conveying the beauty, the sweep and the revolutionary significance of Cantor's creation.

The reasons why Cantor's work has an uncommon attraction for the mathematical amateur are apparent. It is the first rigorous study of the infinite as such; it is singularly free from technical matter and formulas; it is the groundwork of other mathematical disciplines—scarcely any branch of mathematics has been left unaffected by its ideas; it is the closest to logic and to philosophy. Then, too, it is only through an acquaintance with the theory of aggregates that one can appreciate the nature of the modern controversies concerning the infinite, which have divided mathematicians into two or more opposing camps.

The following topics are discussed: aggregate (naïve definition), equivalence,[1] denumerable sets,[2] cardinal number of the continuum and of the set of real functions, calculation with infinite cardinals, ordinal types and ordinal numbers, linear sets, especially with reference to the order types η and λ, and normal order. There follows a consideration of certain paradoxes in logic and in the theory of aggregates, a critique of the concept of set and of the Auswahlprinzip, and a sketch of Zermelo's *Grundlagen*.

On the question of Wohlordnung, Fraenkel takes a decided view—in contradistinction to the doubts in Hobson's recent volume: "There seems to be no good reason for regarding the Auswahlprinzip as less evident than every other axiom

[1] Two sets M and N are said to be equivalent if a correspondence can be set up between them in such a way that to every element of M there corresponds just one element of N, and to every element of N, just one element of M. The mating of the positive integer n with the positive even integer $2n$ shows that if M is the set of positive integers, and N the set of positive even integers, then M and N are equivalent.

[2] Sets equivalent with the set of positive integers.

of mathematics." And as to rigor in the theory of aggregates, he has this to say: "We thus have good ground for regarding the Cantor-Zermelo theory of aggregates as a discipline possessing the same rigor and security as every other branch of mathematics; and the transfinite cardinals and ordinals as possessing in the mathematical-logical kingdom, just as any other mathematical-logical concept, the undiminished rights of citizenship."

The author's interest in logical refinement results in greater attention to abstract questions than may be justified on the ground of promoting the reader's understanding. For example, the space spent in showing in detail (p. 11) that if a set M is equivalent to a set N, then N is equivalent to M, might easily have been used for a more stimulating discussion.

The proof of the non-denumerability of the continuum[1] may be made a little more convincing to the general reader if the *reductio ad absurdum* is avoided and the following statement adopted: For every given infinite sequence S of real numbers, there is a real number not contained in S. (The same remark applies to the proof of the equivalence of the set of real numbers and the set of real functions.) The proof would also be a little simpler if the digits 1 and 2 were used for $\alpha_1, \beta_2, \gamma_3, \cdots$ (p. 34) instead of all the possible digits; subsequent explanations (p. 35) would thus be rendered largely unnecessary. There would be, however, a slight loss in naturalness.

In connection with the definition of cardinal number, it is safer to avoid telling just what a cardinal number is and to state merely when two sets "have the same cardinal number." The author's definition (p. 41) of cardinal number as the common characteristic of equivalent sets is obviously objectionable.

On p. 135 occurs the statement: "We choose as axioms propositions that are as evident as possible." If properly understood, this statement is acceptable; but if the reader is not told also of the definitional character of an axiom, he may not see that an axiom as such is neither true nor false. On p. 136 occurs the statement: "Then each of these three axioms is free from contradiction; nevertheless, when taken together, they contain a contradiction." This is not true. The axioms are consistent but they are not simultaneously valid in the case of Euclidean geometry.

There is one other point in connection with foundations that deserves mention. A number of mathematicians, following Hilbert,[2] have taken the view that the so-called *axiomatic method*—as contrasted with the so-called *genetic method*—is the method *par excellence* for the attainment of rigor.[3] This view is, of course, not justified. The proof of the Wohlordnungssatz, for example, does not gain in convincingness simply because an axiom—the multiplicative axiom—is formu-

[1] This asserts that no sequence $a_1, a_2, a_3, \cdots$ exists such that every real number equals at least one a_n.

[2] Cf. Hilbert, *Grundlagen der Geometrie*, third edition (1909), p. 256.

[3] In the case of the foundations for the system of real numbers, the genetic method would start with the positive integers and successively define the negative integers, the rational numbers, and the irrational numbers. The axiomatic method would start with an aggregate S and two undefined operations $+$ and $\times$, and would postulate enough about $(S, +, \times)$ to prove all other known facts concerning the continuum of real numbers.

lated; our conviction depends rather on the way in which this axiom appeals to us intuitively. The *Grundlagen* of Zermelo constitute a more rigorous basis for the theory of aggregates than the treatment of Cantor, not simply because they are formulated as axioms, but mainly because they restrict the definition of the term aggregate. In this matter, it would be hard for the reader to get the right point of view from Fraenkel's book.

There is one further question the reviewer wishes to raise. To obtain clearness of exposition, the author has adopted the method of long explanations. Does clearness, however, necessitate long explanations? Perhaps so, in an oral discourse; but a reader has more time to think than a listener. Here and there, in the reviewer's opinion, Fraenkel might have conserved the reader's energy by being more brief; and the space thus gained might have been utilized for applications.

HENRY BLUMBERG.

La Composition de Mathêmatique dans l'examen d'admission a l'École Polytechnique, 1901–1921. By F. MICHEL and M. POTRON. Paris, Gauthier-Villars. 1922. 452 pages. Price 40 francs.

As the title indicates, this book is primarily a collection of the mathematical problems set at the examinations for admission to l'École Polytechnique, 1901–1921. Complete solutions are given, many of them with bibliography. A second part of the book is a collection of exercises proposed by the authors. These are chosen with the purpose of illustrating the principles of the problems of the examinations. References to the examination problems on which they bear are given.

As to the examinations themselves: they seem to be in no way questions about mathematical facts nor do they call for proofs of well-known theorems. Problems requiring genuine ability are set. These are chosen from fields that are varied but are, nevertheless, such as should be familiar to a good first-year graduate student in an American university. The majority are of graduate difficulty although some easier ones occur.

The book, apart from its avowed purpose of aiding candidates in their preparation for a particular examination, should be of interest to Americans as a study in French education and the problems as suggesting, through analogy and generalization, comparatively simple fields for investigation.

TOMLINSON FORT.

A Course in Analytic Geometry. By P. P. BOYD, J. M. DAVIS and E. L. REES. New York, D. Van Nostrand Co., 1922. Price $2.40.

The authors, who are all professors in the University of Kentucky, state in their preface:

"The arrangement of the material of analytics is usually somewhat artificial and not in accordance with any underlying idea or principle. In this course the two fundamental problems, the locus of an equation and the equation of a locus, are given their due prominence and made the basis for arrangement of the work in both plane and solid analytics. This conduces to clarity,

The chapters dealing with the equation of a locus and the analytic treatment of geometric properties follow the usual plan. However, it is doubtful whether enough is said in Chapter 7 regarding empirical loci to make the introduction of the topic worth while.

The arrangement of the material of solid analytics follows closely that of plane analytics. Quadric surfaces are treated as loci of points, lines and conics which move subject to suitable conditions.

On the whole, the text presents the subject in a satisfactory manner and, if properly supplemented by informal lectures, should form a good basis for the work in analytics.

Analytic Geometry, Brief Course. By L. P. SICELOFF, G. A. WENTWORTH and D. E. SMITH. Boston, Ginn & Co. Price $1.80.

This text furnishes rather a contrast to the one discussed above. In fact, in the preface we read:

> "This text has not been prepared for the purpose of exploiting any special theory of presentation; it aims solely to set forth the leading facts of the subject clearly, succinctly, and in the same practical manner that characterizes the other textbooks of the series. . . . Throughout the work a special effort has been made to present the material in so natural and simple a manner that the student can comprehend it largely through his own reading."

The locus of the equation is discussed briefly, 10 pages, and simply. Transformation of coördinates is given a short chapter as is also polar coördinates. The analytic treatment of the simple geometric properties of the conics is given at the close of the chapters on each of the conics. The general conic and the reduction of the general quadratic to standard form are discussed briefly in Chapter 10. The two chapters on solid analytics are short and to the point.

The plan of the text, mechanically, is typical of the other texts of the Wentworth-Smith series, with the free use of italics for theorems and important problems. The text is very readable and, in the opinion of the writer, should need little supplementing by lectures for the average class.

S. E. FIELD.

La Théorie de Bohr (*Publications de la Société de Chimie-Physique, X*). By E. BAUER. Paris, Hermann, 1922. 8vo. 52 pages. Price, paper cover, 4.50 francs.

This is an interesting summary of Bohr's theory of atomic structure as developed up to 1921. It includes brief explanations of the fundamental facts underlying the idea of the nuclear atom, its nuclear charge and dimensions, atomic number, etc. The relation of the electronic configurations to the physical and chemical properties is explained in some detail. Spectral series are developed in terms of the theory of quanta, with emphasis upon the Balmer series of hydrogen, the Stark effect and the X-ray spectra of the heavier elements.

E. F. BARKER.

NOTE ON A RECENT PUBLICATION.

We have already referred (*1921*, 79) to the publication of volume 14 of the magnificent edition of *Oeuvres Complètes* of Christian Huygens now being produced by the Dutch Society of Sciences. Professor D. J. Korteweg of the University of Amsterdam has for many years dedicated his life to the editing of the *Oeuvres*. Pages 1–163 of volume 15, edited by D. J. Korteweg and A. A. Nijland, have just been printed off. They contain "Recueil des observations astronomiques, 1657–1694." The usual avertissement by the editors occupies pages 3–53.

ARTICLES IN CURRENT PERIODICALS.

Acta Mathematica, volume 42, 1922: "Surface transformations and their dynamical applications" by G. D. Birkhoff, 1–119; "Mémoire sur les polynômes de Bernoulli" by N. E. Nörlund, 121–196; "Sur les fonctions hypersphériques et sur l'expression de la fonction hypergéométrique par une dérivée généralisée" by J. K. de Feriet, 197–207; "Über Konvergenz unendlicher Kettenbrüche mit durchweg reellen Elementen" by O. Szasz, 209–237; "A proof that every aggregate can be well-ordered" by P. E. B. Jourdain, 239–261; "Über die konforme Abbildung endlich und unendlich vielfach zusammenhängender symmetrischen Bereiche" by P. Koebe, 263–287; "Über die automorphen Functionen zweier Veränderlichen" by P. J. Myrberg, 289–318; "Développements asymptotiques des solutions d'une classe d'équations différentielles linéaires" by T. Carleman, 319–336.

Journal de Mathématiques Pures et Appliquées, ninth series, volume 1, part 4, 1922: "Nuove osservazioni su alcuni metodi d'approssimazione dell' analisi" by M. Pincone, 335–393; "Differential properties of functions of a complex variable which are invariant under linear transformations" by E. J. Wilczynski, 393–436.

L'École Normale Supérieure, Annales, volume 57, December, 1922: "Sur les formules d'interpolation de Stirling et de Newton" by N. E. Nörlund, 343–403.

Messenger of Mathematics, volume 52, July, 1922: "Electromagnetic potentials and radiation" by R. Hargreaves, 34–38; "Partial fractions associated with quadratic factors" by E. H. Neville, 39–42; "Polygons inscribed in one circle and circumscribed to another" by E. C. Titchmarsh, 42–45; "On curvature, tortuosity and higher flexures of a curve in flat space of n dimensions" by R. F. Muirhead, 45–48.

Quarterly Journal of Mathematics, volume 49, October, 1922: "Developable surfaces through given pairs of guiding curves" by E. H. Neville, 193–220; "On plane unicursal quintic curves with a triple point" by H. Hilton and Grace D. Sadd, 220–226; "Abstract definition of the symmetric and alternating groups and certain other permutation groups" by R. D. Carmichael, 226–283; "On the elementary treatment of the theory of least squares" by L. V. Meadowcroft, 283–288.

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PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and Norman Anning.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3017. Proposed by F. W. Perkins, Jr., Cambridge, Mass.

M. Edouard Lucas, in *Récréations Mathématiques* (vol. 1, pp. 41–51), gives a proof of the following theorem: Any labyrinth can be traversed in such a way as to cover each path twice, and only twice, returning finally to the starting point (provided merely that one has some means of marking paths as they are traversed) by following the three rules given below:

(1) On arriving at a new vertex (*i.e.*, one not visited before), leave by a new path if possible; otherwise return by the same path.

(2) On arriving at an old vertex by a new path, return by the same path.

(3) On arriving at an old vertex by an old path, leave by a new path if possible; otherwise by a path marked just once. It can be shown that it is always possible to do this until every path of the labyrinth has been marked twice.

Assuming Lucas' results show that, if his rules are observed, the two trips on each path are in opposite directions.

Note: We may represent any labyrinth geometrically by a collection of points ("vertices") not necessarily in a plane which are joined by plane or twisted curves ("paths") in any manner whatever, provided merely that it is possible to pass from any point to any other along the curves and that the curves do not meet except at vertices.

3018. Proposed by J. G. Coffin, New York City.

A platform 1000 feet long can move in either direction at any speed up to 2000 feet per second. A man is stationed on one end of the platform and is provided with a horn whose note-frequency is 100 vibrations per second. Assuming that the speed of sound is 1000 feet per second,

explain what occurs to an observer stationed on the other end of the platform when the horn is sounded under the following conditions: (*a*) Platform at rest; (*b*) moving to the right with a speed of 500 ft./sec.; (*c*) 1000 ft./sec.; (*d*) 1500 ft./sec.; (*e*) with the same three speeds but to the left.

What happens under similar conditions when the man with the horn is on the ground and the listener is on the platform and *vice-versa*?

3019. Proposed by J. B. REYNOLDS, Lehigh University.

Find the equation of the curve on the cylinder $x^2 - y^2 = a^2$ such that the tangent to it cuts the xy-plane in a lemniscate as the point of tangency moves along the curve.

3020. Proposed by JOHN NICHOLS, Portland, Oregon.

Rationalize $\alpha^{1/2} + \beta^{1/2} + \cdots + \mu^{1/2} + \nu^{1/2} + p = 0.$

3021. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Prove that the two lines joining the points of intersection of two orthogonal circles to any point of one of them meet the other circle in two diametrically opposite points, and conversely.

3022. Proposed by M. J. SPINKS, Wilmington, Ohio.

Given that ABC is an equilateral spherical triangle right-angled at C, prove that $\sec A = 1 + \sec a$.

3023. Proposed by E. T. BELL, University of Washington, Seattle, Wash.

The equation $x^p + y^p + z^p = 0$ is possible in integers x, y, z prime to the odd prime p, if

$$\frac{1}{2}\left[\frac{1}{2}N_2(p) - \frac{1}{3}N_3(p) + \frac{1}{4}N_4(p) - \cdots + \frac{1}{p-1}N_{p-1}(p)\right] + 1$$

is divisible by p, where $N_r(n)$ is the number of representations (order essential) of n as a sum of r square integers with roots not equal to zero.

3024. Proposed by H. F. MACNEISH, College of the City of New York.

The angles of a triangle ABC are divided into n equal parts ($n = 3, 4, 5, \cdots$) and the two n-sectors of angles B and C which are adjacent to side BC intersect in A_1, the next two n-sectors in A_2 and so on to A_{n-1}. Points $B_1, B_2, B_3, \cdots C_1, C_2, C_3, \cdots$ are similarly determined. Which of the triangles $A_iB_iC_i$ ($i = 1, 2, 3, \cdots n - 1$) are equilateral?

SOLUTIONS

2907 [1921, 277]. Proposed by G. E. RAYNOR, Princeton University.

Sum the following series:

$$(a)\quad \sum_0^\infty \frac{1}{(2n+1)(3n+1)}; \qquad (b)\quad \sum_0^\infty \frac{1}{(3n+1)(3n+2)}.$$

SOLUTION BY OTTO DUNKEL, Washington University.

These two series are special cases of

$$\sum_{i=0}^{i=\infty} \frac{1}{(a_1 i + b_1)(a_2 i + b_2)} = \frac{1}{a_1 b_2 - a_2 b_1} \sum_{i=0}^{i=\infty} \left(\frac{1}{i + p_1} - \frac{1}{i + p_2}\right), \tag{1}$$

where the a's and b's are positive integers, *i.e.*, > 0; $a_1b_2 - a_2b_1 \neq 0$; and $p_1 = b_1/a_1$, $p_2 = b_2/a_2$.

Set

$$\sum_{i=0}^{i=m} \frac{1}{i+p} - \log\left(\frac{m+1+p}{p}\right) = C(m, p); \tag{2}$$

then, since $C(m, p)$ is positive and less than $\sum_{i=0}^{i=\infty} 1/2(i+p)^2$, it converges to a limit $\gamma(p)$ and we have

$$\sum_{i=0}^{i=m_1} \frac{1}{i+p_1} - \sum_{i=0}^{i=m_2} \frac{1}{i+p_2} = \log\left[\frac{1 + \dfrac{1+p_1}{m_1}}{1 + \dfrac{1+p_2}{m_2}} \cdot \frac{a_1 m_1}{a_2 m_2} \cdot \frac{b_2}{b_1}\right] + C(m_1, p_1) - C(m_2, p_2). \tag{3}$$

If we take $m_1 = m_2$, then we have as the value of (1)

$$\frac{1}{a_1b_2 - a_2b_1}\left[\log\frac{a_1b_2}{a_2b_1} + k\right], \tag{4}$$

where $k = \gamma(p_1) - \gamma(p_2)$.

The value of k may be found as follows: we have

$$\Sigma\left(\frac{1}{i+p_1} - \frac{1}{i+p_2}\right) = a_1\Sigma\frac{1}{a_1i+b_1} - a_2\Sigma\frac{1}{a_2i+b_2}.$$

Set

$$f(x) = a_1\sum_{i=0}^{i=\infty}\frac{x^{a_1i+b_1}}{a_1i+b_1} - a_2\sum_{i=0}^{i=\infty}\frac{x^{a_2i+b_2}}{a_2i+b_2}, \qquad 0 \leqq x < 1,$$

then

$$f'(x) = a_1\sum_{i=0}^{i=\infty}x^{a_1i+b_1-1} - a_2\sum_{i=0}^{i=\infty}x^{a_2i+b_2-1} = \frac{a_1x^{b_1-1}}{1-x^{a_1}} - \frac{a_2x^{b_2-1}}{1-x^{a_2}},$$

and hence

$$f(x) = a_1\int_0^x\frac{x^{b_1-1}}{1-x^{a_1}}dx - a_2\int_0^x\frac{x^{b_2-1}}{1-x^{a_2}}dx,$$

where the constant of integration is zero, since $f(0) = 0$.

Now $f(x)$ may be written as a *single* power series for $x < 1$ instead of the sum of two power series as above, the terms of the second series being combined with corresponding terms of the first or inserted in order between pairs of terms of the first. Thus, if $a_2m_2 + b_2$ is the exponent in a term of the second series, we can find a positive integer m_1 so that

$$a_1m_1 + b_1 \leqq a_2m_2 + b_2 < a_1(m_1+1) + b_1$$

if we assume, as we may, that $b_1 \leqq b_2$. Hence we see that

$$\operatorname{limit}\frac{a_2m_2}{a_1m_1} = 1,$$

as m_1 and m_2 become infinite. Now (3) shows that this power series for $x = 1$ converges and we have

$$\operatorname{limit}\left[\sum_{i=0}^{i=m_1}\frac{1}{i+p_1} - \sum_{i=0}^{i=m_2}\frac{1}{i+p_2}\right] = \log\left(\frac{b_2}{b_1}\right) + k.$$

But for $x < 1$ the power series has the value given above in integral form, and hence by a well-known theorem (see Goursat-Hedrick, *Mathematical Analysis*, vol. 1, p. 378)

$$\operatorname*{limit}_{x=1}\left[a_1\int_0^x\frac{x^{b_1-1}}{1-x^{a_1}}dx - a_2\int_0^x\frac{x^{b_2-1}}{1-x^{a_2}}dx\right] = f(1) = \log\left(\frac{b_2}{b_1}\right) + k. \tag{5}$$

It now follows from (1) and (4) and this result that

$$\sum_{i=0}^{i=\infty}\frac{1}{(a_1i+b_1)(a_2i+b_2)} = \frac{1}{a_1b_2 - a_2b_1}\left[f(1) + \log\left(\frac{a_1}{a_2}\right)\right],$$

where $f(1)$ is the limit given in (5). This shows that the parts of the integrals in (5) which become infinite for $x = 1$ cancel. This may also be shown directly by an independent method. This same method of summation may be applied whatever the number of factors, $ai + b$, in the denominator of (1) provided that all the determinants such as $a_1b_2 - a_2b_1$ are different from zero.

Applying this result to the given special cases, we have

$$(a)\quad \frac{\sqrt{3}}{6}\pi + \log\frac{3\sqrt{3}}{4}, \qquad (b)\quad \frac{\pi}{3\sqrt{3}}.$$

2924 [1921, 393]. Proposed by FLORENCE P. LEWIS, Goucher College.

Given a triangle and a conic. Through each vertex of the triangle there pass two lines harmonic to the tangents through that point and to the sides of the triangle. Prove that the six lines so found pass by threes through four points.

Solution by Otto Dunkel, Washington University.

The equations of the three sides of the triangle and of two straight lines through the vertices A and B, distinct from the sides, may be written in abridged notation as $\alpha = 0$, $\beta = 0$, $\gamma = 0$,

$\beta + \gamma = 0$, $\gamma + \alpha = 0$, by a suitable choice of α, β and γ. Hence we may write the equations of the six lines, separating the sides of the triangle harmonically, as

$$\beta + \gamma = 0, \qquad \gamma + \alpha = 0, \qquad \alpha + h\beta = 0,$$
$$\beta - \gamma = 0, \qquad \gamma - \alpha = 0, \qquad \alpha - h\beta = 0.$$

Then each of the three pairs of the following lines:

$$\beta + p\gamma = 0, \qquad \gamma + q\alpha = 0, \qquad \alpha + rh\beta = 0,$$
$$p\beta + \gamma = 0, \qquad q\gamma + \alpha = 0, \qquad r\alpha + h\beta = 0,$$

are also separated harmonically by the corresponding pair of the first set, for $2(\beta + p\gamma) = (1 + p)(\beta + \gamma) + (1 - p)(\beta - \gamma)$ and $2(p\beta + \gamma) = (1 + p)(\beta + \gamma) - (1 - p)(\beta - \gamma)$.

If the last set of six lines are tangent to a conic, then the hexagon obtained by taking them in the order written, *i.e.*, $\beta + p\gamma$, $p\beta + \gamma$, $\gamma + q\alpha$, etc., will have the diagonal lines

$$q\gamma - rh\beta = 0, \qquad p\beta - q\alpha = 0, \qquad r\alpha - hp\gamma = 0,$$

and these must meet in a point by Brianchon's theorem. This gives readily $h^2 = 1$ and on inserting either value of h in the first set, it will be seen at once that the six lines meet as stated in the problem.

Also solved by WILLIAM HOOVER, R. M. MATHEWS, J. R. MUSSELMAN, and H. L. OLSON.

2941 [1922, 28]. Proposed by W. D. CAIRNS, Oberlin College.

$1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2$ is a function of n. Find its derivative with respect to n.

SOLUTION BY S. A. COREY, Des Moines, Iowa.

The given function of n, when developed by Euler's summation formula, reduces to the form $(2n^3 - 3n^2 + n)/6$, the derivative of which is $n^2 - n + 1/6$.

NOTE BY THE EDITORS—The sum $\sum_{i=1}^{i=n-1} i^2$ is a function of n which is defined only for integral values of n. It has then no derivative. In order to speak of a derivative, the function must be defined for other values of n. This may be done arbitrarily by finding the sum as a polynomial of the third degree in n and then defining the values of the function for all values of x as the values of this polynomial, $(2x^3 - 3x^2 + x)/6$; this the above solution does. But this is only one of many ways of defining a function $f(x)$ such that when $x = n$, a positive integer, $f(x) = \sum_{i=1}^{i=n-1} i^2$; for example, we might write

$$f(x) = A \sin^p \pi x + (2x^3 - 3x^2 + x)/6, \qquad p > 0,$$

and hence the derivative would vary with the definition of the function for non-integral values.

Also solved by T. M. BLAKSLEE, MICHAEL GOLDBERG, C. E. NORWOOD, E. J. OGLESBY, W. J. THOME, J. B. REYNOLDS, and F. L. WILMER.

2942 [1922, 28]. Proposed by L. E. DICKSON, University of Chicago.

I am dealt 13 cards at whist. What is the chance that all my cards will be diamonds?

SOLUTION BY E. W. WOOLARD, U. S. Weather Bureau, Washington, D. C.

The number of ways in which 52 cards may be dealt to four players giving a single card to the first, second, third, fourth and then repeating in this order until the cards are exhausted is 52! The number of ways in which a specified player may receive 13 diamonds and the other three no diamonds is 13! 39!. Hence the probability of the player receiving 13 diamonds is 13! 39!/52! or 1 in 635,013,559,600. Here the problem is treated as a problem in permutations, but it may also be considered by combinations giving the same result.

Also solved by A. BOGARD, S. A. COREY, E. J. OGLESBY, and J. B. REYNOLDS.

2943 [1922, 29]. Proposed by L. E. DICKSON, University of Chicago.

In the game of bridge, what is my chance: (*a*) that my hand will contain 4 aces? (*b*) that some hand will contain 4 aces? (*c*) that my hand will be a Yarborough, *i.e.*, contain no honor?

SOLUTION BY E. J. OGLESBY, Flushing, N. Y.

There are 52! ways in which 52 cards may be dealt one by one to four players in order. A specified player may receive four aces in $4!\ ({}_{13}C_4)$ or $13!/9!$ ways, and there are 48! ways in which the other three players may receive the remaining 48 cards. The probability for (*a*) is then

$$\frac{13!\,48!}{9!\,52!} = \frac{11}{4165},$$

while for (*b*) it is 44/4165.

(*c*) Webster's *New International Dictionary*, 1910, defines "Yarborough" as a hand containing no card higher than a nine. This requires that my hand shall be selected from the 32 cards which are nine or lower. There results the probability:

$$({}_{32}C_{13})/({}_{52}C_{13}) = (32!\,39!)/(52!\,19!) = 0.00054.$$

Also solved by S. A. COREY, J. B. REYNOLDS, and E. W. WOOLARD.

NOTE BY THE EDITORS—Part (*c*) was interpreted differently by some of the solvers with a corresponding difference in the numerical result.

2944 [1922, 29]. Proposed by S. A. COREY, Des Moines, Iowa.

A particle of mass m, starting from rest, is drawn by a string over a smooth horizontal plane, the other end of the string moving in the plane with uniform acceleration n along a line perpendicular to the initial position of the string. Prove that the tension of the string is $3mn \cos \theta$, where θ is the angle which the string makes with the given line. Also prove that the motion of the particle is vibratory.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

The acceleration of the particle may be considered as made up of three components, n parallel to the given line, $a\ddot{\theta}$ perpendicular to the string and $a\dot{\theta}^2$ along the string. Resolving perpendicular to the string, we have $n \sin \theta + a\ddot{\theta} = 0$, and upon integrating,

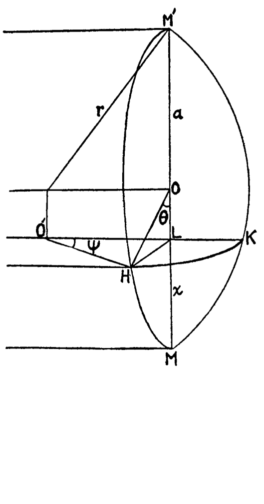
$$-n \cos \theta + \tfrac{1}{2}a\dot{\theta}^2 = c = 0.$$

Therefore, $\dot{\theta} = \mp \sqrt{\dfrac{2n \cos \theta}{a}}$ and the string oscillates between $\theta = \pm \dfrac{\pi}{2}$. Again, resolving along the string, we have, T being the tension, $T = m(a\dot{\theta}^2 + n \cos \theta) = m(2n \cos \theta + n \cos \theta) = 3mn \cos \theta$.

Also solved by F. L. WILMER.

2947 [1922, 29]. Proposed by D. H. MENZEL, Princeton University.

An oil tank has the shape of a cylinder with ends which are segments of a sphere and with horizontal axis. The diameter of the cylinder being given, and the radius of the spherical segments, derive a formula that will express the volume of the liquid contained in the tank in terms of its depth.



SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let l be the length of the cylindrical portion of the tank, a its radius, and r the radius of the spherical caps. Let the distance of a horizontal section from the bottom of the cylinder be $x = ML = a(1 - \cos \theta) = 2a \sin^2 \theta/2$, where MOM' is the vertical diameter of a cylinder end. The horizontal section intersects this circular end in a chord HLH' which subtends the angle 2θ at the center O, such that $\theta = \angle LOH$, and the spherical cap in a segment $HLH'K$ with the radius $O'K$. Let $\psi = \angle KO'H$, then $O'L = \sqrt{r^2 - a^2} = a\lambda$, $O'K = \sqrt{r^2 - a^2 \cos^2 \theta} = a\lambda \sqrt{1 + \sin^2 \theta/\lambda^2}$, and $\tan \psi = \sin \theta/\lambda$.

The section consists of a rectangle of area $2al \sin \theta$ and two segments of a circle each of area $a^2(\lambda^2 + \sin^2 \theta)\psi - a^2\lambda \sin \theta$. Set-

ting $dx = a \sin\theta d\theta$, the volume is given by the integral

$$V = 2a^2 \int_0^\theta \left[(l - a\lambda) \sin^2\theta + a(\lambda^2 \sin\theta + \sin^3\theta) \tan^{-1}\left(\frac{\sin\theta}{\lambda}\right) \right] d\theta$$

$$= a^2 \left(l - 2\lambda a - \frac{4}{3}\lambda^3 a \right) \theta + a^2 \left(\frac{2}{3} a\lambda - \frac{l}{2} \right) \sin 2\theta$$

$$+ \frac{a^3}{3} \cos\theta [\cos 2\theta - 6\lambda^2 - 5] \tan^{-1}\left(\frac{\sin\theta}{\lambda}\right) + \frac{4a^3}{3} (\lambda^2 + 1)^{3/2} \tan^{-1}(\sqrt{\lambda^2+1} \tan\theta/\lambda)$$

where $\theta = 2\sin^{-1}\sqrt{x/2a}$, $0 \leqq \theta \leqq \pi$, and $0 \leqq \tan^{-1}\left(\frac{\sqrt{\lambda^2+1}}{\lambda} \tan\theta\right) \leqq \pi$. When $\theta = 0$ or π, $\tan^{-1}(\sin\theta/\lambda) = 0$. When $\lambda = 0$, $\tan^{-1}(\sin\theta/\lambda) = \pi/2$, $\tan^{-1}(\sqrt{\lambda^2+1}\tan\theta/\lambda) = 0$.

Also solved by Moe Buchman, P. Fitch, Michael Goldberg, and H. W. Reddick.

2948 [1922, 29]. Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of the normal planes to the curve,

$$x = a\cos t, \qquad y = a(1 - \cos t), \qquad \text{and} \qquad z = a \sin t.$$

Solution by Maurice Baudin, Miami University.

Translating the coördinate axes to $(0, a, 0)$ as origin and rotating them about the z-axis through $-\pi/4$, we write the equations of the given curve (a plane curve)

$$x = \sqrt{2}a\cos t, \qquad y = 0, \qquad z = a\sin t.$$

The equation of the normal is

$$Z - \sqrt{2}\tan t\, X + a \sin t = 0.$$

Eliminating t from this equation and $-\sqrt{2}\sec^2 t\, X + a\cos t = 0$, we obtain the equation of a cylinder

$$(\sqrt{2}\, X)^{2/3} + Z^{2/3} = a^{2/3}.$$

2950 [1922, 29]. Proposed by T. M. SIMPSON, JR., Randolph-Macon College, Ashland, Va.

Determine the curve which cuts the radius vector at an angle proportional to the radius vector.

Solution by A. H. Wilson, Haverford College.

The differential equation of the curve is

$$\frac{r d\theta}{dr} = \tan kr,$$

where k is the factor of proportionality, and the angle in question is measured in the positive direction from the radius vector to the tangent.

On expanding $\tan kr$, an integration in series may be effected:

$$\theta = c + kr + k^3 r^3/9 + \cdots;$$

which is convergent when $r < \pi/2k$.

There are infinitely many limiting or singular curves of the solutions, namely, the concentric circles $r = (2n + 1)\pi/2k$.

These circles are included among the solutions.

Inside such a circle and near it $dr/d\theta > 0$; while outside $dr/d\theta < 0$. Hence r is increasing with θ just inside the circle and decreasing as θ increases just outside; that is, a curve in any one of the rings defined by the concentric circles wraps around (approaches) both the inner and the outer circle for increasing θ.

It is clear from the series solution that any curve in the first circle cuts a radius infinitely many times; and the like is true for the curves of any ring. The curves therefore approach the circles asymptotically.

On the circles $r = n\pi/k$ the curves are radial in direction, and the sense of variation of r changes at these points.

The origin is a "ray-point."

These results are for the most part also clear from geometric considerations.

Also solved by WILLIAM HOOVER who studied the pedal curve.

2951 [1922, 81]. Proposed by B. F. FINKEL, Drury College.

What is the average linear velocity of a point on the periphery of a locomotive driving wheel 2 feet in diameter and making 100 revolutions per minute?

SOLUTION BY H. S. UHLER, Yale University.

The following solution is offered because it introduces generality but does not involve any more kinematical principles than the given numerical problem.

Let a, v, and ω denote respectively the radius, the rectilinear velocity, and the angular velocity of the wheel. Let 2θ denote the angle which the radius to the point on the periphery makes, at any instant, with the upper half of the vertical diameter. Then the angle which the tangential velocity $a\omega$ makes with the rectilinear velocity v equals 2θ. By any rule for compounding vectors it will be found that the magnitude of the resultant velocity of the moving point is represented by

$$(v^2 + a^2\omega^2 + 2va\omega \cos 2\theta)^{1/2}.$$

To find the mean value V_m of this expression it is sufficient to integrate 2θ only over a vertical semicircle, since the magnitude of the resultant velocity is the same for the two points of any pair on the same horizontal line or level. Hence

$$\begin{aligned} V_m &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} (v^2 + a^2\omega^2 + 2va\omega\cos 2\theta)^{1/2} d\theta \\ &= \frac{2}{\pi}(v + a\omega)\int_0^{\frac{1}{2}\pi} d\theta\sqrt{1 - k^2 \sin^2\theta}, \end{aligned}$$

where $k^2 \equiv 4va\omega/(v + a\omega)^2$.

When v does not equal $a\omega$ (slipping, skidding, etc.), k^2 is less than unity and V_m equals

$$(v + a\omega)\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right].$$

See B. O. Peirce's "*A Short Table of Integrals,*" No. 525, elliptic integral E.

When $v = a\omega$ the motion is pure rolling, $k^2 = 1$, and $V_m = 4a\omega/\pi$.

In the numerical problem $a = 1$ ft., $\omega = 10\pi/3$ radians per sec., hence $V_m = 40/3$ ft. per sec. $= 100/11$ mi. per hr.

Also solved by PHILIP FITCH, MICHAEL GOLDBERG, WILLIAM HOOVER, A. PELLETIER, W. J. THOME, J. B. REYNOLDS, F. L. WILMER and the PROPOSER.

NOTE BY THE EDITORS.—If the problem is interpreted "What is the average of the velocities represented as vectors?" it is evident that the average is v since diametrically opposite velocities $a\omega$ cancel.

Professor Hoover and F. L. Wilmer assumed the number of points proportional to the length of the path of the point considered and got an answer of $\frac{1}{2}(5\pi^2)$ feet per second.

An interesting point about this problem is the fact that a very similar one is proposed in a text-book of physics by a very prominent physicist. The book does not apparently assume any knowledge of calculus on the part of the student. What method did the author have in mind?

2955 [1922, 81]. Proposed by the late L. G. WELD.

Find the proportions of an anchor ring such that its section by a plane parallel to its axis and tangent to its inner circle (circle of the gorge) shall be a lemniscate.

SOLUTION BY H. C. BRADLEY, Massachusetts Institute of Technology.

Let r be the radius of the generating circle of the anchor ring (called also a *tore* or *torus*), and R the radius of the inner circle. The tangent plane to the surface at a point 0 of this latter circle

cuts it in a curve having the form of the figure 8. The indicatrix of the surface at this point must be an hyperbola and the two principal sections are those of the meridian plane at 0 and the plane of the inner circle. Hence the principal radii of curvature are r and R. Also the two tangents to the curve at the double point 0 are the tangents to the asymptotic lines at 0 and it follows from the equation of the indicatrix that the slopes of the two tangents are $\pm\sqrt{r/R}$.[1] In order to be a lemniscate the two tangents must be at right angles, and hence $r = R$.

[1] See *Mathematical Analysis*, Goursat-Hedrick, vol. 1, pages 501–507.

To show that this condition is both necessary and sufficient, we now derive the equation of the curve. Let the x-axis be the tangent to the inner circle, with the origin at 0, and take any point $P(x, y)$ on the curve. The distance of P from the axis of the anchor ring is $\sqrt{x^2 + R^2}$, and P lies on a circle of radius r in the meridian plane through P. Hence from the equation of this circle follows $[\sqrt{x^2 + R^2} - (R + r)]^2 + y^2 = r^2$ or

$$(x^2 + y^2)^2 = 4(R + r)(rx^2 - Ry^2).$$

If $r = R = a/2$, the equation becomes

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

which is that of a lemniscate.

Also solved by A. BOGARD, MICHAEL GOLDBERG, WILLIAM HOOVER, A. PELLETIER and F. L. WILMER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor E. G. Bill, of Dartmouth College, has been appointed acting dean during the absence of Dean Laycock.

At New Hampshire State College, the following have been appointed instructors of mathematics: Mr. W. E. Wilbur, Mr. Hubert Huntley, and Mr. P. F. Howard. Professor H. L. Slobin, head of the department of mathematics, has been reappointed Director of the Summer School for the 1923 session.

Miss Eleanor P. Cushing has been made professor emeritus at Smith College.

Mr. R. S. Kimball has been appointed a member of the department of mathematics at the State Normal School at Worcester, Mass.

Mr. W. F. Shields has been appointed professor of freshman mathematics at Fordham University. He succeeds Rev. L. J. McGarry, who is now at Woodstock College, Maryland.

At Columbia University, Professor T. S. Fiske, of the department of mathematics, and Professor C. L. Poor, of the department of celestial mechanics, have been granted leave of absence for the present semester.

At Rutgers College, Professor A. A. Titsworth, formerly of the department of civil engineering, has been transferred to the department of mathematics as a full professor. Mr. W. B. Campbell, a graduate of the University of Pennsylvania, has been appointed instructor of mathematics. Professor Richard Morris read a paper on "The Cyclic Quadrilateral, a Recreation," at the October meeting of the Mathematical Association of New Jersey.

Miss Ruth Thompson has been appointed instructor of mathematics at the New Jersey College for Women.

Instructor J. T. Fairchild, of Ohio Northern University, has been promoted to a professorship of mathematics.

Associate Professor Mary E. Sinclair, of Oberlin College, is on leave of absence during the second semester of 1922–23, as holder of the Pratt fellowship of the National Association of University Women, and is studying at the University of Chicago.

Rev. Paul Muehlman, formerly head of the department of mathematics at Marquette University, has been transferred to Loyola University, Chicago, where he is holding a similar position.

Assistant professor B. F. Dostal, of the University of Denver, has been appointed assistant professor of mathematics at the Bradley Polytechnic Institute.

Mr. H. I. Mayes has been appointed a member of the department of mathematics at Blackburn College, Carlinville, Ill.

Mr. J. B. Hobbs has been appointed a member of the department of mathematics at the State Normal School, Oshkosh, Wisconsin.

At St. Mary's College, St. Mary, Kentucky, Mr. Charles McCarthy and Mr. Joseph Walsh have been appointed professors of mathematics.

Professor J. N. Mallory, of Union University, Jackson, Tenn., has returned to his work after a leave of absence of one and a half years, which he spent at Peabody College.

Mr. A. L. Hill has been appointed head of the department of mathematics at the State Normal School and Teachers College, Peru, Nebraska.

Mr. L. V. Robinson has been appointed head of the department of mathematics at Meridian College, Meridian, Texas.

Miss Paula Henry has succeeded her sister, Miss Phillis Henry, as director of the department of mathematics at Kidd-Key College, Sherman, Texas. Miss Phillis Henry has accepted a position as instructor at the University of Texas.

Miss Emma F. Barbe has been appointed instructor of mathematics at Westmoorland College, San Antonio, Texas.

At the Texas Agricultural and Mechanical College, Assistant Professors D. C. Jones and W. L. Porter have been promoted to associate professorships of mathematics, Mr. W. L. Hughes and Mr. P. K. Smith have been appointed assistant professors, and Mr. C. E. McCurry instructor.

At the University of Denver, Professor G. W. Gorrell of the Colorado State School of Mines has been appointed associate professor, and Dr. E. Frances Seiler of the University of Illinois instructor.

Mr. V. F. Morse, a graduate of the California Institute of Technology, has been appointed instructor of mathematics and mechanical drawing at Occidental College, Los Angeles, California.

Mr. P. M. Iloff has been head of the mathematics department of the State Teachers and Junior College at Chico, Cal., since January, 1922.

The Royal Astronomical Society has awarded its gold medal to Professor

A. A. Michelson, of the University of Chicago, for his application of the interferometer to astronomical measurements.

ANDREĬ ANDREEVICH MARKOV, professor of mathematics in the University of St. Petersburg from 1886 till 1907 when he was made professor emeritus, died July 27, 1922. He was born at Rjasan, Russia, June 2/14, 1856, and was a student at the University of St. Petersburg where he received the degree of doctor of mathematical sciences in 1884. He was a privatdocent in the University 1880–1886. His brilliant achievements won him election to the Academy of Sciences in St. Petersburg when he was only thirty years of age. He was one of the two members of this Academy which edited the *Oeuvres* (published in 1899 and 1907) of his former master, Tchebychef. His popular work on the calculus of probability first appeared at St. Petersburg in 1900, and a German translation, by H. Liebmann, of the second edition (1908), appeared at Leipzig in 1912. His calculus of finite differences first published at St. Petersburg in 1891 was also translated into German (Leipzig, 1896). Long lists of his published papers, beginning in 1879, may be found in "Poggendorff"; in the Royal Society *Catalogue of Scientific Papers,* volumes 10, 12, and 17; and in *Revue Semestrielle des Publications Mathématiques.* Brief biographical notes and a portrait are given in *Acta Mathematica, 1882–1912. Table Générale des Tomes 1–35,* Upsala, 1913.

ANTONIO FAVARO, distinguished professor of graphical statics and history of mathematics in the University of Padua since 1872, died September 30, 1922. He was born at Padua, May 21, 1847. He was the author of over 400 papers, memoirs, and books and was the acknowledged authority on the life and works of Galileo whose work he edited in the great national edition (20 volumes, Florence, 1890–1909). For this he was made "nobile." His most recent volume was the finely illustrated *L'Università di Padova* (Venice, 1922) distributed to delegates to the celebration of the seven hundredth anniversary of the founding of the University of Padua (*1922,* 318). His *Lezioni di statica Grafica* (Venice, 1877) was translated into French (2 volumes, Paris, 1879, 1885).

(*Published July 31, 1923.*)

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should bè addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Summer Meeting of the Association, Vassar College, September 5–6, 1923

Eighth Annual Meeting, University of Cincinnati, December, 1923

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Knox College, Galesburg, May 4–5

IOWA, Cornell College, Mount Vernon, April 27–28

KANSAS, Topeka, January 20

KENTUCKY, University of Kentucky, Lexington, April

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Baltimore, May 12

MINNESOTA, St. Paul, May 27

MISSOURI, University of Missouri, Columbia, November 30–December 1

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April

SOUTHEASTERN, Agnes Scott College, Decatur, March 10

TEXAS, Houston, December 1–2

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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EIGHTH REGULAR MEETING OF THE OHIO SECTION.

The eighth regular meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on March 30, 1923, in connection with the meetings of the Ohio College Association and allied societies. Chairman H. L. Coar presided, being relieved by Professor J. H. Weaver for an interval.

Sixty-three persons were registered, the following forty-two being members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace M. Bareis, W. S. Beckwith, R. D. Bohannan, R. L. Borger, J. R. Brandeberry, W. D. Cairns, V. B. Caris, E. H. Clarke, H. L. Coar, O. L. Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, G. P. Harmount, H. W. Kuhn, H. B. Lemon, Anna D. Lewis, L. P. Loomis, E. S. Manson, C. N. Mills, C. N. Moore, C. C. Morris, M. A. Nordgaard, J. R. Overman, A. D. Pitcher, S. E. Rasor, P. L. Rea, Bernice Sanders, Hazel E. Schoonmaker, W. G. Simon, S. A. Singer, C. E. Stout, M. O. Tripp, J. H. Weaver, R. N. Wildermuth, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

At the business session, the secretary reported a membership of eighty-five and ten institutional members as against seventy-eight and nine, respectively, last year. Officers elected for this year are: Chairman, Professor W. E. ANDERSON, Miami University; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; Third member of the executive committee, Professor V. B. CARIS, Ohio State University. A program committee with continuity of organization was created, Professor Kuhn being chosen for three years service, Professor C. N. Moore, two years, and Professor R. B. Allen, one year. Following an inquiry from the University of West Virginia, it was voted to invite the mathematicians of that state to join in the meetings of the Ohio section. A collection amounting to $19.75 was taken for the uses of the section. The financial situation of the section is satisfactory.

The section dinner, with fifty present, served in the dining room of Campbell Hall, was very successful. The evening session was held in the same room.

The following eleven numbers were presented at the two sessions:

(1) Chairman's address: "Freshman mathematics in the liberal arts college" by Professor H. L. COAR.

Discussion: Professor W. D. CAIRNS.

(2) "On the tangency of circles" by Professor C. N. MILLS.

(3) "Some short methods for the solution of certain differential equations" by Professor R. D. BOHANNAN.

(4) "Origin and development of our present method of extraction of cube and square roots" by Professor M. A. NORDGAARD.

(5) "Industrial mathematics" by Professor J. B. BRANDEBERRY.

(6) "Formulas pertaining to friction on helical gears" by Professor C. O. WILLIAMSON.

(7) "Collegiate Euclidean geometry" by Professor M. O. TRIPP.

(8) "The DuBois-Reymond versus the Cantor-Dedekind theory of the irrational number" by Professor S. E. RASOR.

(9) Reports of the Yanney committees on the mathematical situation in Ohio. Professor C. N. MOORE, general chairman.

(10) Discussion: "Why elect mathematics?"

(11) "Number of hours required for a mathematical major in Ohio colleges" by Professor C. N. MILLS.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In his paper on freshman mathematics Professor Coar said: Recent years have seen considerable progress in the teaching of mathematics in college, especially along the line of bringing mathematics into touch with science, business, and life in general. That freshmen in our liberal arts colleges are expected to continue at once the task of acquiring further mathematical technique, without having an opportunity to pause and first secure some mathematical perspective, seems psychologically unsound and is one of the chief reasons why such a small and apparently decreasing per cent. of students elect mathematics. There seems, in this respect, to exist a gap between high school and college mathematics. Lack of some clear knowledge of our elementary number system, too little familiarity with the language of mathematics and its interpretation, a rather hazy conception of how to think analytically, are a few of the elements that constitute this gap. The first semester of the freshman mathematics should be largely devoted to the bridging of this gap. Little, if any, new subject matter should be taught, but the subject matter of the elementary schools will serve as a background for the course. New interest can thus be aroused, and the student will be able to pursue advanced work with greater intelligence and hence with enhanced pleasure and more success.

Differing from Professor Coar in his estimate of the present status, Professor Cairns held that while there is relatively too much purely formal work in high school courses it does not follow that the whole of the present trend and the present experimenting is wrong. This plan would probably clear up the meaning and the bearings of secondary mathematics for the general student, but this would be at the expense of some of the subject matter of the freshman courses and would entirely shut out one of the commendable elements in present-day progress, viz., an introduction to the calculus. Teaching pupils how to think (which, to have any significance, must mean how to think in some particular fields), how to study, and how to express their reasoning clearly is now being done if teachers are at all proficient, and will not be notably improved by the substitution of partially familiar material. One great difficulty is to teach students to avail themselves of the great advantage of any symbolic language and yet to make sure that they think logically and state their arguments clearly. In the face of these criticisms, Professor Cairns expressed his conviction that the proposed plan holds large values potentially and his belief that Professor Coar's

concrete and complete formulation of his plans will offer a valuable contribution to the enlivening of college teaching.

2. Professor Mills presented a method to find the radius of a circle tangent internally to three given circles. Using the cosine theorem to find the cosine of the sum of two angles, the author arrived at a fourth degree equation. The positive root of this equation is the result required.

Note: The author also gave a solution which was being presented on the same day at the Colorado meeting by Mr. J. Q. McNatt.

3. In discussing some short methods in certain types of differential equations, Professor Bohannan showed that the "principle of continuity" as used by D'Alembert in handling linear differential equations with constant coefficients, when the roots are equal, can be extended to all the failing cases when solutions are attempted by the "symbolic method"; as also to the failing cases when the solutions in the general case take the form of series, the failing cases occurring here also when the roots of certain equations are equal or differ by a multiple of the difference of the exponents of consecutive terms of the series.

4. Professor Nordgaard stated that: Our present method of extracting arithmetical roots by the orderly evolving of the digits of the root (evolution) depends upon the inverse of the expression $a^2 + 2ab + b^2$ and $a^3 + 3a^2b + 3ab^2 + b^3$ and upon our decimal place value notation including zero. By illustrations from the source materials from Theon of Alexandria, Aryabhata, Brahmagupta, Māhāvira, Srīdhara, Bhāskara, Leonardo of Pisa, Peurbach, Chuquet, Gemma Frisius, and Cardan, he traced the gradual crystallization of our present method, including the periodization of the radicand, the trial and complete divisor, and the setting-up *schema.*

5. Professor Brandeberry's paper dealt with certain non-credit courses in mathematics given in evening school at Toledo University. These courses are offered in connection with courses given by other departments, such as the departments of drawing, electricity, etc., and are designed especially for men engaged in the industries. Certain experimental courses were described, and it was stated that as a result of the experiences at Toledo University, the best program to follow consisted of an introductory course in shop arithmetic; a course in formula work, which would enable the men to use formulas ordinarily encountered in their handbooks; a course in trigonometry, wherein is taught the solution of right and isosceles triangles, and the solution of triangles by the law of sines and cosines, by naturals; a course in practical mechanics, with emphasis on graphic solutions; a course in strength of materials given in connection with machine and tool designs.

6. The formula of Reuleaux for the work lost by friction when two gears are meshed together,

$$W_f = m\pi f\left(\frac{1}{N} + \frac{1}{N_1}\right)\frac{\epsilon}{t},$$

where $W_f =$ work lost by friction, $m =$ pressure coefficient, $f =$ coefficient of

friction, N and N_1 are the number of teeth on the two wheels, ϵ = angle of contact, t = pitch, is transformed for helical gears ($N = N_1$) into

$$W_f' = K \frac{(N + 2\cos^3\alpha)\cos\left[\gamma + \sin^{-1}\left(\dfrac{N\cos\gamma}{N + 2\cos^3\alpha}\right)\right]}{\cos\alpha},$$

where $K = mf/\cos^2\gamma$, γ = constant angle of the involute at the pitch circle, α = angle of the helix. A plot of this shows that W_f' decreases as α increases.

7. The object of Professor Tripp was to offer suggestions for emphasizing geometric instruction in college in such a way as to be of direct benefit to the prospective high school teacher. The writer insisted that geometry is of great educational value because of the splendid training it gives in the use of language; spherical trigonometry and descriptive geometry were mentioned as subjects well adapted to the development of space intuition. A course in synthetic geometry which would serve as a sequel to euclidean plane geometry and an introduction to projective geometry was recommended.

8. Professor Rasor's paper introduced the theorem on the necessary and sufficient condition for the convergence of a sequence of numbers. For the DuBois-Reymond theory this theorem was proved and the real number system considered as identical with the totality of all infinite decimal fractions. If the decimal sequence is periodic, it was shown that the number is rational and conversely, and if the sequence is non-periodic the number is irrational. On the contrary, it was pointed out that the Cantor theory starts with the consideration of sequences of rational numbers and then lays down as a definition of a regular sequence what was taken as a theorem in the other theory.

9. At the 1922 meeting, following the suggestion of Professor B. F. Yanney, chairman, six committees to study and report on the mathematical situation in Ohio were formed, with Professor C. N. Moore general chairman. Four of the committees presented reports covering much painstaking work: "College entrance requirements," Professor E. H. Clarke, chairman; "Teacher training in mathematics," Professor C. L. Arnold, chairman; "College courses," Professor W. E. Anderson, chairman; "High school courses," Professor R. B. Allen, chairman. The section expressed its thanks to these committees for their work and requested them to continue their investigations and report further. The general chairman and sub-chairman were requested to edit the reports and summarize the findings for the files of the secretary so that they may be useful to the section and also to the state department of education.

10. In response to a request from the executive committee, many members had sent in reasons why high school freshmen and college freshmen should consider electing mathematics. These were canvassed briefly and a committee consisting of Professors Armstrong, Arnold and Pitcher was appointed to prepare statements addressed to the two classes named and to undertake to place these before the public school pupils concerned before the close of this school year.

11. The basis of this report was a set of questionnaires sent to twenty-six of

the Ohio colleges and universities. The main point of interest was that the number of hours required for a mathematical major ranges from fifteen to thirty-eight hours. So wide a range indicates that the colleges are not certain as to the basic values of many courses. The author urges that more interest should be taken in geometrical subjects, such as modern geometry and descriptive geometry.

G. N. ARMSTRONG, *Secretary-Treasurer.*

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The seventh annual meeting of the Rocky Mountain section was held at the University of Colorado, Boulder, Colorado, on March 30 and 31.

The attendance was twenty-four, including the following fourteen members of the Association: I. M. DeLong, G. W. Finley, P. Fitch, J. C. Fitterer, G. W. Gorrell, C. A. Hutchinson, Claribel Kendall, O. C. Lester, G. H. Light, F. H. Loud, S. L. Macdonald, J. Q. McNatt, W. J. Risley, H. E. Russell.

The section voted to accept the invitation of the Colorado Fuel and Iron Co. to hold the next meeting at the Steel Plant in Pueblo, Colorado. Mr. J. Q. McNATT was elected chairman for this meeting.

At the close of the Friday session, the section was favored with a talk by Dr. SAUL EPSTEEN on certain phases of the Einstein Theory.

The following nine papers were read:

(1) "Certain associativity conditions in linear algebras" by Assistant Professor CLARIBEL KENDALL. (Asst. Professor G. W. SMITH, Collaborator.)

(2) "The curve of the price of lumber" by Professor S. L. MACDONALD.

(3) "The intrinsic equation for a family of curves possessing a certain property" by Professor G. H. LIGHT.

(4) "The bearing which the work in the grades has upon college mathematics" by Professor G. W. GORRELL.

(5) "Parabolic grouping of pythagorean triangles" by Professor W. J. HAZARD.

(6) "To calculate the radius of a circle inscribed in a concave triangle" by Mr. J. Q. McNATT.

(7) "A nomographic perpetual calendar" by Mr. W. K. NELSON.

(8) "Graphical methods for complex roots" by Professor W. J. HAZARD.

(9) "A game of solitaire for an arithmetician" by Dr. F. H. LOUD.

In the absence of the author, a paper by Professor C. H. SISAM was read by title only. All the papers led to considerable discussion. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

(1) In this paper it was shown that the associativity conditions

$$A_{ijkm} \equiv \Sigma(\gamma_{ijk}\gamma_{kim} - \gamma_{ikm}\gamma_{jik}) = 0 \qquad (i, j, k, m = 1 \cdots n)$$

for a linear algebra of order n are highly redundant. For $n = 2$ eight independent

syzygies among A_{ijkm} have been found. Four of these are linear and four quadratic. In the general case there will be at least $n^4 - n^3$ such relations.

(2) Taking the price of all commodities for the year 1913 as a standard, lumber included, the ratio of the price of lumber to this standard was computed for each year, beginning with 1865. Plotting a curve from the results thus obtained gave as a "most probable" equation that of a straight line. It was found that a more accurate equation was that of a parabola. Upon further investigation it was found that a logarithmic curve was a trifle better.

(3) Professor Light derived the general equation of the curves whose polars with respect to a fixed circle cut off a segment on the normal proportional to the radius of curvature. When the circle becomes a point, the intrinsic equations for logarithmic spirals, cardioids, parabolas, circles, lines and hyperbolas are obtained.

(4) It is the belief of a number of college teachers of mathematics, judged by the expressions at association meetings, that the instruction in high schools is of inferior quality. The writer believed this to be largely unfounded. High school teachers are usually well prepared to do their work. The amount of required work in high school is not sufficient to give to the high school teacher the time necessary to make much impression upon his students.

Grade teachers are not specially prepared in mathematics and are unable to give clear explanations in many instances. They are likely to fall into the practice of mere formal work in arithmetic.

A test was given to college freshmen in which problems were taken from a grade text in arithmetic. Several students did not attempt certain problems and very few attempted any analysis, though explanation was requested. The writer was inclined to believe that the student's customary approach to a problem follows his practice in the grades rather than that required in high school.

(5) In his paper Professor Hazard showed that when such triangles are plotted in the first quadrant with one acute vertex at the origin and with the base and altitude of the triangle as the coördinates of the other acute vertex, then the triangles having a common property have their vertices on a parabola. Triangles with a common unit difference between the hypotenuse and one side lie on one parabola; those with a difference of 9 lie on another parabola, etc. All possible integral differences represent, also, integral sums of the hypotenuse and one side. There are four families of parabolas, confocal at the origin, being the loci of the vertices of triangles having a constant integral odd difference, odd sum, even difference, and even sum of the hypotenuse and one side. Possible odd differences and sums are given by the squares of the odd numbers. Even differences and sums are given by twice the squares of all the integers.

(6) A unique method was shown in this paper for computing the radius of a circle inscribed in the area enclosed by three mutually tangent circles, when the radii of these circles were known.

(7) This paper showed how some equations of three or more variables may be represented very simply by the methods of Nomography, while the cartesian

representation is hard to construct and hard to read. A method was given of constructing a chart for an equation which represents the day of the week as a function of the year, month and day of the month, thus producing a graphical perpetual calendar.

(8) Professor Hazard commented on the fact that most text books of algebra give graphical methods for the real roots of quadratic and other equations, but either state or imply that the graphical method fails in the case of complex roots because the plotted curve does not cross the axis of X. When the roots of a quadratic are represented by $(r \pm ni)$ the vertex of the graph is the point (r, n^2), so it is only necessary to find the square root of the ordinate to have the roots determined. This may be done by the geometrical method. Reference was made to the straight line and circle construction for the real roots of a quadratic, used by L. E. Dickson and attributed by him to D'Ocagne and Lill. It was shown how this construction may be extended to find the complex roots of quadratics. Reference was also made to a construction given by Schultze for complex roots.

(9) This paper dealt with problems in the theory of numbers and considered the examination of primes with a view to the relation as modulus sustained by each to smaller numbers, the exponents to which the latter belong, and similar inquiries bearing upon the properties of repetends in both the decimal and binary systems. The author's examination, beginning with the smallest primes, has reached the prime, 2161.

PHILIP FITCH, *Secretary.*

MARCH MEETING OF THE SOUTHEASTERN SECTION.

The second meeting of the Southeastern Section of the Mathematical Association of America was held at Agnes Scott College, Decatur, Georgia, March 10, 1923. The meeting was held in the college chapel with Professor FIELD presiding.

There were eighty-five present, including the following twenty-four members of the Association:

D. F. Barrow, S. M. Barton, J. B. Coleman, T. R. Eagles, Floyd Field, Tomlinson Fort, Miss Leslie Gaylord, J. P. Hill, A. W. Hobbs, J. W. Hinton, Miss Ruby Hightower, J. F. Messick, Miss Fannie S. Mitchell, R. E. Mitchell, A. B. Morton, I. C. Nichols, M. T. Peed, W. W. Rankin, Jr., D. Rumble, David Eugene Smith, D. M. Smith, R. P. Stephens, A. H. Stevens, Miss Rose Wood.

The following officers were elected: R. P. STEPHENS, chairman; TOMLINSON FORT, vice-chairman; W. W. RANKIN, Jr., secretary-treasurer. The executive committee decided to hold the next meeting at the University of Georgia, Athens, Ga.

Agnes Scott College entertained the Southeastern Section at lunch after the regular program had been completed.

Professor DAVID EUGENE SMITH of Columbia University was the principal speaker. Professor Smith was brought to Agnes Scott College by the College Lecture Association. He addressed the faculty and students and many visitors

on Friday night, March 9, on the subject: "Ten Great Epochs in the Human Development of Mathematics." Preceding the address a banquet was held in the Alumnæ House in honor of Professor Smith and Professor H. E. Slaught; the latter, however, was unable to attend the banquet or the meeting of the Association.

The following program was presented:

(1) "Teaching the History of Mathematics in College" by Professor DAVID EUGENE SMITH.

(2) "Correlation of Mathematics" by Professor J. B. COLEMAN.

(3) "Some Reasons Why I Am a Devotee of Mathematics" (by title) by Professor H. E. SLAUGHT.

(4) "Unified Mathematics for Freshmen" by Professor A. W. HOBBS.

(5) "Present Reforms in College Entrance Requirements in Mathematics" by Professor DAVID EUGENE SMITH.

Abstracts of the papers 1, 2 and 4 are given below:

1. The purpose of such courses are (1) cultural, for those who are interested in mathematics and wish to know something of the growth of the subject; (2) educational, for those who propose to teach mathematics, particularly in the secondary schools and colleges. They tend to show mathematics as a steadily advancing subject and they give encouragement to those who look to its continued improvement. At Columbia University the work is open to seniors and graduates and extends over two years. The first half year is devoted to a general survey; the second half year to the history of the leading topics generally taught, and it extends through the history of the calculus. There are this year seventy enrollments in this class. The second year is devoted to individual work, the class being limited to about eight or ten. Each one in this second course works upon his subject of major interest, and the class period is devoted to reports and discussions. The libraries and private collections of material are unusually good in New York, and the speaker showed some of the equipment available for instruction.

2. Professor Coleman presented a general discussion of the vital subject of correlation, or degree of causality, with particular reference to the Pearson "product-moment" coefficient, where straight line relationship is assumed, and correlation ratios, for non-linear relationship. The Pearson coefficient between mathematical average and general average was calculated for students at each of three institutions. With liberal samples at the University of South Carolina, the University of Virginia, and Columbia University, the coefficients were found to be .475, .412 and .697, respectively, indicating a close relationship. Similar coefficients for those with high mathematical averages were found to be .542, .538 and .604, respectively. Any further inference with respect to students of exceptional mathematical ability would hardly be warranted, because of the limited data and lack of uniform variation of the coefficient with the data furnishing these results.

4. There has been a great deal of trouble among college teachers over the

question of good texts in college algebra. The trouble is due not so much to the texts nor to the poor preparation of the students as to the fact that the subject cannot be justified as a required course for college freshmen. The courses in college algebra as given add very little that is essentially new to the student's training. A great deal of it is review of the work done in the High School with the view to perfecting algebraic technique. Algebraic technique cannot be perfected in this way. There is no purpose nor utility evident in these courses and the students are "fed up" on this type of study. The best thing at this stage, according to Professor Hobbs, is a course in general mathematics that will let them see mathematics at work and not always on parade. To this end the function idea should be emphasized and illustrated in problems of finance and mechanics. These problems should not be introduced in the artificial way in which they appear in the traditional texts. Some elementary mechanics should be a part of the course, so that the students will get the physical significance of their work. The derivatives and integrals of some of the functions should be given along with elementary applications to geometry and mechanics. There is no claim here that students taking such courses will be better prepared to go ahead with advanced courses than if they take the traditional courses but there is the claim that if we are to preserve what mathematics has to contribute to a liberal education we must offer the freshmen something that is closer related to life than are theory of equations and Determinants.

W. W. RANKIN, Jr., *Secretary-Treasurer.*

GRAFTING OF THE THEORY OF LIMITS ON THE CALCULUS OF LEIBNIZ.[1]

By FLORIAN CAJORI, University of California.

Leibniz issued his first publication on the calculus in 1684, Newton in 1693. At that time both used infinitely small quantities which were dropped when comparatively small. It is one of the curiosities in the history of mathematics that this rough procedure was adopted, even though before this time theories of limits had been worked out in geometry by Giovanni Benedetti, S. Stevin, L. Valerius, Gregory St. Vincent and Tacquet, and in higher arithmetic by John Wallis. At one time Leibniz, when hard pressed by one of his critics, approached closely to the theory of limits; even to him the theory of limits seemed to be the nearest bomb-proof place of resort. But he never actually founded his calculus upon that theory.

The earliest prominent worker of continental Europe to suggest the founding of the calculus of Leibniz on the limit concept was D'Alembert. The question arises, from what source did D'Alembert derive his inspiration for that project?

[1] Read at a joint session of the Association with the American Mathematical Society and Section A of the American Association for the Advancement of Science at Cambridge, Mass., December, 1922.

He himself had been brought up on the text books of Reyneau and De l'Hospital; he was familiar with the papers of Leibniz and the Bernoullis. All these sources were based on the use of infinitely small quantities. What predecessors influenced him in the study of limits? We find an answer in D'Alembert's great article "Différentiel" in Diderot's *Encyclopédie,* 1754, where he mentions two publications on limits. One of these is Newton's *Quadratura curvarum* of 1704: D'Alembert interprets Newton's "Prime and ultimate ratios" as being really limits. The second publication is De la Chapelle's *Institutions de géométrie,* 1746, a text-book which enjoyed great popularity in France. Through De la Chapelle, D'Alembert connected with the pre-Newtonian theory of limits, as found in Stevin, Gregory St. Vincent and others.

From De la Chapelle, D'Alembert quotes the theorems that "if two magnitudes are the limits of one and the same quantity, the two magnitudes are equal to each other," also the theorem that the limit of a product equals the product of the limits.

Newton's prime and ultimate ratios do not contemplate primarily one constant which one variable approaches; the prime and ultimate ratios are ratios of two quantities just springing into being or else vanishing. Only secondarily does Newton, in applying his theory, consider in the right member of his equations what we would call the limit of a ratio.[1] Accordingly Newton in his *Quadratura curvarum* of 1704 did not quite present what is ordinarily called the method of limits. He considered the ratio of two quantities each of which was approaching the limit zero, rather than the limit of one quantity that was the ratio of the two quantities. In England it required the talent of Benjamin Robins to transform Newton's fluxions of 1704 into a calculus founded on limits; on the continent this transformation exercised the ability of D'Alembert who laid the emphasis upon the limit of a ratio, and gave only secondary consideration to the ratio of limits that were zero. D'Alembert conceives the relation of infinitesimals to limits in this manner: "One feels that the supposition made here of infinitely small quantities is only for abridging and simplifying the argument, but that at bottom the differential calculus does not necessarily postulate the existence of these quantities; that this calculus really consists only of the algebraic determination of the limit of a ratio."

In the article "Limite" D'Alembert states that one calls a magnitude a limit of another magnitude "when the second may approximate to the first closer than by a given magnitude, however small one may suppose it, without however that the approaching magnitude may surpass the magnitude that it approaches, in such a manner that the difference of such a quantity and its limit is absolutely unassignable."

From the explanation that follows it is evident that D'Alembert's variables did not reach their limits. Let us proceed to consider one of the weak points in D'Alembert's exposition of limits. He takes the parabola $y^2 = ax$ and forms the difference-quotient which in modern notation is $\Delta y/\Delta x = a/(2y + \Delta y)$. As Δy

[1] See Newton's *Tractatus de Quadratura curvarum,* Introductio, where he finds the fluxion of x^n.

approaches zero, the limit of the right member $a/(2y + \Delta y)$ is $a/2y$, which must also be the limit of $\Delta y/\Delta x$. Thus dy/dx is the limit of the ratio of $\Delta y/\Delta x$. "But, it will be said," one must make both $\Delta y = 0$ and $\Delta x = 0$, which gives $0/0 = a/2y$. "What does that mean? I answer (1) that there is nothing absurd in this, for the reason that 0/0 may be equal to anything you wish and therefore equal to $a/2y$; I answer (2) whatever the limit of the ratio of Δy to Δx, when $\Delta y = 0$, $\Delta x = 0$, this limit is not properly the ratio of 0 to 0, because this does not present a clear idea; one cannot form a ratio of two terms of which both are zero." D'Alembert suggests that the ratio of Δy to Δx be considered in the neighborhood of zero, without letting Δy and Δx become actually zero. Thus in the right member of $\Delta y/\Delta x = a/(2y + \Delta y)$, Δy is zero, but on the left side Δy and Δx do not quite reach zero. Here we have the incongruity which troubled conscientious mathematicians during more than a century. It is the same difficulty as that in the theory of prime and ultimate ratios which Newton saw full well, for in 1687 he said, "it is objected that there is no ultimate ratio of evanescent quantities, because the ratio, before the quantities have vanished, is not ultimate; and when they have vanished, is none." [1] How does Newton meet this, his own unanswerable argument? He does so by an appeal to sensuous intuition: "It might as well be maintained that there is no ultimate velocity of a body arriving at a certain place, when its motion is ended. . . . But the answer is easy; for by ultimate velocity is meant that . . . at the very instant when it arrives." [2] If "instant," as used here, is not an infinitesimal constant, the passage is difficult or impossible of interpretation. "That method," said Lagrange, "has the great inconvenience of considering quantities in the state in which they cease, so to speak, to become quantities; for though we can always well conceive the ratios of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become both nothing at the same time."

Recently the hypothesis has been advanced that Newton's definition is "logically sound, for a fluxion is defined by a *Schnitt*." [3] This explanation does not seem valid, for in effecting the Dedekind cut, the commentator uses a range of ratios which is different from the range actually described by Newton.

It seems to the present writer that Newton's ultimate ratios and D'Alembert's 0/0 left students of the calculus still in the clutches of the infinitely little. But there was no dropping of quantities because they were infinitely small as compared with others. For that reason both Newton and D'Alembert declared that the infinitesimal constant was not necessary in developing the calculus.

During the thirty years between 1754 and 1784 there appeared on the Continent more than 28 publications on the calculus.[4] It is not always easy to

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[1] Newton, *Principia*, Book I, Section I, Lemma XI, Scholium.

[2] The Latin passage is as follows: Per velocitatem ultimam intelligi eam, qua corpus movetur; nèque antequam attingit locum ultimum et motus cessat, nequa postea, sed tunc cum attingit.

[3] J. M. Child, *Mathematical Gazette*, vol. 11, 1922, p. 27.

[4] We have made extensive use of G. Vivanti's "Infinitesimalrechnung" in M. Cantor's *Vorlesungen über Geschichte der Mathematik*, Bd. 4, 1908, Abschnitt XXVI; also of Vivanti's *Il Concetto d'Infinitesimo*, Naples, 1901.

classify these with respect to the fundamental principles used. Sometimes there is a fusion of methods. Of the 28 texts a rough approximation to the facts assigns 15 to the infinitesimal method of Leibniz and De l'Hospital, 6 to limits not divorced from infinitesimal constants, 2 to Newtonian fluxions[1] expressed in the notation of Leibniz, 4 to the treatment of dx and dy as zeros, 1 (Lagrange), in 1772, to the use of algebra and function theory. The six authors avowedly using limits are Kästner (1761), Karsten (1760), Tempelhoff (1769), Martini (1761), Frisi (1782–5) and Cousin (1777). Of these Kästner and Cousin were the more important. Kästner did not sever all connection with the infinitely small. He used the limit concept for the purpose of justifying the rules relating to the infinitely small of different orders. It will be noticed that these six names are not among the great leaders of mathematical thought of that time. Euler published in 1755 his very influential *Institutiones calculi differentialis* which looked upon dy and dx as absolute zeros. Lagrange in 1761 defended infinitesimals on the theory of the compensation of errors, and in 1772 first based the calculus on algebra and Taylor's theorem. Nor were the six texts using limits popular and widely used; the most popular were those using infinitesimals. During these 30 years the theory of limits did not attract students of the calculus on the Continent as affording a haven of refuge against the assaults of logicians. And why should it? Was not 0/0 as mysterious as anything that Leibniz and De l'Hospital presented? Nor did the theory of limits, as then presented, altogether free the calculus from the infinitesimal constant.

In 1784 Lagrange induced the Berlin academy, of which he was president, to offer a prize for a satisfactory exposition of the calculus. Evidently he was not fully satisfied with his own production of 1772. The academy demanded "a clear and rigorous theory of what in mathematics is called infinity"; "the higher geometry," continued the academicians, "frequently uses infinitely great and infinitely small quantities; but the ancient scientists have carefully evaded the infinite, and a few celebrated analysts of our day admit that the words infinite magnitude are contradictory. Therefore the Academy demands that it be explained, how from an inconsistent assumption so many correct theorems have been deduced, and that a secure and clear fundamental concept be given which can take the place of the infinite without making the calculus too difficult or too long."

The prize was won by S. A. J. Lhuilier who in an earlier book had used indivisibles, but now (1786) proceeded to modify the ancient method of exhaustion so as to make it a sound foundation for the calculus. We pause a moment to consider the Greek method of exhaustion. This method employed all the machinery of the modern theory of limits as ordinarily presented. Antiphon originated the idea of exhausting the circular area by means of inscribed regular polygons with an ever-increasing number of sides. Upon this Eudoxus founded

[1] The only text on the calculus known to me as written and printed on the continent, which uses the notation of Newton exclusively and develops the subject in the manner of the English mathematicians of the eighteenth century, is the *Fluxie-Rekening* of Arnoldus Bastiaan Strabbe, Amsterdam, 1799.

his method of exhaustion, that was employed later by Euclid and Archimedes. Euclid proves (XII, 2) that two circles are to each other as the squares on their diameters, by inscribing a square, then successively doubling the number of sides a finite number of times until the difference between the circle and the last regular polygon constructed is less than a pre-assigned value. This procedure is part of our modern process of showing that a constant is the limit of a variable. But, strange to say, the Greeks do not pass, as we do, from the variable to its limit. They are like a man who undertakes to cross a ditch by a running broad jump, but who abruptly stops short at the brink and finally arrives at his destination by a tedious circuitous route. They had all the modern machinery, but failed to make the most direct use of it. Certain recent writers [1] declare that the ancients did not possess the idea of a variable capable of taking any one of an unlimited number of values, that their process did not exhaust the area, and that the method of exhaustion is erroneously attributed to the Greeks.[2] This view seems to rest upon a misapprehension. With only a finite number of polygons to choose from, Euclid could not have proved his theorems. For in the epsilon-test the pre-assigned value may be any small value and it might be less than the difference between the circle and the largest of the given polygons; in such a case the *reductio ad absurdum* proof of Euclid would become impossible. Only with an unlimited number of available polygons could Euclid complete his argument. But in that case he is exhausting the circle; the method of exhaustion was truly a Greek possession.

Returning to Lhuilier, we observe that his definition of a limit makes the variable always greater or always less than the limit; the variable could not oscillate to values above and below the limit. However, Lhuilier is more liberal than D'Alembert in disregarding the mooted question whether the limit is reached or not. Lhuilier establishes a procession of theorems on limits with a care never before displayed on the Continent, but some of his demonstrations involving infinite series would now be considered insufficient. He avoids the infinitely small. He starts with an algebraic treatment of differentiation and introduces geometric figures later. He considers lim $\Delta y/\Delta x$ as the limit of one variable, not the ratio of the zero-limits of two variables, and thereby takes an important step in advance of D'Alembert. Lhuilier's diffuse monograph did not greatly impress the mathematicians of the time. There is nothing to indicate that it was read as widely as it deserved to be read.

Eleven years passed. The year 1797 brought three events of importance: Lagrange's publication of his *Théorie des fonctions*, Lazare Carnot's *Réflexions sur la métaphysique du calcul infinitésimal*, and S. F. Lacroix's two treatises on the calculus. All three authors were Frenchmen. Lagrange rejects infinity and infinitesimals and, using only the imperfect algebra of his day, endeavors to

[1] C. R. Wallner, *Bibliotheca Mathematica*, Series 3, vol. 4, 1903, p. 250; H. Suter, *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, vol. 63, 1918, pp. 226–228.

[2] Wallner, *loc. cit.*, p. 252, says: "Es ist dabei wohl zu beachten, dass dieser Begriff des Ausschöpfens vor Gregorius nirgends vorkommt, weshalb das Verfahren der Alten ganz mit Unrecht so häufig als Exhaustionsmethode bezeichnet wird."

establish Taylor's theorem and base the calculus upon it. Interesting is Lagrange's characterization of the methods used by his predecessors. He [1] says that "one knows the difficulties offered by the supposition of the infinitely little," that Euler's interpretation of dx and dy as zeros "reduces their ratio to the vague and unintelligible expression zero divided by zero." [2] In one place Lagrange objects to limits as applied to sub-tangents, for the reason that "the sub-tangent is not strictly the limit of the sub-secant, since nothing prevents the sub-secant from continuing to increase after it has become a sub-tangent." [3] Evidently Lagrange had here in mind a more general law which governs the variable than those allowed in the treatment of limits, for he proceeds to state that "veritable limits are quantities which one can not pass, although one can approach them as near as one wishes." He acknowledges that after the manner of Maclaurin, D'Alembert and others, one can rigorously demonstrate the principles of the calculus with the aid of "limits considered in a particular manner," but he adds that "the kind of metaphysics that one is to employ, if not contrary, is at least foreign to the spirit of analysis which does not need any other metaphysics than that which consists of the first principles and the fundamental operations of calculation." [4] As Camille Jordan [5] expressed it, arithmetic and algebra allow four fundamental operations: addition, subtraction, multiplication and division. Passing to the limit, where a variable is replaced by its limit, is a fifth operation. To Lagrange this fifth operation seemed superfluous, if not actually foreign to the spirit of analysis. Hence, his program to eliminate limits from analysis altogether.

Still more unfriendly to the theory of limits was Lazare Carnot. His famous *Réflexions* were written at a period full of political turmoil. That circumstance explains perhaps why he did not familiarize himself with the work of all of his predecessors—that of Berkeley, Robins, Maclaurin and others. The main point of the first edition of his tract is the "compensation of errors" in the system of Leibniz, an explanation which he believed to be new. As a matter of fact, he was anticipated by Berkeley, Maclaurin, and Lagrange. Carnot discusses limits more from the standpoint of D'Alembert than of Lhuilier; he fails to eliminate that troublesome 0/0. He decides in favor of the infinitesimals of Leibniz. Referring to limits he asks, "shall we prefer a thorny path in which it is difficult to avoid being bewildered, to the plain and easy road by which this analysis conducts us to discoveries?" Carnot's book was widely read, but it was not a great contribution to the better understanding of the philosophy of the calculus.

Friendly toward the theory of limits were Lacroix's two treatises, his large *Traité du calcul différentiel et du calcul intégral* (1797–1800) in three ponderous volumes, and the briefer treatise, the *Traité élémentaire de calcul différentiel et*

[1] Lagrange, *Oeuvres*, vol. 7, p. 325.

[2] *Ibid.*, p. 325.

[3] "parce que rien n'empêche la sous-sécante de croître encore lorsqu'elle est devenue sous-tangente." *Ibid.*, p. 325.

[4] *Ibid.*, p. 326.

[5] C. Jordan, *Cours d'analyse*, vol. I, 3. éd., Paris, 1909, p. 15.

de calcul intégral. In the larger treatise the author gives great prominence to Lagrange's point of view as found in his paper of 1772, but in the introduction Lacroix gives also the method of limits. In the briefer treatise he begins with the theory of limits, a theory which, as he says, "best reconciles brevity with rigor in reasoning." [1]

The briefer text enjoyed much greater popularity than the larger text. It was at once translated into German by Grüson; in 1816 it was translated into English. In France the briefer treatise was used during three fourths of a century; the eighth edition appeared in 1874, edited by C. Hermite and J. A. Serret. In explaining the derivative, Lacroix dwells upon the limit of the ratio, but unfortunately also upon the ratio of the limits when the two limits are zero. He discusses 0/0 also in his *Élémens d'Algèbre* and thus unnecessarily stresses this symbol which was casting disrepute upon the logical foundations of the calculus.

At the opening of the nineteenth century the conceptions of limits, as held by the mathematical public, labored under at least seven defects.

The first criticism is that the fixed infinitesimal frequently appeared in the early nineteenth century treatment of limits. Cauchy,[2] who gave a great impetus to the use of limits, defined in 1821 a limit thus: "When the successive values attributed to a variable approach a fixed value indefinitely, so as to end by differing from it as little as wished, this fixed value is called the limit of all others." The words "approach . . . indefinitely" would be objected to by Whitehead [3] on the ground that they apparently suggest infinitely small quantities. But in the actual determination of limits Cauchy uses (p. 29) the epsilon mode of proof. Cauchy was perfectly clear and avoided infinitesimal constants. The same cannot be said of many other writers. The phrase "approaches indefinitely near" ran through many texts of the last century,[4] a dark stream rendering turbid the courses of mathematical reasoning. B. Williamson [5] uses the theory of limits and at the same time defines the differential dx as "less than any assigned quantity, however small." In this interpretation, Williamson follows L. M. H. Navier's *Leçons d'analyse.* A. N. Whitehead [6] and Bertrand Russell [7] both declare that it was K. Weierstrass who banished the infinitely little from the calculus. Although one must acknowledge the wide influence wielded by Weierstrass, this statement is not historically correct; for as early as the eighteenth century Robins, Maclaurin and Lhuilier demonstrated that the calculus can be built up without the use of the fixed infinitesimal. It is true that there were nineteenth-century texts before Weierstrass that used the theory of limits and yet fondled the mystic infinitesimal, just as there were texts after Weierstrass that yielded to the enticing simplicity of these creatures of more than electronic minuteness.

[1] S. F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, Paris, 1810, vol. I, p. xxxv.
[2] Cauchy, *Cours d'Analyse*, Paris, 1821, p. 4.
[3] A. N. Whitehead, *Introduction to Mathematics*, New York, 1911, p. 226.
[4] See, for instance, W. Whewell, *The Doctrine of Limits*, Cambridge, 1838, p. 18.
[5] B. Williamson, *Differential Calculus*, New York, 1884, p. 10.
[6] A. N. Whitehead, *op. cit.*, p. 226.
[7] Bertrand Russell, *Introduction to Mathematical Philosophy*, Cambridge, 1919, p. 107.

A second defect in the early nineteenth-century treatment of limits was the failure to eliminate 0/0 from the explanation of the fundamental concept of a derivative. Bledsoe,[1] a West Point graduate and Professor at the University of Virginia, pronounced this symbol "the most formidable of all the symbols or enigmas in the differential calculus $\cdots$ which has kept thousands from adopting that method." The failure to exclude division by zero and the consequent failure to observe the proper limitations of the assumption that the limit of a quotient of two variables is equal to the quotient of the limits prevailed in many books that appeared even later than the third quarter of the nineteenth century. The Leibnizian notation for the first derivative was peculiarly adapted to encourage this mysticism. Bledsoe in his *Philosophy of Mathematics* criticizes Charles Davies for considering two kinds of zeros and gives this matter prolonged consideration without being able himself to throw much light. The treatment of the first derivative in modern books considers simply the limit of the quotient.

A third defect was the absence of a scientific number-system which made itself felt in the course of the arithmetization of mathematics. From this standpoint a variable involved a series of *numbers*, not of *quantities*. A scientific treatment of variables and limits presupposes a satisfactory theory of real numbers, including the irrational. Such number-systems were worked out through the labors of Méray, Weierstrass, Dedekind and others, since the year 1869.

A fourth defect was a failure to consider the question of the existence of a limit. Fortunately, the early studies related to functions of comparatively simple character, "vernünftige Funktionen," so that correct results were usually obtained.

A fifth imperfection of long standing was the adherence by many writers to the famous but faulty doctrine that "what is true *up to* the limit is true *at* the limit." English and American writers usually attribute this doctrine to William Whewell,[2] but De Morgan[3] ascribes it to S. D. Poisson. It was used in the eighteenth century, without being at that time specifically formulated.[4] It was accepted by DeMorgan and in the United States by Davies and Peck.[5] It was discussed by Bledsoe[6] and defended by Simon Newcomb[7] against the attacks upon it made by C. H. Judson,[8] but Newcomb excluded from it the consideration of descriptive relations and limited it, as did Poisson, to quantitative relations.

A sixth defect of the theory of limits, as it existed in the first half of the nineteenth century, was the failure to observe uniform convergence. The

[1] A. T. Bledsoe, *Philosophy of Mathematics*, Philadelphia, 1867, p. 215.

[2] W. Whewell, *Doctrine of Limits*, Cambridge, 1838, p. 21.

[3] A. De Morgan, *Transactions of the Cambridge Philosophical Society*, vol. 8, Part 2, 1844, p. 192.

[4] See, for instance, Guido Grandi, *Quadratura circuli et hyperbolae*, Pisa, 1703; G. W. Leibniz, *Acta eruditorum*, 1713, vol. 5, Suppl. pp. 266, 267, 269. These writers discuss $1 \div (1+x) = 1 - x + x^2 - x^3 + \ldots$, when $x = 1$.

[5] Davies and Peck, *Mathematical Dictionary*, Art. "Infinity."

[6] Bledsoe, *op. cit.*, pp. 35, 36.

[7] S. Newcomb, *Analyst*, vol. 9, Des Moines, Iowa, 1882, pp. 114–115.

[8] C. H. Judson, *Analyst*, vol. 8, 1881, p. 109.

removal of this subtle blemish is due to the keen insight of G. G. Stokes [1] in 1847, P. L. Seidel [2] in 1848 and A. L. Cauchy [3] in 1853.

The last imperfection in the treatment of limits that we mention here was the most serious to the rank and file of mathematicians. There were conflicting opinions on the question whether a variable could reach its limit. There is involved in this a conflict between imagination and thought. Can any thing be true which we cannot imagine? Can there be coherent thinking on processes which transcend the imagination? The question is brought to a sharp issue in Zeno's arguments against motion. Upon it depends the race-track controversy as to whether Achilles ever caught the tortoise. If we admit that there are logical processes of the mind which transcend our imagination, then the tortoise is caught after a brief race, then we can explain motion, we have a mind process which is able to interpret what goes on in nature, we can say with Newton, "these generations really take place in the nature of things and are daily seen in the motion of bodies." [4]

As expressed by Couturat, the nerve of the argument that the limit cannot be reached consists in the axiom which has been accepted as evident by many good heads: "The actual infinite cannot be realized." Any movement must contain an actual infinity of parts, which some thinkers have declared impossible.

In the eighteenth century the question whether the limit can be reached was hotly debated by B. Robins and J. Jurin. D'Alembert declared that the limit was not reached and thereby assumed an attitude which has pedagogical advantages over that which transcends the imagination and thereby perplexes the young mind. In the nineteenth century the matter continued to be debated. Cauchy put no restriction upon variables reaching their limits. In 1817 Bolzano,[5] whose writings did not at the time receive the attention they deserved, was concerned with the limits of continuous functions which attain their limits. In Germany, Klügel [6] places no restriction in his definition, but, in the comments which follow, the variable is pictured as not reaching its limit. In 1871 Hermann Hankel [7] starts out in his article "Grenze" by defining limit so that it is not reached; he uses infinitely small quantities. But to remove certain contradictions, he proceeds to develop a new definition which is free from restriction as to the attainment of the limiting value. In France, J. M. C. Duhamel [8] held that the limit is never actually attained. This same view was adopted in England by De Morgan [9] and Todhunter,[10] and in the United States by Bledsoe.

[1] G. G. Stokes, *Transactions Cambridge Philosophical Society*, vol. 8, 1847, p. 533.

[2] P. L. Seidel, *Abhandlungen der Math.—Phys. Classe d. Bayer. Akademie*, München, vol. 5, 1850, p. 381–393.

[3] A. L. Cauchy, *Comptes Rendus*, vol. 36, Paris, 1853, pp. 454–459.

[4] ". . . hae geneses in rerum natura locum vere habent et in motu corporum quotidie cernuntur." From Newton's *Quadratura curvarum*, 1704, Introduction.

[5] P. E. B. Jourdain, "The Development of the Theory of Transfinite Number," *Archiv der Mathematik und Physik*, vol. 14, 1909, p. 297.

[6] G. S. Klügel, Art. "Grenze" in *Mathematisches Wörterbuch*, Leipzig, 1805, vol. 2.

[7] H. Hankel, art. "Grenze" in *Ersch u. Gruber, Encyklopaedie*, Series I, vol. 90, 1871.

[8] Duhamel, *Éléments de calcul infinitésimal*, vol. 1.

[9] A. De Morgan, "Limit," *Penny Cyclopaedia*, 1839.

[10] I. Todhunter, *Differential Calculus*, 7th Ed., London, 1875, p. 6.

On the other hand, William Whewell[1] in England and Charles Davies,[2] another West Point graduate and professor at that institution and at Columbia, held that the variable does reach its limit. Davies says, "if these two quantities are thus to be separated, how can they be brought under the dominion of a common law, and enter together in the same question?"

I illustrate the difficulties encountered by reciting views expressed in this country only forty years ago. If half a debt is paid, then half the remaining debt, and so on, Simon Newcomb argued that "since the debt is halved at every payment, if there was any payment which discharged the whole remaining debt, the half of a thing would equal the whole of it which is impossible." "This is fallacious by proving too much," says De Volson Wood,[3] "Thus, by precisely the same argument, it may be proved that the minute hand of a watch can never overtake the hour hand, that two intersecting right lines can never intersect, that bodies at rest can never move, etc." Newcomb[4] taught that the limit cannot be reached; hence, he said, uniform motion "entirely removes the problem from the class to which the method of limits applies." De Volson Wood declares that regular polygons inscribed in a circle reach their limit under the following conditions: "If the apothem of an inscribed square should grow in length at a uniform rate, and if polygons of double the number of sides be conceived to be instantly described as the growing line becomes the apothem of an octagon, then of a polygon of 16 sides, and so on, the inscribed polygons will reach the circle in a finite time." These are difficult questions on which not all mathematicians agree even at the present time. These are real difficulties that perplex students, and add mystery to the theory of limits.

Modern authors relieve the student's mind of this perplexity by ignoring altogether the question whether the limit is reached or not. It is now seen that the question is one of pure assumption. If we choose to make the limit a term in the range of the variable, the limit is reachable; otherwise it is not reachable. Thus what was a brain-harrowing question for centuries now causes no more mental strain than does the deciding upon an extra lump of sugar for our tea. But the advances in the theory of limits due to Weierstrass and some of his followers have brought changes far more profound. The very idea of a variable has been remodeled. While capable of performing certain acts, there is one feat it can not perform: *the variable cannot vary.* Of a certain number of prescribed values, it takes, not *every one* collectively, but *any one* disjunctively. In general the concept of a variable moving from one value or position to another is abolished. The modern variable has been stripped of every dynamic idea. No longer does it have to pass over an infinite number of steps in a finite time, or else wearisomely keep on stepping forever and ever. This modern theory of limits does not consider directly the question how the minute hand catches up with the hour hand. There is no attempt to reproduce in thought the image of

[1] W. Whewell, *Doctrine of Limits*, Cambridge, 1838, p. xx, 18.
[2] Charles Davies, *Nature and Utility of Mathematics*, New York, 1873, p. 326.
[3] De Volson Wood, *Analyst*, vol. 9, 1882, p. 80.
[4] S. Newcomb, *Analyst*, vol. 9, 1882, p. 115.

what actually transpires in the motion of physical bodies. Some modern pure mathematicians listen with impatience when attempts are made to harness their theory of limits with time and space, with velocity and acceleration and other motions relating to the physical universe. They do not care whether Achilles won the race against the tortoise or not, for this paradox is due to a lack of precision of statement as to what constitutes the range of the variable. Theirs is a beautiful, but very abstract theory, possessing great logical perfection, involving concepts of "class," "range of variable," "neighborhood," "difference," "sequence." We fear that this great gain has been made at considerable sacrifice of objectivity. These modern pure mathematicians cannot say with Newton: "These generations really take place in the nature of things and are daily seen in the motion of bodies." This modern limit-theory cannot be recommended to beginners of the calculus. The physical notion of "velocity" and the "slope of a curve" must be retained as great aids to the young student.

Some defects of the old theory of limits were truly thorns in the flesh. For that reason it is not strange that the infinitely little constants of Leibniz, admitting of great simplicity of manipulation, were actually preferred by many able nineteenth-century mathematicians. S. D. Poisson [1] used infinitely small quantities in his *Traité de mécanique* and he gave what seemed to him valid reasons for the real existence of such quantities. A. A. Cournot in 1841 gave what has been called the clearest philosophy of infinitesimal analysis that has ever been written.[2] Augustus De Morgan in London, and Bartholomew Price at Oxford believed in infinitesimals. Benjamin Peirce, the greatest pure mathematician of America before 1880, was an enthusiastic champion of the fixed infinitesimal. He said "With all the boasted rigor, the ancient Geometry can indeed lead to no result more accurate, none more to be depended upon, than those of the infinitesimal theory: and I doubt if any well constituted mind, well constituted at least for mathematical investigations, ever reposes with more confidence upon the one than the other." However, the consensus of mathematicians now favors the method of limits as more nearly meeting the ideals of logical rigor. While it has saved the calculus from being considered only a crude system of approximation, we can not deny that the treatment involving limits, even in its modern perfected form, lacks directness and simplicity, as compared with the system of infinitesimals. In fact all experience testifies that it would be a pedagogical mistake for us to confine ourselves to the exclusive use of the derivative and to discard the differential. Lhuilier is the only prominent Continental writer who pursued that course. Nineteenth-century writers using the method of limits have habitually employed differentials as well. That is to say, from the derivative $dy/dx = f'(x)$, they passed freely to the differential expression $dy = f'(x)dx$. But great diversity prevailed on the explanation of the differential. How can it be brought into consistent relation with the derivative? The view that dx

[1] S. D. Poisson, *Traité de mécanique*, 2. Éd., vol. 1, Paris, 1833, p. 14.

[2] Max Simon, *Abhandlungen zur Geschichte der Mathem. Wissenschaften*, vol. 8, Leipzig, 1898, p. 128.

and dy were zeros was not satisfactory. With many writers dx and dy were fixed infinitesimals, the very kind of quantities which the method of limits aimed to eliminate. Some looked upon dx as an arbitrary finite increment of x, and dy as the corresponding increment of $f(x)$—a view that was in conflict with the definition of the first derivative. Only in this century have our text books reached substantial agreement by accepting as satisfactory an explanation given about a century ago by Lacroix and Cauchy. In 1797 Lacroix, after defining a derivative as a limit, defined a differential dy as being only a portion of the increment of the function; namely, that portion represented by the product of $f'(x)$ and dx. Cauchy[1] brings out very clearly that the differentials thus defined have a ratio that is equal to the first derivative; that dy is completely determined when one has found the first derivative and agreed upon the value of dx; that dx is purely arbitrary and may be taken to be a finite constant or a variable infinitesimal.

In brief retrospect, we see on the European continent the grafting of the Newtonian and pre-Newtonian concepts of limits upon the calculus of Leibniz; we see during the nineteenth century the gradual elimination of the mystic features and the hampering restrictions that clustered about the limit concept; we see the changes from a naïve treatment to severe arithmetization. Enough of the modern movement is embodied in our elementary text books to relieve the mind of the embarrassment which the older defective theory engendered.

THE FOUR-COLOR PROBLEM.[2]

By H. R. BRAHANA, University of Illinois.

1. Origin. Of the origin of the problem, A. B. Kempe[3] in 1879 says: "It has been stated somewhere by Professor De Morgan that it has long been known to map-makers as a matter of experience—an experience however probably confined to comparatively simple cases—that four colors will suffice in every case. . . . Whether this statement was one merely of belief, or whether Professor De Morgan, or anyone else, ever gave a proof of it, or a way of coloring any given map, is, I believe, unknown; at all events, no answer has been given to the query to that effect put by Professor Cayley to the London Mathematical Society on June 13, 1878." A later reference to an earlier date is made by Guthrie[4] in which he says, "Some thirty years ago . . . my brother," who had been attending Professor De Morgan's class, "showed me the fact that the greatest necessary number of colors . . . is four." Professor De Morgan was pleased with the result and was in the habit of acknowledging to his classes the source of his information. The brother's proof, however, "did not seem altogether satisfactory to himself."

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[1] Cauchy, *Leçons sur le Calcul Différentiel*, Paris, 1829; *Oeuvres*, 2. Serie, vol. 4, p. 289.

[2] For additional references, see *Encyklopädie der Math. Wiss.*, III, AB3; also Errera (*cf.* reference § 5).

[3] *American Journal of Mathematics*, 2 (1879), p. 193.

[4] *Proceedings of the Royal Society of Edinburgh*, 10 (1880), p. 727.

The problem is still unsolved. It has afforded many mathematicians experience and very little else.

2. Statement of the problem. The problem may be stated as follows: Given on the surface of a sphere any map, each region of which is such that any two points on its boundary can be joined by an arc made up wholly of points of the boundary, is it possible to color the map with four colors so that no two regions which touch along an arc have the same color?[1]

3. The problem in general. It is obvious that there would be a similar problem for a surface of genus 1, the anchor ring; another for a surface of genus 2, and so on. A natural conclusion to draw off-hand would be that if the problem in the case of the sphere is difficult, the similar problems would become increasingly more difficult as the value of the genus increases. This conclusion is contrary to experience. The problem has been completely solved for each value of the genus p, between 1 and 6, and the way pointed out for $p > 6$. Heawood[2] solved the problem for $p = 1$, and indicated the method of procedure for each other case. His method is given here because the point of difference between the problem on the sphere and that on a surface of higher genus is interesting. First we need only consider maps on which there are three and only three lines at every vertex—such a vertex is said to be of *degree* 3. For, suppose P (Fig. 1) is a point at which there are n ($n > 3$) lines, we may move one end of a along b to the position indicated by the dotted line. This introduces a new vertex of degree 3 and reduces to $n - 1$ the degree of P. It is evident that if the second map M_1 can be colored with k colors, the first one M may be given the same colors, for on returning from M_1 to M we introduce no new contacts between regions. Now if M_1 contains a vertex of degree n greater than 3, we may obtain an M_2 that has one more vertex of degree 3 and has a vertex of lower degree in place of the vertex of degree n in M_1. This may be continued until there are no vertices of degree greater than 3. The system of vertices and lines joining pairs of vertices is called a *linear graph*. If the vertices are all of the same degree the graph is *regular*, and the degree of a regular graph is the same as the degree of one of its vertices. We may note that in a regular graph of degree 3, $2E = 3V$, where E is the number of lines and V is the number of vertices.

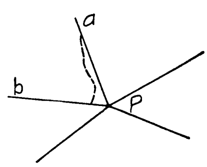


FIG. 1.

Next Heawood obtains a number that he calls the *average number of contacts* for the map. This is $2E/F$ where F is the number of regions in the map. Since each line is on the boundary of two regions, it is evident that if we count the number of sides[3] of each region and add all these numbers, the sum will be $2E$. Now, making use of the generalized Euler formula $V - E + F = 2 - 2p$, where p is the genus of the surface, and of the relation $2E = 3V$, we get an expression

[1] We do not consider the problem of giving the same color to two distinct regions; we consider no countries of more than one region.

[2] *Quarterly Journal of Mathematics*, 24 (1890), p. 332.

[3] By *side* of a region we mean an arc along which it borders a single region.

for the average number of contacts in terms of p and F as follows:

$$\frac{2E}{F} = 6 + 12\frac{(p-1)}{F} \cdot$$

For each value of p it is possible to find a value for F that makes $2E/F$ a maximum. The smallest integer n_p greater than this maximum is evidently a maximum for the number of colors required for any map on a surface of genus p. For, consider the map M of k regions where k is a number such that any map of fewer than k regions can be colored with n_p colors; M must contain at least one region of fewer than n_p sides since n_p is greater than the average. Let a_1^2 be such a region and a_1^1 one of its sides with ends at a_1^0 and a_2^0.[1] Let a_2^1 and a_3^1 be the other two lines with ends at a_1^0, and a_4^1 and a_5^1 the lines with ends at a_2^0. Then by removing the side a_1^1 replacing $a_2^1a_1^0a_3^1$ by a line a_6^1, and $a_4^1a_2^0a_5^1$ by a line a_7^1 we get a new map whose linear graph is regular and which contains one fewer regions than the original map. This map may therefore be colored with n_p colors; only $n_p - 1$ of them can appear about the region a_1^2, and so we may return the region a_1^2 giving it the n_pth color. Obviously $k > n_p$, so our induction is complete.

The following are values of n_p for small values of p. $n_0 = 6$, $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, $n_4 = 10$, $n_5 = 11$, $n_6 = 12$, $n_7 = 12$, $n_8 = 13$, $n_9 = 13$.

Next Heawood sought a minimum m_p for the number of colors that would be required that one might be assured that any map on a surface of genus p could be colored with m_p colors. His method is to determine a number m_p which is such that the average number of contacts is $m_p - 1$ (in general, a map for which this is true does not have a regular graph). The proof of the validity of m_p for a minimum is complete only when a map of m_p regions each touching each other region on a surface of genus p is given. He gave the map for $p = 1$.[2] The value of m_0 is 4, $m_1 = 7$, $m_2 = 8$, and he showed that this number m_p is the same as n_p for $p > 0$. The sphere is the only surface for which the two do not agree.

4. Kempe's solution. Not only did Heawood treat rather successfully the problem for surfaces of higher genus, but he pointed out an error in the solution of the problem for the sphere that had been given by A. B. Kempe (*cf.* § 1) and that had been generally accepted as correct. Kempe showed that we need consider only maps whose graphs are regular and of degree 3. He introduced the notion of *chains* which has probably come in for consideration by everybody who has worked on the problem since.

The work which follows uses the same considerations as Kempe used but is in a different form. We will use a method of induction. Four colors suffice for any map of four regions. Then either four colors suffice for any map, or there exists a map of k regions that cannot be colored with four colors, where k is such

[1] By this notation we designate with a superscript the dimensionality of the element,—thus, a_i^0 is a point, a_i^1 is an arc, a_i^2 is a region.

[2] Heffter gave the maps for $p = 1, 2, \cdots 6$. *Mathematische Annalen*, 38 (1891), p. 477. Another set of "maps of verification" is given for $p = 1, 2, 3$, by the author in a paper soon to appear.

a number that any map of fewer than k regions can be colored. We investigate this hypothetical map M of k regions.

Now M contains no two-sided region, also no triangular region, for if it does contain one of either kind, we remove one side and combine it with one of the surrounding regions; the resulting map can be colored. No more than 3 colors are used around the region. We introduce it again giving it the fourth color and so get M colored in four colors.

Likewise, it can have no four-sided region. Here we make use of the notion of *chains*. Suppose M has a four-sided region. We remove one side of it combining it with one of the surrounding regions. The resulting map can be colored. The regions surrounding the four-sided one may be colored in order A, B, A, and C, in which case we give the fourth color D to the four-sided region. Or they may be colored A, B, C, and D (Fig. 2). In this case let us consider the group of regions, each region having the color A or the color C, that can be reached starting with region a_1^2 of the figure and adding to it every C region which touches it, then adding each A region that touches one of these C regions, and so on. This group of regions Kempe called an *AC chain.* Two AC chains can have no vertices in common, and no regions in common. No BD chain can cross an AC chain.

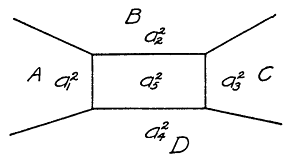


FIG. 2.

Now either the AC chain beginning at a_1^2 contains region a_3^2 or it does not. If it does, then the BD chain beginning at a_2^2 does not contain a_4^2, and if we interchange B and D on that chain we have but three colors about a_5^2 and may give it the color B. In the opposite case we interchange colors A and C on the chain starting at a_1^2 and give a_5^2 the color A.

Next, as Kempe showed, there must be five-sided regions in M. Let a_i be the number of i-sided regions in the map, and suppose there is no region with more than l sides. Then $\sum_{i=2}^{l} a_i$ is the number F. Also, $2E = \sum_{i=2}^{l} i a_i$, for the latter is the sum of the sides of countries, which gives every line twice. Also $2E = 3V$. Using the Euler formula, we get

$$4a_2 + 3a_3 + 2a_4 + a_5 = 12 + \sum_{i=6}^{l} (i - 6)a_i.$$

We have already shown that $a_2 = a_3 = a_4 = 0$, so a_5 is at least 12.[1]

Let us make a short digression to observe that five colors are sufficient for any map on a sphere. It is evident that the minimum map which cannot be colored in five colors must contain a five-sided region. Let one such region a_1^2 be removed as above and let the map be colored. The only possibility that need engage our attention is the case where the regions $a_2^2 \cdots a_6^2$ about a_1^2 have the colors A, B, C, D, and E respectively. The AC chain beginning at a_2^2 contains a_4^2 or does not. In the first case the BD chain beginning at a_3^2 does not

[1] By considering the number of lines, Wernicke (*Mathematische Annalen*, 58 (1901), p. 413) shows that there must be a pentagon adjacent to another pentagon, or else to a hexagon.

contain a_5^2, and after permuting B and D on this chain we may give the color B to a_1^2; in the opposite case we interchange A and C on the AC chain beginning at a_2^2 and give a_1^2 the color A.

Returning to our problem, let a_1^2 be the five-sided region (Fig. 3) surrounded by $a_2^2 \cdots a_6^2$. The only case to consider is the one where $a_2^2 \cdots a_6^2$ take the colors A, B, A, C, and D respectively.

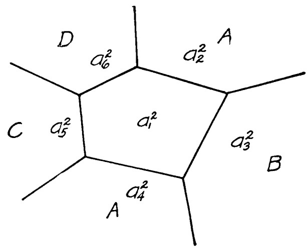


FIG. 3.

First let us notice that the BC chain beginning at a_3^2 must include a_5^2, otherwise we may interchange colors B and C on this chain and then give the color B to a_1^2. Also, the BD chain beginning at a_3^2 must include a_6^2, otherwise we may interchange B and D on it and give a_1^2 the color B. As a result of the first chain it follows that the AD chain which contains a_4^2 does not contain a_6^2 and that by interchanging A and D on it we have two D countries and only one A country touching a_1^2. Or, as a result of the second chain we may interchange A and C on the chain beginning at a_2^2, without changing the color of a_5^2. Kempe interchanged the color on the AD chain, giving a_4^2 the color D, and leaving the others unchanged, *and then* he interchanged the colors on the AC chain, giving a_2^2 the color C and leaving the others unchanged. Thus he was able to give a_1^2 the color A. As Heawood pointed out, he failed to notice that after the first change it is no longer necessary that the BD chain beginning at a_3^1 shall contain a_6^2. If this chain does not contain a_6^2, then an interchange of A and C on the chain beginning at a_2^2 would put C at a_2^2 and A at a_5^2 and there would still be four colors about a_1^2. Heawood not only pointed out this fault in the argument but he presented a map where the difficulty suggested above was actually realized. This map did not, however, disprove the theorem that four colors suffice, for he gave a way of coloring it.

5. The results obtained by use of chains. In a paper entitled *du Coloriage des Cartes et de quelques Questions d'Analysis Situs* [1] Errera carries to its conclusion another method of attack. The conclusion is, as often happens with this problem, an example which shows that the method will not solve the problem. His method is as follows: Make the first interchange of colors as in the paragraph above. This gives the color D to a_4^2. Then we may suppose that there is an AC chain containing both a_2^2 and a_5^2. Interchange B and C on the chain beginning at a_2^2. He continues in this manner; each step, once the first change has been made, is determined. If at any time any of these chains joining certain pairs of regions are missing, the map can be colored. Either that must happen or else the process continues indefinitely. Errera hoped to prove that the latter was impossible.

There are but a finite number of ways of putting four colors on a finite number of regions, so the operation he describes must lead at some stage to the original

[1] Paris, Gauthier-Villars et Cie. (1921).

map. He notes that only after the twentieth operation do the same colors occupy the same positions around a_1^2 as at the beginning. This does not mean that the whole map has taken its original coloring, but rather that the number of operations required to return the map to its original state is a multiple of 20. He hoped to show that no map could have twenty substitutions as the whole or part of its period, and thus that the process would always lead to a map that could be colored. He ended by constructing a map of 17 regions which took on its original coloring after 20 operations. Also he gave a way of coloring the map, thus leaving the problem as much a problem as ever.

In a paper "On the Reducibility of Maps"[1] Birkhoff uses the notion of chains. He defines a *ring* as a set of k regions $a_1a_2 \cdots a_k$ "such that each of these regions has a boundary line in common with the one preceding and following it in cyclical order, but with no other region of the set. A ring R of regions of this kind divides the map into two sets of regions M_1 and M_2 which together with R make up all the regions of the map." He then attempts to color the partial maps $M_1 + R$ and $M_2 + R$ in such a way that the regions of R will have the same colors in both cases or in such a way that by the use of the Kempe chains the regions of R can be given the same colors in both cases. Then putting the partial maps together, one could get the original map colored. He is able to show that the minimum map M (§ 4) contains no rings of four regions; no ring of five regions except about a single region; no ring of six regions surrounding only five-sided regions where the number of such regions is greater than 3; and no ring of n regions within which are n 5-sided regions surrounding a n-sided region; and no ring of $4n$ regions within which are $2n$ 6-sided regions about a single region.

6. Other forms of the problem. There are other interesting forms of the problem. We are considering only the problem of regular maps—maps whose linear graphs are regular and of the third degree, and which contain no 1-, 2-, 3-, or 4-sided regions.

One way to treat the problem is to consider only the linear graph of the map. Tait[2] was the first to notice that the problem was identical with the problem of assigning numbers 1, 2, and 3 to the lines of the graph in such a way that a line bearing each number would be at each vertex.

To see that the two problems are equivalent suppose first that a given map has been colored in four colors and consider the countries at any vertex. If we number the lines according to the following scheme,

$$1 = \frac{A}{B} = \frac{C}{D}$$

$$2 = \frac{A}{C} = \frac{B}{D}$$

$$3 = \frac{A}{D} = \frac{B}{C},$$

[1] *American Journal of Mathematics*, 35 (1913), p. 115.

[2] *Proceedings of the Royal Society of Edinburgh*, 10 (1880), p. 501.

by which we mean that a 1-line is between an A country and a B country or between a C country and a D country, and so on, we see that every vertex will have a 1-line, a 2-line and a 3-line.

Conversely, suppose we have the numbers 1, 2, and 3 attached to the lines in the above-described manner. We may assign the color A to any region a_i^2 and the color of every other region is determined.

To see that no two adjacent regions are given the same color, consider two consecutive sides of the region a_i^2 which was given the color A. One of those sides must be either a 1-side or a 2-side. Such a 1-line or a 2-line will be included in some $(1 - 2)$ circuit—*i.e.*, a set of lines $l_1l_2\cdots l_n$ where l_1 and l_2 have a vertex in common, l_2 and l_3, and so on to l_n and l_1 and where l_1 is a 1-line, l_2 is a 2-line, l_3 is a 1-line, and so on. Such a circuit must contain an even number of lines and an even number of vertices; there must be a 3-line at every vertex of the circuit. All of the 1-lines and 2-lines constitute a set of circuits of the above type. A $(1 - 2)$ circuit divides the sphere into two regions, one of which may be called the inside of the circuit and the other the outside. There may or may not be $(1 - 2)$ circuits inside of a given $(1 - 2)$ circuit C; if there is a single $(1 - 2)$ circuit C_1 inside of C, then C_1 is connected to C by an even number of 3-lines. For C_1 has an even number of vertices and an even number (which may be zero) of those are joined in pairs by 3-lines. The remaining vertices, of which there are an even number, must be joined to C by 3-lines. If C is a circuit with n circuits $C_1, C_2, \cdots, C_n$ inside, no one of which contains more than one of the others inside it, then C is joined to the system by an even number of 3-lines; for each C_i $(1 \leq i \leq n)$ has an even number of vertices joined by 3-lines to some circuit outside it, and so there will be an even number for the whole system, and each 3-line joining two of them reduces by 2 the number of 3-lines joining the system to C. Thus we see that there are an even number of 3-lines going outward from C.

Now let the circuit which touches the region a_i^2 be C. The region a_i^2 is given the color A. Then color every region that touches the circuit C and that can be reached from a_i^2 by crossing 3-lines. If this method ever leads us back to a_i^2 we must have crossed an even number of 3-lines and so a_i^2 would not be given a color different from A. If this set of regions which has just been colored touches any other $(1 - 2)$ circuit, we may continue crossing 3-lines and coloring regions A and D without ever coming to a contradiction. When this has been carried as far as possible we will have obtained a Kempe chain. Now cross a 1-line or a 2-line from any region of this chain. According to the table above, this leads to a B region or a C region. From this continue crossing 3-lines as before and assigning colors according to the table. This gives a BC chain. Continuing in this way as long as there are any uncolored regions, we finally arrive at a coloring of the map. Thus we have shown that the two problems are equivalent.

We will now introduce the following definitions:

The number of vertices of a linear graph is called the *order* of the graph.

A regular graph is said to be *factorable* if it is possible to select a set of lines of the graph that constitutes a regular graph of the same order but of lower degree; the set of lines selected is one *factor*, and the remaining set of lines is the other.

A *leaf* is a part of the graph that is connected with the remainder by a single line provided it contains no such part itself.

An affirmative to the map problem is equivalent to the following theorem:

Any regular graph of degree 3 which contains no leaves and which can be put on a sphere can be factored into three first degree factors.

Tait, accepting Kempe's solution and believing that he had solved the problem in several ways himself, stated this theorem as a corollary. (*Cf.* reference above.)

The most that has been accomplished in this direction is the statement and proof by Petersen [1] of the following theorem:

Any linear graph containing less than three leaves can be factored into a first degree factor and a second degree factor.

It follows *a fortiori* that the linear graph of a regular map on a sphere can be factored into a linear and a quadratic factor. This second degree factor is made up of a number of circuits, which may, however, contain an odd number of vertices and lines.

We may go a step further in refinement. The problem of attaching numbers 1, 2, and 3 to the lines of the linear graph so that one of each kind is at every vertex is equivalent to the problem of attaching $+1$ or -1 to each vertex so that the sum of the numbers on the vertices of any circuit is equal to zero (mod 3). Heawood [2] was the first to put the problem in this form. Veblen [3] applied modular equations to the problem and so translated it into a problem in finite geometry.

Let the number of vertices be α_0, the number of lines be α_1, and the number of regions be α_2. Suppose we were to attach a number to each region, each number to represent a single color and no two distinct numbers to represent the same color. The condition that two adjacent regions have the same color could be expressed by the modular equation

$$y_a + y_b = 0 \pmod 2, \tag{1}$$

where y_a and y_b are the numbers attached to the regions a and b which touch along a particular line. For every line we may write an equation of the above type. The problem may then be considered as the problem of finding a set of α_2 values $(y_1, y_2, \cdots y_{\alpha_2})$, where y_i may have any one of four values, which does not satisfy any of the α_1 equations (1).

He then takes the field $GF(2)$ consisting of 0 and 1 combined modulo 2 and extends it by the Galois imaginaries [4] satisfying the relation $i^2 + i + 1 = 0$.[5]

[1] *Acta Mathematica*, 15 (1891), p. 193. A second proof was given by the author, *Annals of Mathematics*, (2) 19 (1917), p. 59; and later a proof was given by Errera (*loc. cit.*).

[2] *Quarterly Journal of Mathematics*, 29 (1898), p. 270.

[3] *Annals of Mathematics*, 25, 14 (1912), p. 86.

[4] For a discussion of Galois imaginaries see Dickson: *Introduction to the Theory of Algebraic Equations*, p. 42 ff.

[5] i is no longer used as an index.

The extended field $GF(2^2)$ has four elements 0, 1, i and $i + 1$. A set of values $(y_1, y_2, \cdots y_{\alpha_2})$ may be considered as a point on a finite projective space of $\alpha_2 - 1$ dimensions provided we exclude the set $(0, 0, \cdots 0)$ and consider $(ky_1, ky_2, \cdots ky_{\alpha_2})$ to be the same point as $(y_1, y_2, \cdots y_{\alpha_2})$. If the variables y_i range over the $GF(2)$ there will be $2^{\alpha_2} - 1$ points in the space, if they range over the $GF(2^2)$ there will be $(4^{\alpha_2} - 1)/3$ points. The first space is included in the second and the points of the second not included in the first may be regarded as *imaginary* with respect to the first space.

Each of equations (1) represents an $(\alpha_2 - 2)$ space. *Any point which does not lie on any one of these* $(\alpha_2 - 2)$*-spaces represents a solution of the problem.* In general, no real point can represent a solution because then the map would require but two colors which is impossible when the map contains a vertex at which there are an odd number of regions. Every imaginary point is on one and only one real line. But if $(y_1 + iy_1', y_2 + iy_2', \cdots y_{\alpha_2} + iy_{\alpha_2}')$ satisfies one of the above equations, so must $(y_1, y_2, \cdots y_{\alpha_2})$ and $(y_1', y_2', \cdots y_{\alpha_2}')$, and conversely. Hence a solution is given by *each real line which does not lie on any of the* $(\alpha_2 - 2)$*-spaces which are represented by equations* (1).

He also states the problem in terms of modular equations in a form equivalent to the theorem on linear graphs, and still again in a form equivalent to the problem of attaching numbers $+1$ and -1 to the vertices. The only question remaining is whether or not the systems of equations have solutions. Similarly, Birkhoff[1] gives a formula for $P_n(\lambda)$, the number of ways in which a given map of n regions may be colored in λ colors. This, however, does not settle the question, for we have yet to find out whether or not $P_n(4)$ is positive for all values of n.

Finally, Franklin[2] arrives at the rather disheartening conclusion that the most we can say about the minimum uncolorable regular map, taking account of all the reductions that have been made and many more which he makes himself, is that it contains at least 26 regions. He gives a map of 42 regions to which none of the reductions apply. He establishes by a method similar to that used by Birkhoff (see § 5) that the minimum uncolorable regular map does not contain:

(1) A side of a hexagon surrounded by the hexagon and three pentagons;
(2) A pentagon in contact with three pentagons and a hexagon;
(3) A pentagon surrounded by two pentagons and three hexagons;
(4) An even sided region completely surrounded by hexagons and (some or no) pairs of pentagons, the two of each pair being adjacent;
(5) A pentagon surrounded by four pentagons and two hexagons;
(6) A $2n$-sided region surrounded by $2n - 2$ pentagons and two other adjacent regions;
(7) A $(2n - 1)$-sided region surrounded by $2n - 2$ pentagons and one other region.

[1] *Annals of Mathematics*, 25, 14 (1912), p. 42.

[2] *American Journal of Mathematics*, 24 (1922), p. 225.

By a method similar to that used by Wernicke (see footnote, § 4), he shows that, as a result of the above and other like restrictions on the minimum uncolorable map, it contains one of the three arrangements which follow:

(1) A pentagon adjacent to two other pentagons;

(2) A pentagon adjacent to a pentagon and a hexagon;

or

(3) A pentagon adjacent to two hexagons.

A GRAPHICAL TREATMENT OF MIXTURES.

By F. E. WOOD, Northwestern University.

1. Introduction. Mixtures of Two Substances. Let the points of a line segment P_1P_2 of unit length represent mixtures of two materials P_1 and P_2, the mixture of a_1 parts of P_1 and a_2 parts of P_2 being represented by the point whose distance from P_1 toward P_2 is $a_2/(a_1 + a_2)$. Evidently its distance from P_2 will be $a_1/(a_1 + a_2)$, and the points P_1, P_2 will represent 100 per cent. of P_1, P_2 respectively. We take as the *coördinates* of the general point P the values $a_1/(a_1 + a_2)$, $a_2/(a_1 + a_2)$. In order to find the point corresponding to a mixture of a_1 parts of P_1 and a_2 parts of P_2 we divide the segment P_1P_2 by a point M such that $P_1M/MP_2 = a_2/a_1$, the point M being the desired point.

Corresponding to each point between P_1 and P_2 there is a possible mixture of the two materials P_1 and P_2, and conversely for each possible mixture there will correspond a point between P_1 and P_2. To find the mixture corresponding to a point M between P_1 and P_2 we determine the ratio $P_1M/MP_2 = r$; then the mixture is composed of $\rho/(1 + r)$ parts of P_1 and $\rho r/(1 + r)$ parts of P_2 (ρ being a proportionality factor). Two mixtures will have the same corresponding point if and only if they have the same value of a_2/a_1, so that the point corresponding to a mixture depends upon the composition only, and is not affected by the amount of the mixture. Due to this correspondence between mixtures and points on a segment, we shall in the sequel speak of the point $M(a_1, a_2)$ where the coördinates of M are a_1 and a_2 respectively ($a_1 + a_2 = 1$), and the mixture $M(a_1, a_2)$ where the mixture M is made up of a_1 parts of P_1 and a_2 parts of P_2 and $a_1 + a_2 = 1$; between the point M and the mixture M there is a unique correspondence. To a point upon the line P_1P_2, but not between P_1 and P_2, there would correspond an impossible mixture, since a negative amount of one material would have to be taken.

Consider two mixtures of P_1 and P_2, $M_1(a_1, a_2)$ and $M_2(b_1, b_2)$. Let M denote a mixture of k_1 parts of A and k_2 parts of B, where $k_1 + k_2 = 1$; the mixture $M(m_1, m_2)$ contains $k_1a_1 + k_2b_1$ parts of P_1 and $k_1a_2 + k_2b_2$ parts of P_2. Since $P_1A = a_2$, $P_1M = m_2$, $P_1B = b_2$ we find that $M_1M/MM_2 = k_2/k_1$; therefore *the point corresponding to a mixture of any two mixtures of P_1 and P_2 may be found in the same way as the point corresponding to a mixture of P_1 and P_2, the required*

point being the one which divides the segment corresponding to the two mixtures in the inverse ratio to the parts of each mixture taken.

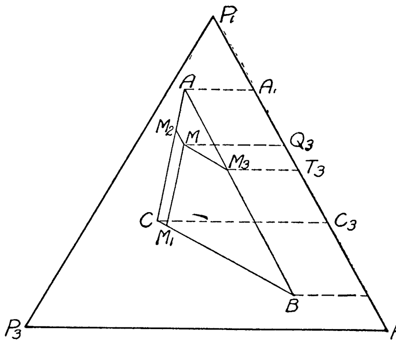
To each point upon the segment M_1M_2 there corresponds a possible mixture M_1 and M_2, and for each point on the line M_1M_2 but not on the segment M_1M_2 there corresponds no possible mixture.

2. Mixtures of Two Diluted Solutions. Some mixtures contain two valuable or important substances and a third unimportant or valueless one (as an ore containing lead, silver and refuse, or a stock food containing protein, carbohydrates and "roughage") where the percentage of the amount of the valuable constituents to the amount of the whole mixture is relatively unimportant. We will denote the substances regarded as valuable "active" and the one which dilutes them "inactive"; in such mixtures only the ratio of the one active substance to the other active substance is important, so that *ratio* mixtures only are to be considered. Two ratio mixtures will be said to be *equivalent* when the percentages of the active substances which enter are in the same ratio for each mixture. The *concentration* of a ratio mixture is defined as the ratio of the amount of active substances to the total amount of the mixture. It is the sum of the percentages of the active substances. The concentration of a mixture containing two undiluted substances is unity. It is evident that an equivalent mixture to any ratio mixture which can be obtained from two substances of concentration unity can also be obtained when the concentrations of the active substances have been altered by the addition of an inactive substance, and conversely. The only changes which arise will be in the formulas.

Let the concentrations of P_1 and P_2 be denoted by α_1 and α_2 respectively; then to a mixture of a_1 parts of P_1 and a_2 parts of P_2 we make to correspond a point M on the unit segment P_1P_2 such that $P_1M = a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$, whence $MP_2 = a_1\alpha_1/(a_1\alpha_1 + a_2\alpha_2)$. For, suppose P_1' and P_2' to be ratio mixtures made from P_1 and P_2 by the addition (or extraction) of the inactive substance to such an extent that the concentrations of P_1' and P_2' are each .01, then one part of P_1 will have as much of the active substances as $100\alpha_1$ parts of P_1', and one part of P_2 as much as $100\alpha_2$ parts of P_2'; so a ratio mixture M' equivalent to M can be made $100a_1\alpha_1$ parts of P_1' and $100a_2\alpha_2$ parts of P_2' whence the coördinates of M' would be $a_1\alpha_1/(a_1\alpha_1 + a_2\alpha_2)$, $a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$. Since the point M' corresponds to all mixtures equivalent to the mixture M', it will coincide with M, whence $P_1M = a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$ as stated.

To determine the point M corresponding to a ratio mixture M of a_1 parts of P_1 of concentration α_1 and a_2 parts of P_2 of concentration α_2 we divide the segment P_1P_2 by a point M such that $P_1M/MP_2 = a_2\alpha_2/a_1\alpha_1$. To determine the ratio mixture which corresponds to a point M upon the segment P_1P_2 we determine the ratio $P_1M/MP_2 = r$, then $a_2\alpha_2/a_1\alpha_1 = r$, $a_1 + a_2 = 1$; so we find $a_1 = \alpha_2/(\alpha_2 + r\alpha_1)$, $a_2 = r\alpha_1/(\alpha_2 + r\alpha_1)$. As in paragraph 1, we can show that a mixture of any two diluted mixtures has its corresponding point upon the segment determined by the diluted mixtures, and the position of the point for a given mixture can be similarly obtained.

3. Mixtures of Three Substances.[1] If mixtures of three substances P_1, P_2, P_3 are to be considered, we take an equilateral triangle whose sides are each of unit length and whose vertices are P_1, P_2, P_3. A mixture of a_1 parts[2] of P_1, a_2 parts of P_2, and a_3 parts of P_3 will be represented by a point Q within the triangle, which is determined as follows: upon the segment P_1P_2 determine a point Q_3 such that $P_2Q_3 = a_1$. Similarly upon the segment P_2P_3 determine the point Q_1 such that $P_3Q_1 = a_2$; draw a line through Q_3 parallel to P_2P_3 and a line through Q_1 parallel to P_3P_1; these two lines meet in Q, the required point. We denote by (a_1, a_2, a_3) the *coördinates* of Q, which lies within the triangle when the coördinates are all positive.



Through Q draw a line parallel to P_1P_2 meeting P_1P_3 in Q_2 and P_3P_2 in M, and a line through Q_1 parallel to P_1P_2 meeting P_1P_3 at N. Then $P_3Q_1 = P_3N$; $P_1Q_2 = P_1Q_2$; $P_2Q_3 = QM = QQ_1 = Q_2N$ so that $P_3Q_1 + P_1Q_2 + P_2Q_3 = P_1P_3 = 1$. Since $P_3Q_1 = a_2$, $P_2Q_3 = a_1$, $a_1 + a_2 + a_3 = 1$, it follows that $P_1Q_2 = a_3$, so that the same point Q would be obtained by using *any* two of the three values a_1, a_2, a_3. This justifies calling (a_1, a_2, a_3) the coördinates of Q. In the sequel we shall call Q_1, Q_2, Q_3 the *projections* of Q upon the sides of the triangle $P_1P_2P_3$ in the proper order. The vertices of the triangle $P_1P_2P_3$ correspond to the substances P_1, P_2 and P_3; the points on each of the three sides correspond to mixtures of two substances only, and the points within the triangle to mixtures of all three substances.

To determine the point $Q(q_1, q_2, q_3)$ corresponding to a mixture $Q(q_1, q_2, q_3)$ we proceed as already indicated;[3] to determine a mixture corresponding to point Q within the triangle we find the projections of Q, namely Q_1, Q_2, Q_3, upon the three sides of the triangle, then q_1, q_2, q_3 are obtained from the equations $P_2Q_3 = q_1$, $P_3Q_1 = q_2$, $P_1Q_2 = q_3$, and give the required parts.

4. The Use of a General Triangle for Mixtures of Three Substances. It is

[1] Methods of setting up a correspondence between the points of a triangle and mixtures of three substances have been given by Gibbs, *Transactions, Connecticut Academy*, vol. 3 (1876), p. 176, and by Roozeboom, *Zeitschrift für physikalische Chemie*, vol. 15 (1894), p. 145. See also Bancroft, *The Phase Rule*. The method used here is capable of easy generalization (see paragraph 4) and leads to some results (see paragraphs 5–8) which the writer believes to be new. The coördinate system set up is the one usually known as Areal; see C. A. Scott, *Modern Analytic Geometry*, and Askwith, *Analytic Geometry of the Conic Sections*, chapter XIV. A recent treatment of the applications of this method can be found in Haskell, *How to Make and Use Graphic Charts*, pp. 30–36.

[2] In the remainder of this paper by "parts" will be meant percentages of the whole mixture.

[3] In this representation of mixtures of three substances by points within an equilateral triangle, one may start in either of two ways. The intersection of the line through Q parallel to P_2P_3 with P_1P_3 instead of P_1P_2 might have been taken. In either case the three lines through Q must be taken parallel to the three sides of the triangle in a cyclic order, and the distance to the projection on each of the three sides must be measured from the vertex which is on the side parallel to the projecting line.

evident that the equilateral triangle with unit side, which has been used in the previous paragraph, might be replaced by an equilateral triangle with side equal to ρ, provided the distances measured upon the sides were ρa_1, ρa_2, ρa_3 instead of a_1, a_2, a_3 respectively, as this would merely magnify (if $\rho > 1$) the original configuration. In such a case one measures a_1 along the side of the triangle, regarding the side of the triangle as of unit length.

If one wishes to consider a mixture of three substances, each of which is a mixture of three fundamental substances, difficulty is encountered because the three points which correspond to the three mixtures will not in general form an equilateral triangle; so that we need a method for determining the point corresponding to a mixture of three substances when the corresponding triangle is arbitrary (except that the three vertices must not be collinear). This method follows: Let $P_1P_2P_3$ be a triangle with sides $P_1P_2 = \lambda$, $P_2P_3 = \mu$, $P_3P_1 = \nu$. The point Q corresponding to a mixture Q whose percentages of P_1, P_2, P_3 are a_1, a_2, a_3 respectively is determined as follows. On the segment P_1P_2 determine a point Q_3 such that $P_2Q_3 = \lambda a_1$, upon P_2P_3 determine Q_1 such that $P_3Q_1 = \mu a_2$, draw a line through Q_3 parallel to P_2P_3, and a line through Q_1 parallel to P_3P_1; these two lines meet in Q, the required point.

In order to justify this method, one needs to show that if a line through Q parallel to P_1P_2 meets P_1P_3 in Q_2, then $P_1Q_2 = \nu a_3$, for then the point Q is the same whichever pair of the coördinates (a_1, a_2, a_3) is taken. To show that this is true, prolong QQ_2 to meet P_2P_3 at M, and draw a line parallel to P_1P_2 through Q_1 meeting P_3P_1 at N. Then using similar triangles

$$\lambda a_1 = P_2Q_3 = QM = \frac{\lambda}{\nu} QQ_1 = \frac{\lambda}{\nu} Q_2N,$$

whence $Q_2N = \nu a_1$,

$$\mu a_2 = P_3Q_1 = \frac{\mu}{\nu} P_3N,$$

whence $P_3N = \nu a_2$, so that $P_1Q_2 = \nu - \nu a_1 - \nu a_2 = \nu a_3$.

The method in this case is merely to measure from the proper vertices the distances a_1, a_2, a_3 upon the three sides regarding in each case the length of the side as unity, and to draw lines through these points parallel to the sides of the triangle in the proper order, the three lines so drawn meeting in the required point.

5. Mixtures of Mixtures of Three Substances. Let A, B, C denote mixtures of P_1, P_2, P_3 and also denote the points corresponding to those mixtures; let the coördinates of A, B, C be $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. Let M denote a mixture of A, B, C with coördinates $M(\mu_1, \mu_2, \mu_3)$ with respect to the triangle ABC (see Figure), the point M corresponding to the mixture M being determined with respect to the triangle ABC by the method described in paragraph 4. We wish to show that the point M so determined corresponds to a ratio mixture M of the substances P_1, P_2, P_3 which is made up of μ_1 parts of A, μ_2 parts of B and μ_3 parts of C. We denote the composition of A by the

symbolic equation $A = a_1P_1 + a_2P_2 + a_3P_3$; similarly $B = b_1P_1 + b_2P_2 + b_3P_3$, $C = c_1P_1 + c_2P_2 + c_3P_3$, $M = \mu_1A + \mu_2B + \mu_3C$, whence

$$\begin{aligned} M = & (\mu_1a_1 + \mu_2b_1 + \mu_3c_1)P_1 \\ & + (\mu_1a_2 + \mu_2b_2 + \mu_3c_2)P_2 \\ & + (\mu_1a_3 + \mu_2b_3 + \mu_3c_3)P_3. \end{aligned}$$

We wish to show, then, that the coördinates of M with respect to the triangle $P_1P_2P_3$ are $(\mu_1a_1 + \mu_2b_1 + \mu_3c_1,\ \mu_1a_2 + \mu_2b_2 + \mu_3c_2,\ \mu_1a_3 + \mu_2b_3 + \mu_3c_3)$. Let A_3, B_3, C_3, Q_3 be the projections of A, B, C, M upon the side P_1P_2; draw MM_3 parallel to BC and meeting AB at M_3, and let T_3 denote the projection of M_3 on the side P_1P_2. Then $P_2A_3 = a_1 - b_1$, and by similar triangles

$$\frac{A_3B_3}{AB} = \frac{A_3T_3}{AM_3} \quad \text{or} \quad \frac{a_1 - b_1}{-\lambda} = \frac{A_3T_3}{\lambda(1 - \mu_1)},$$

whence $A_3T_3 = (1 - \mu_1)(b_1 - a_1)$; also

$$\frac{MM_3}{CB} = \frac{Q_3T_3}{C_3B_3} \quad \text{or} \quad \frac{\mu\mu_3}{\mu_3} = \frac{Q_3T_3}{c_1 - b_1},$$

whence $Q_3T_3 = \mu_3(c_1 - b_1)$.

Therefore $P_2Q_3 = P_2A_3 + A_3T_3 + T_3Q_3 = \mu_1a_1 + \mu_2b_1 + \mu_3c_1$, which is the first coördinate of M with respect to $P_1P_2P_3$; a similar method will determine the other two coördinates.

The theorem just proved has several important corollaries:

1. If the points corresponding to three mixtures A, B, C of three substances P_1, P_2, P_3 are plotted and taken as the vertices of a triangle, then a fourth mixture M can be made from the mixtures A, B, C if and only if the point M corresponding to this mixture M lies within or on the boundary of the triangle ABC.

2. If the point M corresponding to a mixture M lies within the triangle corresponding to three mixtures A, B, C, the parts of A, B, C which will give the mixture M may be determined as follows (see Figure): draw the projections of M upon the triangle ABC, viz., M_3, M_1, M_2; then

$$\mu_1 = \frac{BM_3}{BA}, \qquad \mu_2 = \frac{CM_1}{CB}, \qquad \mu_3 = \frac{AM_2}{AC}$$

are the required values.

3. If the point M lies on one side of the triangle ABC, then the mixture M can be made up of two of the three mixtures A, B, C.

4. The various mixtures which result from taking a fixed proportion of A, and varying the proportions of B and C, have their corresponding points upon a line which is parallel to BC.

5. If the points corresponding to three mixtures lie upon a line, then one of the mixtures can be obtained from the other two, and all the mixtures of the three will have their corresponding points on a line.

6. Mixtures of Three Diluted Substances. A treatment of mixtures of four substances would require the use of a tetrahedron in space. However, the previous discussion and results may be extended to include mixtures of three active substances and one inactive substance. If the concentration of each of the substances considered is the same, the extension is immediate; otherwise some additional argument is necessary.

Let A, B, C be three mixtures of the active substances P_1, P_2, P_3 and an inactive substance; let the coördinates[1] of the corresponding points be $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$ and the concentrations be α, β, γ respectively. Let A', B', C' be mixtures equivalent to A, B, C, each having a concentration of .01. Then a mixture M' equivalent to a mixture M made from μ_1, μ_2, μ_3 parts of A, B, C respectively could be made from $100\mu_1\alpha$, $100\mu_2\beta$, $100\mu_3\gamma$ parts of A', B', C' respectively. Hence the coördinates of M', and so of M, are proportional to $\mu_1\alpha$, $\mu_2\beta$, $\mu_3\gamma$. The point M may therefore be found by the method explained in paragraph 4, and will lie within the triangle ABC, or upon some side of it.

Conversely, let the points corresponding to mixtures A, B, C form the vertices of a triangle and let M denote a point within the triangle ABC; then a mixture corresponding to M can be mixed from A, B, C and the relative proportions of each found from the equalities $BM_3/BA = \rho\mu_1\alpha$, etc., where M_3 is the projection of M upon BA, ρ a factor of proportionality, α, β, γ the concentrations of A, B, C and μ_1, μ_2, μ_3 the required proportions.

7. Mixing Fertilizers. The situation studied in paragraph 6 arises when it is desired to mix a given ratio fertilizer from three standard ones, if possible.[2] The method in this case would be to plot the points corresponding to the three standard fertilizers, then to see whether the point corresponding to the desired fertilizer is within or upon the boundary of the triangle formed by the first three. If not, the desired mixture cannot be obtained from the standard ones; if within, the mixture can be obtained, and the proportions found as indicated in paragraph 6. If one imposes the condition $\mu_1 + \mu_2 + \mu_3 = 2000$, the number of pounds of each standard fertilizer required to make a ton of the mixture is obtained. The concentration of the fertilizer must then be obtained from the concentrations of the standard fertilizers and the proportions used. If this for example is .1 when the desired fertilizer should have .2, then twice as much of the mixture should be used as would be used if the strength were .2.

It may also happen that one wishes to form a mixture of two standard fertilizers A and B in such a way as to obtain a fertilizer as nearly equivalent as possible to a given mixture M;[3] the foot of the perpendicular dropped from M upon AB corresponds to the required mixture.

[1] Here we take as the coördinates of a point the values of the percentages taken of the three active substances in a definite order. The sum of the coördinates of a point will equal the concentration of the corresponding mixture. Two equivalent mixtures will have the same corresponding point.

[2] Here P_1, P_2, P_3 are Nitrogen, Potash, Phosphorus, and the inactive substance is usually called filler.

[3] A mixture P will be said to be more nearly equivalent to a mixture Q than a mixture R when the distance PQ is less than the distance QR, the points P, Q and R being plotted as indicated in paragraph 6.

8. Solution of Three Linear Simultaneous Equations by Graphical Analysis. Let three active substances be denoted by P_1, P_2, P_3 and three mixtures of them by $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$. Further let $M(\mu_1, \mu_2, \mu_3)$ denote a mixture made from x_1, x_2, x_3 amounts of A, B, C respectively; then x_1, x_2, x_3 are solutions of the three equations

$$(A) \qquad \begin{cases} x_1 + x_2 + x_3 = \lambda_1 \\ \dfrac{a_1x_1 + b_1x_2 + c_1x_3}{a_3x_1 + b_3x_2 + c_3x_3} = \dfrac{\mu_1}{\mu_3} = \lambda_2 \\ \dfrac{a_2x_1 + b_2x_2 + c_2x_3}{a_3x_1 + b_3x_2 + c_3x_3} = \dfrac{\mu_2}{\mu_3} = \lambda_3 \end{cases}$$

where λ_1, λ_2, λ_3, a_1, $\cdots$ c_3 are positive (or zero except that $\lambda_1 \neq 0$) constants. Conversely, given a set of equations of the form (A), there will correspond a problem in mixtures which can be solved graphically, and so the equations also can be solved graphically.

The methods employed in this paper may be extended to include points with negative coördinates, an extension useful in solving equations but not in making mixtures. Moreover, a set of equations

$$(B) \qquad \begin{cases} \alpha_1x + \beta_1y + \gamma_1z = \delta_1 \\ \alpha_2x + \beta_2y + \gamma_2z = \delta_2 \\ \alpha_3x + \beta_3y + \gamma_3z = \delta_3 \end{cases}$$

where α_3, β_3, γ_3, δ_3 are all different from zero may be put into the form (A) by the substitution $\alpha_3x = u$, $\beta_3y = v$, $\gamma_3z = w$, becoming

$$u + v + w = \delta_3,$$

$$\frac{\dfrac{\alpha_1}{\alpha_3}u + \dfrac{\beta_1}{\beta_3}v + \dfrac{\gamma_1}{\gamma_3}w}{u + v + w} = \frac{\delta_1}{\delta_3},$$

$$\frac{\dfrac{\alpha_2}{\alpha_3}u + \dfrac{\beta_2}{\beta_3}v + \dfrac{\gamma_2}{\gamma_3}w}{u + v + w} = \frac{\delta_2}{\delta_3},$$

and so may be solved except in special cases. These special cases will arise when three of the points $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$, $(\delta_1, \delta_2, \delta_3)$ are collinear. Finally if in set (B) at least one equation actually contains an α, at least one a β, at least one a γ, and at least one a δ, a linear combination of the three equations can be obtained containing no vanishing constant, and this equation can be used in place of

$$\alpha_3x + \beta_3y + \gamma_3z = \delta_3.$$

ON THE CIRCLES OF ANTISIMILITUDE OF THE CIRCLES DETERMINED BY FOUR GIVEN POINTS.

By R. A. JOHNSON, Hamline University, St. Paul, Minn.

1. The following discussion is restricted to the finite real plane, except as imaginary elements are specifically mentioned. By definition, the word *circle* is extended to include straight lines. An *inversion* in a proper circle is a transformation by reciprocal radii. An inversion in a straight line is a reflection. A *circle of antisimilitude* of two circles is a circle, with regard to which they are mutually inverse. It is known that two intersecting circles always have two circles of antisimilitude which intersect orthogonally at their common points, and have their centers at the centers of similitude of the given circles. When two given circles do not intersect, one circle of antisimilitude is real and one is imaginary. If an inversion is performed with regard to a center of inversion which is on a circle of antisimilitude, the given circles invert into equal circles. Hence two circles can always be inverted into equal circles; and three given circles can be so inverted, provided that any circle of antisimilitude of any pair meets a circle of antisimilitude of another pair.

2. It is proposed to study the circles of antisimilitude of the four circles determined by four arbitrary points not on a circle. The figure may be simplified by an inversion whose center is one of the given points; the four circles in question then become the sides and circumcircle of a triangle. We shall consider the simplified figure first.

3. Let the vertices of a triangle be A_1, A_2, A_3; its incenter U, its excenters V_1, V_2, V_3. Let P_i be the mid-point of that arc A_jA_k of the circumcircle which is opposite A_i, and Q_i the mid-point of that which contains A_i. By a well-known theorem,[1] P_i is the center of a circle through A_j, A_k, U, V_i; and Q_i is the center of a circle through A_j, A_k, V_j, V_k. We shall designate these circles by p_i, q_i respectively. The line A_jA_k shall be designated as usual by a_i, and the circumcircle by c.

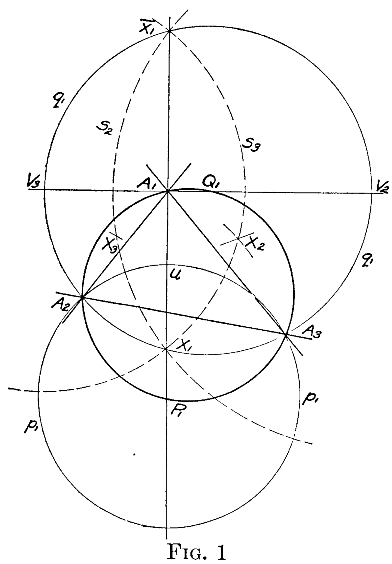


FIG. 1

The circles of antisimilitude of the lines a_j and a_k are the bisectors of the angle A_i. The circles of antisimilitude of the line a_i and the circle c are the circles p_i, q_i. These circles of antisimilitude are concurrent by sixes at the points U, V_1, V_2, V_3. Hence if any one of these points is taken as center of inversion, the inverses of a_1, a_2, a_3, c are equal circles, and the inverses of A_1, A_2, A_3, together with the center of inversion, form an orthocentric system.[2]

[1] For references, see R. C. Archibald, this MONTHLY, *1921*, 229.

[2] R. A. Johnson, this MONTHLY, *1916*, 161; Arnold Emch, *ibid.*, *1916*, 162.

The circle q_i meets the interior bisector of A_i at two points which we will call X_i, $\overline{X}_i$. The circle p_i does not meet the exterior bisector of A_i in real points; we may say for convenience that this line and circle meet in two imaginary points Y_i, $\overline{Y}_i$. These twelve points complete the tally of points of intersection of the twelve circles of antisimilitude.

The six points (X) *lie by fours on three circles* s_1, s_2, s_3 *whose centers are respectively* V_1, V_2, V_3, *and which are mutually orthogonal. Each of them is orthogonal to three angle bisectors and three p- and q-circles.*

For

$$UX_j \cdot U\overline{X}_j = UA_i \cdot UV_i = UX_k \cdot U\overline{X}_k,$$

whence X_j, $\overline{X}_j$, X_k, $\overline{X}_k$ are concyclic. The center of the circle is obviously V_i; there are three such circles, and since

$$\angle V_jX_iV_k = \angle V_jA_iV_k = 90°,$$

any two of them are orthogonal. Obviously also U is their radical center. We designate by s_i the circle through X_j, $\overline{X}_j$, X_k, $\overline{X}_k$.

By an inversion in s_i, the circle $V_iX_j\overline{X}_j$ inverts into the line $X_j\overline{X}_j$; that is, by this inversion A_i, A_j, A_k exchange positions respectively with U, V_k, V_j. It follows that s_i is orthogonal to the circles q_i, p_j, p_k, as well as to those angle bisectors which pass through V_i. Each of the other three bisectors inverts into a p- or q-circle. Thus:

Each circle s_i *is orthogonal to six of the twelve circles of antisimilitude, and the other six are by pairs inverse with regard to it. It is therefore coaxal with each of the three pairs.*

For instance, s_i is coaxal with $V_jA_iV_k$ and p_i, and therefore passes through the imaginary points Y_i, $\overline{Y}_i$ mentioned above. Wé easily see that an inversion in the center U, with radius given by

$$r^2 = \overline{UA}_i \cdot \overline{UV}_i \quad \text{(a negative constant)},$$

interchanges each A_i with the corresponding V_i, and also each X_i with $\overline{X}_i$. (In terms of real elements only, this transformation is an inversion whose radius is $\sqrt{A_iU \cdot UV_i}$, followed by a rotation through 180°.) By this inversion, then, each circle s_i is unchanged, and the circle of inversion is therefore their common orthogonal circle. This same inversion moreover interchanges the two circles which intersect at Y_i and $\overline{Y}_i$; hence the six imaginary points $(\overline{Y})$ lie on this imaginary circle, and are its intersections with s_1, s_2, s_3.

The twelve points (X), (Y) *are the total intersection of a set of four mutually orthogonal circles whose centers are* U, V_1, V_2, V_3. *The six imaginary points* (Y) *lie on the imaginary circle whose center is* U.

We may see incidentally that *the four circles each of which passes through three of the four points* U, V_1, V_2, V_3 *are equal; their twelve circles of antisimilitude are precisely the same angle bisectors and p- and q-circles as those of the given four circles.*

4. By an inversion with regard to an arbitrary center, which does not lie on any of the lines or circles of the figure, we are led to the following more general results.

Let four arbitrary points be A_1, A_2, A_3, A_4; *denote by* c_{ijk} *the circle through* A_i, A_j, A_k, *and designate by* t_{ij}, t_{ij}' *the circles of antisimilitude of* c_{ijk} *and* c_{ijl}. *These twelve circles of antisimilitude are concurrent by sixes at four real points* B_1, B_2, B_3, B_4. *Further, if the primes are suitably assigned, any circle* t_{ij} *meets* t_{kl}' *in two real points* (X), *and the corresponding pair* t_{ij}', t_{kl} *meet in two imaginary points* (Y). *These twelve points are the total intersection of four mutually orthogonal circles* (s); *on one of these four circles are the six imaginary points* (Y), *and on each of the others are two imaginary and four real points.*

Each circle (s) *is orthogonal to six of the circles of antisimilitude* (t), *and the other six are by pairs mutually inverse with regard to it. With regard to any one of the s-circles, the tetrads* $A_1A_2A_3A_4$ *and* $B_1B_2B_3B_4$ *are mutually inverse; incidentally, these two quadrangles are four times in perspective, namely at the centers* S_1, S_2, S_3, S_4 *of the s-circles,*[3] *which latter constitute an orthocentric system. The t-circles are also circles of antisimilitude of the four circles determined by the four points* B_1, B_2, B_3, B_4.

If one of the points B_i *is taken as center of inversion, the A's invert into an orthocentric system, and the remaining B's into the feet of the altitudes of this system; and vice versa, if an* A_i *is taken as center of inversion. If a point* (X) *is taken as center of inversion, each quadrangle inverts into a parallelogram. The two parallelograms are congruent and have common diagonals and center. That is, in the inverted figure* $A_iB_jB_kA_l$ *and* $B_iA_jA_kB_l$ *are similar rectangles, similarly placed with regard to a common center.*

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. SHOULD BOOK REVIEWS BE CENSORED?

By L. E. DICKSON, University of Chicago.

A mathematical manuscript is never accepted by a journal in America until it has been approved by at least one referee. But book reviews appear to escape such a check. This is unfortunate. An error in an original paper reflects only upon the author. But errors in a book review usually do not reflect upon the reviewer as they should, but often do great injustice to the author of the book.

[3] The twelve points (A), (B), (S) are, in other words, the plane projections of a desmic configuration.

Most authors prefer to suffer in silence rather than attempt impracticable redress. Surely book reviews should be submitted to impartial referees prior to publication.

Several years ago the writer found out first hand from those then writing most of the reviews that it was no longer the fashion to write frankly concerning the merits or shortcomings of books, but that one should "damn with faint praise" and carefully conceal from the dear public the reviewer's real opinion of a book. Often what the reviewer said in several pages could have been said in the single sentence: "This is a good book, but I hate like the deuce to admit it."

Unfortunately the American reviewer often does not aim primarily to portray the degree of success attained by a book in the choice of subject matter and in its clarity of presentation, but rather to convince the reader that the reviewer himself is a person who knows a great deal about the subject and how it should have been presented.

If I select a specific case for illustration, I must choose one whose consideration will do no harm to the reviewer, author, or publisher. These requirements are met in the case of the recent favorable review of my *First Course in the Theory of Equations* by Professor Bennett. This review appeared in the MONTHLY in the final number for 1922, published April 9, 1923. The text had appeared a year earlier and was already familiar to nearly all teachers of collegiate mathematics, as shown by the fact that it has had a larger sale than any other mathematical text beyond the calculus. Hence I cannot be accused of attempting here to advertise this text.

The vague remarks comprising the first page of the review imply that the reviewer is dissatisfied with the traditional choice of subject matter in books on the elementary theory of equations, although he admits that this choice is acceptable to most teachers of it. The second page gives a kindly appreciation of the general features of the book. A far more concrete idea of the book was given in the same amount of space in the review by Robert d'Adhémar in the *Bulletin des Sciences Mathématiques* for 1922.

Since the review by Bennett is exceptionally favorable to the book, why do I call attention to it now? Solely because the collection on the final page of ten "minor infelicities" contains instances of amazing carelessness on the part of the reviewer.

In a proof (p. 18) by mathematical induction (a process explained very fully and simply on p. 5), I stated that "if we multiply each member of identity (9) by $x - \alpha_{n+1}$, it is not much trouble to verify that the resulting identity can be derived from (9) by changing n into $n + 1$, so that (9) is proved true by mathematical induction." The reviewer complained that "in connection with mathematical induction the author speaks of 'changing n into $n + 1$,'" with the implication that the author had committed the error often made by a novice in induction.

I am at loss to know why he condemns my phrase "A point on the graph at which the tangent is both horizontal and an ordinary tangent is a bend point . . .," since it is both mathematically exact and expressed clearly.

"The discriminant, after being defined in general on page 47, is redefined for the quartic on page 51." In the earlier case the definition had been applied to write out the discriminant of the cubic only. In the new case there was need of the explicit expression for the discriminant of the quartic and it was justified by repeating the definition. Clearness and continuity of thought should not be a crime.

On page 21 I said that "any number which exceeds all real roots of a real equation is called an *upper limit to the real roots*." The reviewer complained that the italicized phrase is defined as a single concept without any inquiry as to the meaning of the word "limit" or the sense of "upper." Why did he fail to criticize for similar reasons my definition of "integral rational function" as synonymous with "polynomial"? When one desires to clarify the distinction between the right-hand and left-hand pages of an open book, does he need to enter upon the anatomical properties of the hand as distinguished from the foot? The many teachers who are using this elementary text will approve of my abstaining from entering upon the delicacies inherent in the notion of a limit, and my sensible definition of the above entire phrase as a compound name.

Since I spoke of the trigonometric form (in polar coördinates) of $a + bi$, the reviewer says I should have given also a separate name to $a + bi$. But I had called $a + bi$ a complex number and now found need to speak of its *trigonometric form.*

"Immediately after emphasizing the fact that a complex number has many amplitudes differing by integral multiples of 360°, the term 'the amplitude' is used without a hint as to which one is 'the amplitude.' The facts are that, on the last two lines of page 3, I said, 'For the amplitude we may select, instead of θ, any of the angles $\theta \pm 360°$, $\theta \pm 720°$, etc.,' while in the succeeding fifteen lines I used the term 'an amplitude' twice and spoke of 'the indicated amplitudes' (i.e., amplitudes 120° and 240° indicated in the figure), but never employed the imputed term 'the amplitude.'"

"In the exercises on page 74 the letters must denote only real quantities although no such warning is given." But the theorem under which these exercises fall, as well as the heading of the chapter, explicitly stated that we are dealing with equations with real coefficients. Moreover, all of the literal coefficients which appear in the exercises are explicitly squares, with the evident object of implying that these coefficients are not merely real, but positive.

It happens that I am now sending eight minor corrections to the plates in anticipation of a new printing; a careful examination of the above "infelicities" convinced me that no correction is needed on account of them.

Since the reviewer, himself an expert, has gone patiently through the text seeking faults, with no better success than the mentioned groundless infelicities, I conclude that the text is remarkably free from faults, thanks partly to the minute criticisms of the manuscript and proof-sheets by the specialists cited in the preface. For the trouble of writing also this second elementary text on the theory of equations, my real reward has been the knowledge that it is now

yearly inspiring thousands of students with a genuine interest and affection for mathematics.

II. Infinite and Imaginary Elements in Algebra and Geometry.

By Tomlinson Fort, University of Alabama.

Since the things advocated by Professor R. M. Winger in his article bearing the above title and appearing in a recent number of the Monthly (*1922*, 290–296) are at variance with the general practice followed by me in my teaching of freshman and sophomore classes, I am prompted to comment on what he says.

In my opinion one of the duties of the teacher of elementary mathematics is to attempt to rid the subject of mysticism and to make it appear at all times in conformity with the student's experience. Like most teachers I welcome elementary uses for imaginary numbers but can not believe the first course in analytic geometry the place for them. To the freshman, points and lines are more or less the pictures that he sees and are not ideals. His analytic geometry is truly a geometry of reals and for the instructor to confine his teaching to the "real domain" seems to me sound mathematically and to rid the subject of that element of mystery which I do not believe it possible to eliminate when trying to teach geometry with imaginary elements to such immature students. I never feel that I am telling the freshman a half truth when I tell him that (0, 0) is the only point satisfying $x^2 + y^2 = 0$ or that $x^2 + y^2 + 1 = 0$ has no locus. I think that imaginary elements should be introduced at a later time when the whole subject can be subjected to a more critical examination. The same is true of infinite regions of the plane. To bring a class of freshmen to understand and remember their ideal nature seems to me out of the question. I simply never mention the subject: Parallel lines do not meet. The parabola is an open curve and division by zero is never permitted.

It is with a kind of horror that I read where the author advocates the postulation of "*The Number Infinity*" defined by $k/0 = \infty$, $k \neq 0$. How I fight division by zero in my classes! How many times have I told my students that 90° has no tangent and have forbidden this same infinity! To describe the behavior of the roots of an equation,

$$a_0x^n + \cdots + a_n = 0; \qquad a_0 \neq 0,$$

when $a_0, \cdots, a_{r-1}$ approach zero seems to me sound teaching, but to speak of the equation as having r infinite roots when $a_0 = \cdots = a_{r-1} = 0$ and the equation is only of the $(n - r)$th degree, as the author advocates, is to me very bad. It is just that from which I thought American mathematics was growing. The author states that what he advocates is in accordance with "*sound* European tradition." If I understand him correctly, I am sure in particular that his infinity was not in use in the teaching at Göttingen in 1912 or in Paris in 1913 when I was a student at those universities. Also in my opinion he will find that it has been eliminated from the better teaching in England. With reference to infinity,

I think the best policy in elementary teaching is to explain carefully the various unending processes that arise and in particular to make it clear that there is no largest real number. In more advanced courses where infinite elements are introduced care should be taken to explain their ideal character and to show just why it is wise to postulate them in exactly the form that it is done.

III. A Theorem Concerning the Concurrency of Four Planes.

By E. L. Rees, University of Kentucky.

In a previous number of this Monthly (*1920*, 165) the author gave a theorem on the concurrency of three lines by means of which certain theorems of geometry could be proved in a very brief and simple way. In the present note a similar theorem is given, relating to the concurrency of four planes, with applications to illustrate its use.

Let $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ be the position vectors of three points determining a plane, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ those of a second plane, and so on. *A necessary and sufficient condition that four planes be concurrent is that* $\Sigma a\mathbf{A} = \Sigma b\mathbf{B} = \Sigma c\mathbf{C} = \Sigma d\mathbf{D}$, *where* a_1, a_2, a_3, $\cdots$, d_1, d_2, d_3 *are some quantities, not all zero, satisfying the equations* $\Sigma a = \Sigma b = \Sigma c = \Sigma d$.

The proof is omitted, being almost identical with that given for the theorem referred to.

If $\Sigma a\mathbf{A} = \cdots = 0$, and $\Sigma a = \cdots \neq 0$, the planes intersect at the origin.

If $\Sigma a\mathbf{A} = \cdots \neq 0$, and $\Sigma a = \cdots = 0$, the planes intersect at infinity.

If $\Sigma a\mathbf{A} = \cdots = 0$, and $\Sigma a = \cdots = 0$, the vectors of each sum are termino-collinear and the planes are not determined. We exclude this case.

As applications we prove the following theorems.

The six planes determined by the edges of a tetrahedron and points taken on the opposite edges are concurrent if the ratios in which the points divide the opposite edges are such that the product for each face is unity, and conversely.

Proof: Let the vertices of the tetrahedron be A, B, C, D, and let the division points E, F, G, H, I, J be taken in accordance with the equations below. The hypothesis of the theorem is satisfied in the most general way in the following equations.

$$(a+b)\mathbf{E} = a\mathbf{A} + b\mathbf{B}, \quad (a+d)\mathbf{G} = a\mathbf{A} + d\mathbf{D}, \quad (b+d)\mathbf{I} = b\mathbf{B} + d\mathbf{D},$$
$$(a+c)\mathbf{F} = a\mathbf{A} + c\mathbf{C}, \quad (b+c)\mathbf{H} = b\mathbf{B} + c\mathbf{C}, \quad (c+d)\mathbf{J} = c\mathbf{C} + d\mathbf{D},$$

where a, b, c, d are any scalar quantities. Add $c\mathbf{C} + d\mathbf{D}$ to each member of the first equation, $b\mathbf{B} + d\mathbf{D}$ to each member of the second, etc. It results that

$$(a+b)\mathbf{E} + c\mathbf{C} + d\mathbf{D} = (a+c)\mathbf{F} + b\mathbf{B} + d\mathbf{D} = (a+d)\mathbf{G} + b\mathbf{B} + c\mathbf{C}$$
$$= (b+c)\mathbf{H} + a\mathbf{A} + d\mathbf{D} = (b+d)\mathbf{I} + a\mathbf{A} + c\mathbf{C} = (c+d)\mathbf{J} + a\mathbf{A} + b\mathbf{B},$$

the scalar coefficients of any four of which satisfy the conditions of the theorem. Hence the proof is complete.

Corollary: The four lines joining the vertices of a tetrahedron with the

points of intersection of the lines which join the vertices of the opposite faces with the division points of the sides of these faces are concurrent.

This corollary may also be proved independently by the theorem in the article referred to as follows.

Let A', B', C', D' represent the points on the faces opposite A, B, C, D respectively. We easily find

$$(b + c + d)\mathbf{A}' = b\mathbf{B} + c\mathbf{C} + d\mathbf{D}, \qquad (a + b + d)\mathbf{C}' = a\mathbf{A} + b\mathbf{B} + d\mathbf{D},$$
$$(a + c + d)\mathbf{B}' = a\mathbf{A} + c\mathbf{C} + d\mathbf{D}, \qquad (a + b + c)\mathbf{D}' = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}.$$

Adding $a\mathbf{A}$ to each member of the first equation, $b\mathbf{B}$ to each member of the second, etc., we get as before

$$(b + c + d)\mathbf{A}' + a\mathbf{A} = (a + c + d)\mathbf{B}' + b\mathbf{B} = (a + b + d)\mathbf{C}' + c\mathbf{C}$$
$$= (a + b + c)\mathbf{D}' + d\mathbf{D}.$$

Since the sums of the scalar coefficients of the different members are equal the corollary is proved. The point of intersection of these lines is given by

$$\frac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D}}{a + b + c + d}.$$

If the points of division are the midpoints of the edges we then get the center of gravity of the tetrahedron which is given by $\frac{1}{4}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})$. The equations also show that this point divides each of the lines joining a vertex with the center of gravity of the opposite face in the ratio 3 : 1.

Conversely, let b/a, c/b, a/d be the ratios in which AB, BC, and DA are divided by three of the given planes; then, as shown above, the points which divide the edges CA, BD, and DC in the ratios a/c, d/b, c/d respectively determine with the opposite edges planes concurrent with the other three. These planes are therefore among those given. The ratios satisfy the conditions of the theorem, hence the converse is proved.

Very similar to the proof just given is that of the following theorem.

The planes determined by the sides of a space quadrilateral and points taken on the opposite sides are concurrent if the product of the ratios in which these sides are divided is unity.

As a last application we prove the following.

A plane cuts each of the six edges of a tetrahedron; another point is taken on each edge, so as to cut it harmonically; prove that the six planes through these latter points and the opposite edges of the tetrahedron intersect in one point. (From Frost's *Solid Geometry.*)

In view of the theorem of Menelaus (the vector proof of which is quite simple) the hypothesis of our theorem is contained in the assumption that the edges AB, BC, CA, BD, DA, CD are divided respectively in the ratios b/a, c/b $- a/c$, d/b, $- a/d$, $- d/c$ which of course depend on the position of the cutting plane with reference to the tetrahedron.

Denoting by E, F, G, H, I, J the points taken so as to cut the edges harmonically, we have

$$(a-b)\boldsymbol{E} = a\boldsymbol{A} - b\boldsymbol{B}, \qquad (-b+c)\boldsymbol{F} = -b\boldsymbol{B} + c\boldsymbol{C}, \qquad (c+a)\boldsymbol{G} = c\boldsymbol{C} + a\boldsymbol{A},$$
$$(-b+d)\boldsymbol{H} = -b\boldsymbol{B} + d\boldsymbol{D}, \qquad (d+a)\boldsymbol{I} = d\boldsymbol{D} + a\boldsymbol{A}, \qquad (d+c)\boldsymbol{J} = d\boldsymbol{D} + c\boldsymbol{C}.$$

Add the terms necessary in each equation to make the right-hand member equal to $a\boldsymbol{A} - b\boldsymbol{B} + c\boldsymbol{C} + d\boldsymbol{D}$. We then have

$$\begin{aligned}(a-b)\boldsymbol{E} + c\boldsymbol{C} + d\boldsymbol{D} &= (-b+c)\boldsymbol{F} + a\boldsymbol{A} + d\boldsymbol{D} = (c+a)\boldsymbol{G} - b\boldsymbol{B} + d\boldsymbol{D} \\ &= (-b+d)\boldsymbol{H} + a\boldsymbol{A} + c\boldsymbol{C} = (a+d)\boldsymbol{I} - b\boldsymbol{B} + c\boldsymbol{C} = (d+c)\boldsymbol{J} - b\boldsymbol{B} + a\boldsymbol{A},\end{aligned}$$

the scalar coefficients of which satisfy the conditions of the theorem, which completes the proof.

The converse of this theorem may be proved by an argument similar to that given above.

IV. On the Evaluation of an Elliptic Integral.

By Rev. M. F. Egan, University College, Dublin.

The following simple method gives, within narrow limits, the approximate value of the complete elliptic integral

$$K = \int_0^{\pi/2} d\varphi/(1 - k^2 \sin^2 \varphi)^{1/2}$$

when k is nearly equal to unity.

Putting $\sin \varphi = \tanh u$, we get

$$K = \int_0^{\infty} du/(1 + k'^2 \sinh^2 u)^{1/2}.$$

Since $k'^2 (= 1 - k^2)$ is small, the term $k'^2 \sinh^2 u$ is of importance only for large values of u, and for these $\sinh u$ is nearly equal to $\frac{1}{2}e^u$. We are therefore led to consider the integral

$$K_0 = \int_0^{\infty} du/(1 + \tfrac{1}{4}k'^2 e^{2u})^{1/2}.$$

Putting $v = e^{-u}$, we get

$$\begin{aligned} K_0 = \int_0^1 dv/(v^2 + \tfrac{1}{4}k'^2)^{1/2} &= \sinh^{-1}(2/k') \\ &= \log_e (4/k') + k'^2/16 \end{aligned}$$

to the order k'^2. Since $\sinh u < \frac{1}{2}e^u$, $K > K_0$. On the other hand

$$\begin{aligned}\frac{1 + \frac{1}{4}k'^2 e^{2u}}{1 + k'^2 \sinh^2 u} &= 1 + \frac{k'^2(1 - \frac{1}{2}e^{-2u})}{2(1 + k'^2 \sinh^2 u)} \\ &< 1 + \tfrac{1}{2}k'^2 < (1 + \tfrac{1}{4}k'^2)^2,\end{aligned}$$

so that $K < (1 + \frac{1}{4}k'^2)K_0$. Therefore

$$K_0 < K < (1 + \tfrac{1}{4}k'^2)K_0.$$

RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers. By L. E. DICKSON. Volume III.[1] Quadratic and Higher Forms. With a Chapter on the Class Number, by G. H. CRESSE. Carnegie Institution, Washington, D. C. March, 1923. v + 313 pages.

The astonishing rapidity with which Professor Dickson has made the manuscript ready for the printer and has seen through the press the first three volumes (1919, 1920, 1923) of his monumental history of the theory of numbers must be a cause for admiration on the part of every one who has witnessed the performance; and the surprise only increases when the reader finds that each volume is done perhaps even more skillfully than the preceding one. In the preface to the third volume we are told that it has been completed promptly owing to the favorable reception accorded to the first two volumes. The preface to the second volume closed with a sentence referring to the third as the concluding volume; but in the third now before us we have (on page 3) a reference forward to volume IV. The reviewer ventures to predict that the favorable reception of the third volume will give the author still more reason for proceeding promptly with the fourth if his astonishing supply of energy is holding out well enough to leave him still susceptible to such influence.

The present reviewer is not so well acquainted with the subject matter of this volume as he is with that of the two preceding; and consequently his judgment about it is entitled to less confidence, especially with reference to the question of completeness; but from what he knows of this matter for volumes I and II and from the fact that the third volume deals with matter which by its nature is more readily accessible to a full treatment and from the evident care of the author and those who have assisted with the perfection of the manuscript and the proofsheets, he feels confident that little will be found missing, so little in fact that it becomes a matter of importance that every reader respond to the author's request by informing him of every omission or error of any sort that may be noticed. The work comes so near to being entirely complete that it is a matter of importance and common interest to supply to it whatever may yet be lacking. The subject-matter of the volume is such that its publication should give a great impetus to the development of the important arithmetic theory of forms. By a "form" is meant a homogeneous polynomial such as $f = ax^2 + bxy + cy^2$, all of whose terms are of the same total degree in x and y, the constant coefficients a, b, c being integers in the greater part of the theory. The arithmetical theory of forms has an important application to the problem of finding all the ways of expressing a given number m in a given form f, that is, of finding all the integral solutions x, y of the equation $ax^2 + bxy + cy^2 = m$. This problem is best attacked generally by considering with f all the equivalent forms g which can be derived from f by applying to the variables linear substitu-

[1] For reviews of Vols. I and II, see this MONTHLY, *1919*, 396; *1921*, 72.

tions with integral coefficients of determinant unity. It is by a simultaneous consideration of all these forms g that one is able to solve completely the equation $f = m$. It is this type of question that is treated in volume III. The results have obvious applications to many of the problems discussed in volume II and to a few of those in volume I. From these remarks it will be seen that volume III deals mainly with general theories rather than with special problems and special theorems. The investigations in question are largely those of leading experts and deal with some of the more advanced parts of the theory of numbers. All previous reports and treatises on forms are now very inadequate owing to the large number of important recent papers on the subject. These are perhaps some of the reasons which cause the author to speak of this volume as "doubtless the most important one of the series."

Of each of the first two volumes the author gave a summary in the preface, this being possible partly on account of the elementary nature of much of the subject-matter and the fact that many of the results were embodied in definite theorems which could be expressed without the use of difficult technical terms. But, in the present volume, "it is a question not primarily of explicit results, but chiefly of general methods of attacking whole classes of problems, the methods being often quite intricate and involving extensive technical terminology." For this reason the preface does not contain a summary of the contents of the volume; but to each of the longer chapters an appropriate introduction and summary are attached.

For the purpose of describing briefly its contents it is convenient to divide the volume (perhaps mechanically rather than logically) into three nearly equal parts, the first two of which deal with binary quadratic forms. One of these consists of the single chapter VI on class number; another, of the remaining chapters I to VIII, all on binary quadratic forms; and the last, of the further chapters IX to XIX, dealing with ternary and quaternary quadratic forms, quadratic forms in n variables, cubic forms, forms of degree greater than three, Hermitian forms, bilinear forms and related matters, representation by polynomials modulo p, and the congruencial theory of forms.

The part first named (chapter VI, pages 92–197) is by G. H. Cresse who devoted five years to the preparation of this report on the difficult subject of the number of classes of binary quadratic forms with integral coefficients, difficult partly on account of the fact that it involves many branches of pure mathematics. It was written as a thesis for the doctorate at Chicago. L. J. Mordell read critically the manuscript of this chapter and also the proofsheets, comparing the whole with the original papers and making numerous suggestions for its improvement. Much of it was examined also by E. T. Bell. The completeness and accuracy of this important chapter are thus made certain by a collaboration of effort in revising and checking. A similar, if less extensive, collaboration marks the preparation of all parts of the book, the proofsheets of each chapter having been read carefully by at least one specialist in the subject of the chapter.

Chapters II, III, IV, V, VII, VIII deal respectively with the following topics

in the theory of binary quadratic forms: explicit values of x, y in $x^2 + \Delta y^2 = g$; composition of binary quadratic forms; orders and genera, their composition; irregular determinants; binary quadratic forms whose coefficients are complex integers or integers of a field; number of classes of binary quadratic forms with complex integral coefficients. The first chapter (pages 1–54), entitled "Reduction and equivalence of binary quadratic forms, representation of integers," contains a brief introduction and explanation of symbols and a treatment of the topics in the theory of binary quadratic forms not belonging to chapters II to VIII. The nature of the contents is made clear by a brief summary.

The subject-matter of the last third of the volume has already been indicated by a summary of the principal chapter headings. It remains to add a few remarks concerning certain of the chapters. The results recorded in some of these are obviously fragmentary; and this suggests the desirability of a systematic analysis of their subject-matter with the purpose of carrying out further investigations so as to put the whole into a more satisfying form. Thus chapter XIII on cubic forms in three or more variables (together with the corresponding pages 594–595 of volume II) will bring to the reader's attention certain problems which it is desirable to have treated more fully, problems where certain of the methods of attack seem to lie rather close to hand. It may be that this topic is accessible to the amateur who has considerable initiative; it is certainly accessible to a good graduate student interested in the theory of numbers. If the matter of this chapter were brought to a fairly satisfactory state it would probably lead the way to a further treatment of certain related forms of degree higher than 3 and to the association of the results with further relevant matters in volume II.

The last chapter, on the congruencial theory of forms, deals with some of the most recent contributions to the theory of numbers. It is due primarily to Dickson and his students. It is a safe prediction that this subject will receive further effective attention in the near future.

It is pleasant to give credit to those to whom it is due when such an extensive and valuable work is put into our hands. First of all, of course, this goes to Dickson himself. Nor must one forget the work of Cresse in the long chapter VI. A significant service has also been rendered by those who have assisted in perfecting the manuscript or in the critical reading of the proofsheets; and to these Dickson has referred in his prefaces in a generous way. The work of those who help with corrections and additions to be incorporated into the promised supplement may also come to deserve our generous appreciation. And we should also remember gratefully the encouragement which has come from the attitude of the Carnegie Institution toward the printing of such an extensive work. Writing in November, 1918, at the close of the preface to volume I, Dickson said, "Finally, this laborious project would doubtless have been abandoned soon after its inception seven years ago had not President Woodward [of the Carnegie Institution] approved it so spontaneously, urged its completion with the greatest thoroughness, and given continued encouragement." This should be a constant

reminder to us of the need of ample means of publishing comprehensive scientific works when persons of ability and suitable preparation are willing to undertake the arduous and unremunerative task of preparing them.

While all this work is fresh in his mind, Professor Dickson, it would seem, is peculiarly well-fitted to prepare a concise outline history of the theory of numbers, indicating the major historical stages in the development of the theory and showing the growth and progress of ideas as the centuries have passed. If he should even venture into the uncertain domain of prophecy and indicate the probable direction of further progress, his predictions would be a matter of interest both now and hereafter. When a master, with the work of the past well in mind, tries to see the trend of the future, his judgment will be a matter of interest whether or not the direction of progress turns out to be such as he anticipates. It may even throw some light on the difficult question as to the way in which new discoveries arise.

The extraordinarily varied development of modern mathematics has led thoughtful persons to inquire what is to be done to prevent the whole discipline from falling apart into chaos. Two possible solutions to the difficulty are thinkable. We may find new and deep-lying principles unifying great bodies of doctrine: this is the ideal solution, but it is not one to be had simply by asking for it; we have to await the discovery or invention or creation of the unifying conceptions. The other (partial) solution is that afforded by full and complete summaries of existing knowledge in certain chapters of science. This will also be needed even if new and deep-lying conceptions arise to unify our knowledge. Indeed such complete summaries as those of Dickson will be of increasing importance whatever direction the development of science may take.

But the needs cannot all be met by these summaries alone. We require also extended and comprehensive expositions of these subjects. This is the more important in the case of disciplines which are somewhat fragmentary in significant parts, especially if their character in this respect leads (as it often does) to the production of a scattered literature containing many fragments. The great increase in the accessibility of the theory of numbers due to Dickson's extensive work whets our appetite for more, especially as the existence of his work seems to render feasible a concerted effort to put the whole of the present theory of numbers into a connected and complete expository form. The suggested task is a great one—and it requires the coöperation of numerous workers. But what better time is there to suggest the undertaking of a great enterprise than that time which witnesses the bringing well along toward completion of another great enterprise which does much to make the new one feasible? If we cannot enter upon the whole of the extended labor, may we not undertake the performance of certain parts of it—those parts which are most needed and give promise of the largest usefulness for the greatest period of time?

R. D. Carmichael.

In Chapter XVIII we have the coördinates of spheres, both signless and oriented. In connection with the latter we find an acute case of the trouble which arises when an author is not clear as to whether he is operating in the real or the complex domain. If the domain be complex, the distinction (p. 344) that a sphere with positive radius bounds its inside while one with negative radius bounds its outside is meaningless, but if the domain be real, the statement on p. 352 that there are ∞^2 spheres through a minimum line is erroneous. Rather more attention is given to systems of oriented spheres than would be warrantable were it not for the purpose of giving substance to the Line-Sphere transformation with which the chapter closes.

The last chapters deal with projective and euclidean geometry in a space of any number of dimensions. The treatment of the hyperquadric seems to me particularly good, although I note the absence of Sylvester's Law of Inertia which gives valuable information about real hyperquadrics.

It is a mistake for one viewing a landscape not to see the wood for the trees. It is a mistake to fix the attention on the minutiæ of a book, and lose sight of the general sweep. I have meticulously pointed out many small errors; it is the duty of an honest reviewer to do so. But now that I have them out of the way, I am free to say how much I admire the book as a whole. Professor Woods has undertaken the important and difficult task of introducing the student to that rich field of science called Modern Geometry. In my opinion, he has succeeded unquestionably better than any other writer using the English tongue. A book with which one might compare the present is Veblen and Young's *Projective Geometry*[1] which seems to include a good part of geometry, arithmetic, and abstract logic within its compass. But these authors consistently place logical considerations before didactic ones, and he who would approach higher geometry *for the first time* through their two closely written volumes must have much patience and courage, for "Jordan am a hard road to trabbel."

Years ago an enthusiastic French student said to me, in praise of Darboux's lectures:

"Il dit chaque jour quelque chose de nouveau, quelque chose d'intéressant."

This is essentially true of Professor Woods' book, and the student who passes rapidly from one new and interesting topic to another may well say, in current phrase: "Day by day, in every way, it's getting better and better."

J. L. COOLIDGE.

The Mathematical Theory of Probabilities. Volume I, Second edition. By ARNE FISHER. New York, The Macmillan Co., 1922. 8vo. 29 + 289 pages. Price $5.00.

It is especially gratifying that a man with a wide experience in the applications of probability to statistics, life insurance and the telephone business should write in the English language a treatise on theoretical probability and its applications. In an introductory note to the first edition, F. W. Frankland, one of

[1] Ginn and Co., Boston, vol. 1, 1910, vol. 2, 1918.

the foremost actuaries in America, wrote as follows: "While in French, in Italian, in German, in Danish, and in Dutch, scientific works on statistics were available galore, the dearth of such literature in the English language was little short of a national or racial scandal. With such works as those of Yule and Bowley, in recent years, there has been some possibility for the English-speaking student to acquire part of the knowledge needed. But it is hardly necessary to point out what a very large amount of new ground is covered by Mr. Fisher's new book as compared with such works as I have referred to."

These remarks apply, *a fortiori*, to the enlarged second edition. Mention should be made, however, of the recent appearance of J. M. Keynes's *A Treatise on Probability*, which will appeal strongly to those whose interest in probability is primarily philosophical or logical, this treatise being a continuation of the work of Boole and Venn. To those who want a text in English comparable to some extent in scope and general treatment with the standard books in German and French, Fisher's book is a godsend. The need of such a book is felt most keenly in those institutions where the study of Spanish has assumed such exaggerated proportions that a teacher of seniors or first-year graduates can not take for granted that his students can read either French or German. The reviewer has used the first edition of Fisher's book as a text, and found it highly satisfactory. The enlarged second edition should be doubly so. Problems for students, however, must be supplied by the instructor. For statistical experts, too, in certain forms of business, the book will be invaluable, as clearly indicated in an introductory note by Mr. M. C. Rorty, of the American Telephone and Telegraph Co. Mr. Rorty, after mentioning Mr. Fisher's skill in dealing with statistical curves, writes: ". . . Practical experience with these curves soon showed that in spite of minor errors, they were close enough to the real facts to make them of primary importance in traffic studies of all kinds and particularly in the development of mechanical switching devices. Their use for such purposes has now become a commonplace in telephone engineering."

Fisher's book is written in an entertaining style with numerous references, historical and critical. Special mention is made of Czuber's *Wahrscheinlichkeitsrechnung* as being lucid, terse, systematic, and attractive; and Fisher's book in the introductory portions bears some resemblance to that of Czuber. For example, Czuber's definition of mathematical probability is adopted—with quotation marks and reference—by Fisher.

The first five chapters of Fisher's book are devoted to definitions, explanations, and well-selected algebraic probability problems, ending with the St. Petersburg paradox. In Chapter VI inverse probability is discussed at length; absurdities arising from the indiscriminate use of Bayes's Theorem are pointed out, but this theorem is deemed valid when properly used. The exposition of Bayes's Theorem would have been more lucid if it had been divorced at the start from considerations of repeated trials. The law of large numbers is discussed in Chapter VII, and the relation between mathematical probabilities and relative frequencies. In Chapter VIII Stirling's Formula is deduced by Cesàro's elegant method,

which gives control of the error. As preparatory to this, the Wallis expression for π as an infinite product is obtained by the use of J_n, the integral from 0 to $\pi/2$ of $\sin^n x$. The argument on page 92 is, however, inconclusive; because J_n is an infinitesimal. But the desired result, Lim $J_{2m-1} \div J_{2m} = 1$, can be easily obtained from $J_{2m} < J_{2m-1} < J_{2m-2}$ and Lim $J_{2m-2} \div J_{2m} = 1$. In Chapter IX appear the theorems of Bernoulli, Poisson, and Tchebycheff. The next three chapters show how the actual dispersion may be compared with the theoretical dispersion to determine whether the series belongs to the Bernoullian type, to the Poisson type, or to the Lexis type, with normal, subnormal, or hypernormal dispersion, respectively. This is one of the most characteristic sections of Part I.

The next two parts, consisting of 120 pages, are an addition to the first edition. In the introductory historical survey, the theory of Laplace is deemed the proper starting point for statistical theory. Laplace, it is noted, was not a computer; and the less brilliant work of Gauss received the greater attention. The theory of Laplace, however, has been rejuvenated, and brought to greater use by such writers as Charlier, who supplied suitable forms for computation. Fisher mentions the valuable work of Karl Pearson; but regards his method, which involves closed expressions, as less flexible and effective than the developments in series preferred by Scandinavian and German writers. In another publication, Fisher hopes to give some still more advanced probability theories. Here, no doubt, reference will be made to the monumental work of R. von Mises,[1] who, by axioms on sequences, eliminates the mystery of "equally probable cases," and by an elaboration of the Czuber method eliminates the troublesome *a priori* probability used in Bayes's Theorem.

Part II gives the mathematical theory of statistical series. By the use of the Fourier Integral Equation, a very general form of frequency function is set up as an integral involving the semi-invariants of Thiele. If only the first two of these semi-invariants are used, the frequency function is represented by the well-known "normal" probability curve. A general frequency function is then expressed as a series in terms of the Hermite polynomials and the factor $e^{-z^2/2}$, the coefficients c_i being integrals involving Hermite polynomials. The coefficients are also expressed in terms of the semi-invariants, and by proper choice of origin and unit, $c_1 = 0 = c_2$.

Part III is devoted primarily to applications of the theory, with numerical computations in detail. Here appear, also, brief expositions of the method of least squares, of logarithmically transformed frequency functions, of the Charlier B Curve, and of the law of small numbers. Of special interest is the graduation of the American Men Mortality Table (AM$^{(5)}$) by means of a compound Charlier curve.

There is a table of the Laplacean probability function, certain of its derivatives, and its integral. But no index of titles or authors is given. This would have been especially useful, because of the very numerous references appearing throughout the book.

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[1] (1) *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, (2) *Grundlagen der Wahrscheinlichkeitsrechnung*, *Mathematische Zeitschrift* (1919), vol. 4, pp. 1–97; vol. 5, pp. 52–99.

Though statisticians are by no means agreed as to where emphasis should be placed, it may safely be said that Fisher's book covers a wide range of very important topics. And, in the reviewer's opinion, it is the only book in English comparable with the standard works on probability in German, French, and Italian.

E. L. DODD.

Real Mathematics. By ERNEST G. BECK. Froude, Hodder and Stoughton, London, 1922. 306 pages.

The motive of the author is described in the preface as follows:

"One of the principal objects of this book is to offer assistance to the practical engineer and to engineering students in the acquisition of a real, serviceable and sound mathematical equipment. The book is intended to augment the standard textbooks and orthodox methods of study. The desire is rather to bring about a change of attitude towards mathematics than to propose methods which shall merely be different from those in use—to show the thing as an actual, tangible reality, instead of as a collection of rigid and unrelated rules and formulæ: and to persuade students to touch and handle it for themselves with confidence and understanding instead of regarding it from afar as some rather awful and totally incomprehensible abstraction."

In his endeavor to harmonize the modern demands with the best traditions of the various topics, the author has been very successful. He has treated carefully the foundation of each topic, and gives numerous examples to explain further the difficulties involved. In all cases, in order that the reader may learn principles and gain power, fundamental methods are set forth and applied to problems.

The author has shown good pedagogical insight by trying to make the reader feel at the beginning of each topic under consideration that a worth-while problem requiring solution is before him. Interest is aroused immediately, and the reader is more than anxious to follow up the carefully planned explanation.

The book is well illustrated, is written in an interesting style and will be valuable not only to the engineer, but also to any teacher of mathematics.

The subject matter takes up the usual topics given during the first two years of college mathematics. About half of the book is devoted to Arithmetic, Geometry, and Trigonometry, and the remainder to the Calculus.

To sum up, it seems to the reviewer that the author has been very successful in his effort to enliven and enrich the subject and to present it in such a manner as to develop in the reader an exact mathematical imagination.

C. N. MILLS.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 44, no. 3, July, 1922 (published February, 1923): "Plane involutions of order four" by T. R. Hollcroft, 163–171; "Differential equations with a continuous infinitude of variables" by I. A. Barnett, 172–190; "The factorization of the rational primes in a cubic domain" by G. E. Wahlin, 191–203; "On the structure of finite continuous groups with a single exceptional infinitesimal transformation" by S. D. Zeldin, 204–216; "The Laplace-Poisson mixed equation" by K. P. Williams, 217–224; "The four color problem" by P. Franklin, 225–236.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, no. 8, November, 1922: "The twenty-ninth summer meeting of the American Mathematical Society" by R. G. D.

Richardson, 377–387; "A property of certain functions whose Sturmian developments do not terminate" by O. D. Kellogg, 388–389; "Relating to the proof of an existence theorem for a certain type of boundary value problem" by H. I. Davis, 390–394; "Note concerning the roots of an equation" by K. P. Williams, 394–396; "The complete existential theory of Hurwitz's postulates for Abelian groups and fields" by B. A. Bernstein, 397–399; "On Kummer's memoir of 1857 concerning Fermat's last theorem" by H. S. Vandiver, 400–407; Review by D. Jackson of H. Hahn, *Theorie der reellen Funktionen*, Vol. 1 (Berlin, 1921), 408–411, "Shorter notices," 412–417 [Reviews: by E. B. Wilson of J. H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge, 1920) and *Problems of Cosmogony and Stellar Dynamics* (Cambridge, 1919); by E. B. Cowley of G. Loria, *Storia della Geometria Descrittiva dalle Origini sino ai Giorni nostri* (Milano, 1921); by F. M. Morgan of H. Hilton, *Plane Algebraic Curves* (Oxford, 1920); by J. B. Shaw of D. Bhattacharyya, *Vector Calculus* (Calcutta, 1920); by J. W. Young of L. Bolton, *An Introduction to the Theory of Relativity* (New York, 1921); by C. Sisam of L. Berzolari, *Geometria analitica*, Part 1, *Il Metodo delle Coordinate* (2d ed. Milano); by F. Cajori of the Gauthier-Villars reprints of memoirs by Laplace on *Probabilities*, by Lavoisier and Laplace on *Heat*, and by Ampère on *Electromagnetism and Electrodynamics;* and by L. C. Mathewson of L. Baumgartner, *Gruppentheorie* (Berlin, 1921)]; Notes, 418–420; New publications, 421–424— No. 9, December, 1922: "Condition that a tensor be the curl of a vector" by L. P. Eisenhart, 425–427; "A new generalization of Tchebycheff's statistical inequality" by B. H. Camp, 427–432; "Two theorems on multiple integrals" by P. Franklin, 433–435; "Kirkman parades" by F. N. Cole, 435–437; "Impossibility of restoring unique factorization in a hypercomplex arithmetic" by L. E. Dickson, 438–442; "A revision of the Bernoullian and Eulerian functions" by E. T. Bell, 443–450; "On the existence of curves with assigned singularities" by J. L. Coolidge, 451–455; "A generalization of normal congruences of circles" by J. L. Walsh, 456–462; "A correction" by E. Hille, 462; Reviews: by C. N. Moore of A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre* (Vol. 1, 3d part, Leipzig and Berlin, 1921), 463–465, and of A. Wangerin, *Theorie des Potentials und der Kugelfunktionen* (Vol. 2, Berlin and Leipzig, 1921), 465; by G. E. Wahlin of L. Bianchi, *Lezioni sulla Teoria dei numeri algebrici e Principi d'Aritmetica analitica* (Pisa, 1921), 466, and of E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig and Berlin, 1918), 466–467; by H. J. Ettlinger of L. Bieberbach, *Lehrbuch der Funktionentheorie* (Vol. 1, Leipzig, 1921) and *Funktionentheorie* (Leipzig, 1922), 467–468, and of H. Burkhardt, *Funktionentheoretische Vorlesungen* (Berlin: Vol. 1, part 1, *Algebraische Analysis*, 3d edition, 1920; part 2, *Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen*, 5th edition, 1921), 475; by O. D. Kellogg of H. Rothe, *Vorlesungen über höhere Mathematik* (Vienna, 1921), 468–469; by J. W. Young of L. B. Benny, *Plane Geometry* (Glasgow and Bombay, 1922), 469, and of F. F. P. Bisacre, *Applied Calculus* (Glasgow and Bombay, 1921), 471; by W. C. Graustein of G. Bouligand, *Cours de géométrie analytique* (Paris, 1919), 470–471; by C. L. E. Moore of A. Lotze, *Die Grundgleichungen der Mechanik insbesondere der starren Körper* (Leipzig, 1922), 472; by L. C. Mathewson of J. Poirée, *Précis d'arithmétique* (Paris, 1921), 472–473; by L. W. Dowling of O. S. Adams, *Latitude Developments connected with Geodesy and Cartography* (Washington, 1921), and of C. H. Deetz and O. S. Adams, *Elements of Map Projection* (Washington, 1921), 473; by J. N. Van der Vries of F. Schuh, *Lessen over de Hoogere Algebra* (Vol. 1, Groningen, 1920), 474; by A. D. Pitcher of *Encyklopädie der Mathematischen Wissenschaften*, Vol. 2, part 2 (Leipzig, 1901–1921), 474; and by A. Emch of M. Emanaud, *Géométrie perspective* (Paris, 1921), 475; Notes, 476–478; New publications, 479–483; Thirty-first annual list of papers read before the American Mathematical Society and subsequently published, 484–492. [The number of papers listed is 153, by 97 authors, the highest number of papers by one author being 11; 16 authors have contributed 3 or more papers each to a total of 63 papers]; Index of Volume 28, 493–503.

Bulletin des Sciences Mathématiques, series 2, volume 47, January, 1923: "Sur le mouvement d'une planète dans un milieu résistant" by P. Fatou, 19–40; "Une démonstration d'un théorème de Coriolis" by Niewenglowski, 40–42; "Note relative à l'attraction d'un ellipsoide" by J. Mascart, 43–48.

Journal of the Washington Academy of Sciences, volume 11, no. 19, November 19, 1921: "On the correlation between any two functions and its application to the general case of spurious correlation" by L. J. Reed, 449–455.—Volume 12, no. 1, January 4, 1922: "A mathematical note on the annealing of glass" by E. D. Williamson, 1–6.

L'Enseignement Mathématique, volume 22, no. 5 (published January, 1923): "Un chapitre de méthodologie mathématique, les imaginaires de Galois" by M. Stuyvaert, 249–268;

"Sur les radicaux carrés" by B. Niewenglowski, 269–277; "Démonstration du théorème de Liouville par l'élimination du temps entre les équations de Lagrange" by E. Turrière, 277–285; "La section mathématique de l'institut normal supérieur de Bolivie" by C. Lurquin, 286–290; "Chronique," 290–304 [contains among other things a summary of colloquium lectures given before the Société Mathématique Suisse at Bienne, April 23, 1922, by E. Hecke of Hamburg, on Arithmetic and the Theory of Functions; by M. Plancherel of Zürich, on Difference Equations and Differential Equations; and by W. Blaschke of Hamburg, on Selected Chapters in Differential Geometry]; "Notes et Documents," 304–311; "Bibliographie," 311–324 [Reviews: by A. Buhl of P. Appell, *Education et Enseignement* (Paris, 1922), of E. Goursat, *Leçons sur le Problème de Pfaff* (Paris, 1922), and of B. Levi, *Abbaco da 1 a 20* (Parma. 1922), "Une œuvre italienne digne d'être imitée en toutes les langues"; by R. Wavre of H. Bergson, *Durée et simultanéité* (Paris, 1922), of G. Juvet, *Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu* (Paris, 1922), of P. Lévy, *Leçons d'Analyse Fonctionelle* (Paris, 1922), and of Ch.-J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale* (4th ed., 2 vols., Louvain and Paris, 1921–1922); and by D. Mirimanoff of L. Bieberbach, *Lehrbuch der Funktionentheorie* (Vol. 1, Leipzig and Berlin, 1921)]; "Bulletin Bibliographique," 325–336 [short notices of 28 mathematical books, and lists of articles in current periodicals and of doctors' theses].

Mathematische Annalen, volume 88, nos. 1 and 2, December, 1922: "Über die Komposition der quadratischen Formen" by A. Hurwitz, 1–25; "Über die Anzahl der Klassen positiver ternärer quadratischer Formen von gegebener Determinante" by A. Hurwitz, 26–52; "Zur Theorie der Polynomideale und Resultanten" by K. Henzelt and E. Noether, 53–79; "Algebraische Theorie der Ringe" by W. Krull, 80–122; "Zur Theorie der Kleinschen Ergänzungsrelationen" by H. Falckenberg, 123–135; "Zur Parametrixmethode" by O. Haupt, 136–150; "Die logischen Grundlagen der Mathematik" by D. Hilbert, 151–165 [Quotation: "Meine Untersuchungen zur Neubegründung der Mathematik bezwecken nichts Geringeres, als die allgemeinen Zweifel an der Sicherheit des mathematischen Schliessens definitiv aus der Welt zu schaffen"]; "Eine Bemerkung zu der Arbeit des Herrn Bieberbach 'Über die Verteilung der Null- und Einstellen analytscher Funktionen'" by M. Fekete, 166–168.

School Review, volume 30, no. 7, September, 1922: "Using home-made tests in high schools" by L. Byrne, 536–546 [describes certain tests in algebra and geometry, and draws conclusions in harmony with the recommendations of the National Committee on Mathematical Requirements]; "Higher mathematics in high school" [note on N. B. Rosenberger, *The Place of Elementary Calculus in the Senior High School Mathematics* (New York, 1921)] by E. R. Breslich, 552–553.—No. 10, December: Note by E. R. Breslich on G. Wentworth, D. E. Smith and H. D. Harper, *Fundamentals of Practical Mathematics* (Boston, 1922), 792.—Volume 31, no. 1, January, 1923: Note by E. R. Breslich on the Cleveland Board of Education's *Course of Study and Syllabus: Junior High School Mathematics* (Cleveland, 1922), 70–71; Note by C. A. Stone on W. D. Reeve, *General Mathematics, Book Two* (Boston, 1922), 75–76.

Scientific Monthly, volume 15, no. 6, December, 1922: "Easy group theory" by G. A. Miller, 512–519.—Volume 16, no. 1, January, 1923: "The theory of relativity and its influence on scientific thought" by A. S. Eddington, 34–53.—No. 2, February: "The mathematical sciences in the Latin colonies of America" by F. Cajori, 194–204.

Transactions of the Cambridge Philosophical Society, volume 22, no. 18, February, 1920: "The hydrodynamical theory of the lubrication of a cylindrical bearing under variable load, and of a pivot bearing" by W. J. Harrison, 373–388.—No. 19, May, 1920: "On the integers which satisfy the equation $t^3 \pm x^3 \pm y^3 \pm z^3 = 0$" by H. W. Richmond, 389–403.—Nos. 20 and 21, August, 1920: "On cyclical octosection" by W. Burnside, 405–411; "Congruences with respect to composite moduli" by P. A. MacMahon, 414–424.—No. 22, October, 1920: "On the stability of the steady motion of viscous liquid contained between two rotating coaxal circular cylinders" by K. Tamaki and W. J. Harrison, 425–437.—No. 23, May, 1922: "On a general infinitesimal geometry, in reference to the theory of relativity" by W. Wirtinger, 439–448.—No. 24, May, 1922: "On the fifth book of Euclid's Elements" by J. M. Hill, 449–462; [Quotation: "The object of this paper is to endeavor to recover the train of thought which led the writer (supposed to be Eudoxus, but whom I will refer to as Euclid), of the above-mentioned work, to the formulation of his Fifth Definition (the test for the sameness of two ratios), and his Seventh Definition (the test for distinguishing the greater from the smaller of two unequal ratios)."—No. 25, June, 1922: "The influence of electrically conducting material within the earth on various phenomena of terrestrial magnetism" by S. Chapman and T. T. Whitehead, 463–482.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF ALBION COLLEGE, Albion, Mich. [*1921*, 322.]

As officers for the first semester of the year 1922–1923, the following were elected: President, Judson Foust '23; vice-president, Norma Sleight '24; secretary-treasurer, Sybil Davidson '24; member of program committee, Gordon Wheeler '24: as officers for the second semester: President, Gordon Wheeler '24; vice-president, Cleon Richtmeyer '24; secretary-treasurer, Mary Winegar '23; member of program committee, Walter Moore '24.

The following papers were read:

October 17, 1922: "A bibliography of uses and value of mathematics" and "Contribution of mathematics to the world's progress" by Sybil Davidson '24; "Mathematics applied to professions" by Cleon Richtmeyer '24; "The cultural value of mathematics" by Mary Winegar '23.

November 7: "Human worth of rigorous thinking" by Lester Chamberlain '24; "Mathematics in the United States previous to 1800" by Gordon Wheeler '24.

November 20: "Thirteen proofs of the Pythagorean Theorem" by Evelyn Scott '24 and Elvah Dayton '24.

December 24: "Psychology of errors in mathematics" by Evelyn Palmatier '24; "Psychology of the equation" and "Psychology of problem solving" by Norma Sleight '24.

January 8, 1923: "The roots of the quartic equation" by Judson Foust '23.

January 23: "Ruler-compass construction of regular polygons" by Harold Black '23.

February 26: "Development and application of hyperbolic functions" by Walter Moore '24; "A discussion of circular functions" by William Plumb '24.

March 2: Roll call—Who's Who in mathematics. "Selling mathematics" (presidential address) by Gordon Wheeler '24; "Some of the difficulties of calculus teaching" by Professor E. R. Sleight.

March 16: "Confocal conics" by Mary Bachelor '24; "Some common fallacies in mathematics" by Frances Howlett '23.

April 17: Roll call—Theorem with name of discoverer. "Hyperspace" by Helen Camburn '24; Discussion of fourth dimension—Professor E. R. Sleight, leader.

May 1: "The nth root of arithmetical numbers" by Mary Winegar '23; "Magic squares" by Frances Howlett '23.

May 15: A mathematician noted in some other line. "History and transcendency of π" by Cleon Richtmeyer '24; "Astrology" by Sybil Davidson '24.

May 29: Social evening and election of officers.

(Reported by Professor E. R. Sleight.)

THE MATHEMATICAL CLUB, HARVARD UNIVERSITY, Cambridge, Mass. [*1921*, 274.]

The following officers were elected for 1922–1923: President, H. W. Brinkman, Gr.; secretary-treasurer, B. Z. Linfield, Gr.; faculty adviser, Dr. Joseph L. Walsh. Papers were read as follows:

October 11, 1922: "Elementary definitions of the trigonometric functions" by Professor W. F. Osgood.

October 24: "Continuation of analytic functions" by M. H. Stone, instructor.

November 7: "The four color map problem" by Dr. Philip Franklin, instructor.

November 21: "Asymptotic series" by Bernard Koopman, instructor.

December 5: "A mathematician's tour in Europe" by Professor R. C. Archibald, of Brown University.

December 19: "Automorphic functions" by Isador Sheffer '23.

January 5, 1923: "Complex plane analytics in 4-space" by Dr. James Taylor, of the Massachusetts Institute of Technology.

February 20: "Nets and the Dirichlet problem" by Dr. Norbert Wiener, of the Massachusetts Institute of Technology.
March 6: "The equations of elasticity" by Julian Holley, Gr.
March 27: "A method of series in elasticity" by Carl Garabedian, Gr.
April 10: "Some problems in algebraic geometry" by Professor J. L. Coolidge.
May 1: "Squaring the circle" by David Widder, instructor.
May 15: "The mathematical works of Professor Charles Leonard Bouton" by Professor W. C. Graustein.

(Reported by B. Z. Linfield.)

The Mathematics Club of Hood College, Frederick, Md.

On December 7, 1922, the Mathematics Club of Hood College was organized to stimulate interest in Mathematics and to present to the students extra-curriculum topics. All students taking elective courses in mathematics are eligible to membership. Goldae Biser '23 was elected president; Miss Margaret Packer, instructor, is acting secretary; Professor Lillian O. Brown is faculty adviser. The following is the program for the year 1922–1923:
December 7, 1922: "Anecdotes about famous mathematicians" by all members.
January 12, 1923: "Mathematics and life activities" by Professor Clara Bacon of Goucher College.
February 23: "Three classical geometric problems" by Dorothy Eyler '23, Pauline Beachley '23, and Sarah Markley '23.
March 16: "Hindu arabic numerals" by Goldae Biser '23; "Development of mathematical symbols" by Hazel Zimmerman '23.
April 20: "Mathematical prodigies" by Ruth Feaga '24 and Ruth Michael '24; "Curiosities of numbers" by Helen Goodfellow '24 and Grace Allen '24.
May 18: "Women in mathematics" by the sophomore members.
June 2: Picnic.

(Reported by Miss Packer.)

The Junior Mathematics Club of the University of Wisconsin, Madison, Wis.

[*1921*, 391.]

The Junior Mathematics Club, with a membership of thirty, held meetings in 1922–1923 as follows:
October 26, 1922: General discussion and business meeting.
November 9: "Music and mathematics" by Professor A. Dresden.
November 23: "Mathematical logic" by Frank Bruner '25; "Relation of high school mathematics to college algebra" by Reinhard Hein '26.
December 7: "Theory of probability" by Professor L. W. Dowling.
March 8, 1923: "Geometry in the compass plane" by Pearl Anderberg '23.
March 22: "Hyperbolic synthetic geometry" by Elizabeth Baird '23.

(Reported by Miss Viola Jenson, Secretary.)

TO THE COMPLEX VARIABLE.

By Josephine Reddish, Knoxville, Tenn.

As I pore o'er your pages by lamplight,
Oh, wonderful book of my dreams,
I can scarce see the ρ's and the θ's
For the glint of those dazzling beams.

For there on the pages before me,
'Mid the lore of the ancient and wise,
Gleams with wondrous unquenchable brilliance
The light of those marvelous i's.

The x's and y's are commingled,
Confused with the u's and the v's;
And my mind and my heart are a-tremble
With the pangs of some strange new dz's.

When weary from long hours of labor,
I find myself starting to nap,
And my mind refuses to function
In the maze of a conformal map,

'Round the curves of the equipotential,
  Coursing in on the long lines of flow,
From regions divine, holomorphic,
  Comes stealing a quickening glow.

I cast off my growing inertia,
  Fickle goddess in complex disguise,
As I bask with the heart of a lover,
  In the light of your wonderful i's.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and Norman Anning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3025. Proposed by E. H. CLARKE, Hiram College.

Sum the infinite series

$$1 + \frac{2^k x^2}{2!} + \frac{3^k x^4}{4!} + \frac{4^k x^6}{6!} + \cdots \frac{n^k x^{2n-2}}{(2n-2)!} + \cdots,$$

k a positive integer.

3026. Proposed by B. F. FINKEL, Drury College.

A flat board 12 ins. square is suspended in a horizontal position by strings attached to its four corners, A, B, C, D, and a weight equal to the weight of the board is laid upon it at a point 3 ins. distant from the side AB and 4 ins. from AD; find the relative tensions in the four strings.

From Bowser's *Analytical Mechanics*, p. 92.

Is this a determinate problem? If not, why not?

3027. Proposed by C. N. SCHMALL, New York City.

A parabola whose base (double ordinate) is h and altitude k has a circle inscribed of diameter d, and a circle circumscribed, of diameter D. Show that $D + d = h + k$. Note that k can not be $< h/2$.

3028. Proposed by NORMAN ANNING, University of Michigan.

Equilateral triangles are described on the sides of a right triangle. Dissect the triangles on the legs and reassemble the parts to form the triangle on the hypotenuse.

3029. Proposed by J. ROSENBAUM, Milford, Conn.

To locate two points, D and E, on the sides AB and BC of a triangle ABC such that $AD : DE : EC$ shall be equal to $p : q : r$, where p, q, and r are given line segments.

The above is a generalization of problem 2816 (*1920*, 134).

3030. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the envelope of the bisector of the angle that a given segment subtends at a variable point of a given line.

3031. Proposed by S. A. COREY, Des Moines, Iowa.

On page 183 of the 1922 Monthly, Professor Bennett has given an expression for the limit of error in evaluating a definite integral by Simpson's rule. Give a similar expression for limit of error when evaluation is made by Weddle's rule; also when evaluation is made by the similar but somewhat more accurate formula:

$$\int_0^x f(x)dx = \frac{x}{840}[41(y_0 + y_6) + 216(y_1 + y_5) + 27(y_2 + y_4) + 272y_3].$$

(See Monthly, June–July, *1912*, 125, and Feb., *1918*, 88.)

3032. Proposed by OTTO DUNKEL, Washington University.

If $a_1, a_2, \cdots, a_n$ are any real or complex quantities which satisfy the equation

$$x^n - na_1x^{n-1} + {}_nC_2a_2^2x^{n-2} + \cdots + (-1)^i{}_nC_ia_i^ix^{n-i} + \cdots + (-1)^na_n^n = 0,$$

where ${}_nC_i = n!/(n-i)!i!$, prove that $a_1 = a_2 = \cdots = a_n$.

3033. Proposed by A. A. BENNETT, University of Texas.

Prove the existence in the inversion plane, for an arbitrary positive integer n, of a configuration of 2^{n-1} points and 2^{n-1} circles, n of the points lying on each circle and n of the circles passing through each point. Show that an arbitrary set of n points entirely unrestricted save for distinctness may be taken as points of the configuration, and that there will then remain two degrees of freedom in the construction of the figure. State the Euclidean theorem in case one of the points is at infinity. Show also that in case n concurrent circles of the configuration are congruent, all the circles are congruent.

3034. Proposed by J. L. RILEY, Stephenville, Texas.

If every root of the equation $f'(x) = 0$ be subtracted from every root of the equation $f(x) = 0$, find the sum of the reciprocals of the differences.

SOLUTIONS.

2871. [1921, 36]. Proposed by the late L. G. WELD.

Weight being disregarded, a package may be admitted to the parcels post if the length plus the greatest girth, measured transversely to the length, does not exceed 72 inches. What is the size of the smallest square window through which all admissible rectangular boxes can be passed?

SOLUTION BY A. A. BENNETT, University of Texas.

Let a, b, c be the dimensions of the rectangular box where $a \geqq b \geqq c$. Then a is the length, and $2(b + c)$ the girth. Thus $a + 2(b + c) \leqq 72$ is the condition imposed by law for rectangular boxes. A box may be fitted snugly through a square window in each of two ways. The length does not enter this question. In one way the sides of the box are parallel to those of the window and the smallest square window has each edge e equal to b. In the other way the box fits in diagonally and the square window is readily seen to have each edge e equal to $(b + c)/\sqrt{2}$. The most troublesome box is clearly that of the most troublesome cross section so that the length a, by definition not less than b, must be a minimum that is equal to b in the worst case. For boxes fitted in snugly obliquely, the edge of the square window is proportional to the girth so that $b + c$ must be a maximum or, a, a minimum. One arrives then at a cubical box, $a = b = c$. But this does not give the solution, since a cubical box by being put in parallel to the edges of the window will go through a smaller square window. The worst case is, therefore, one in which the box fits snugly in each way, diagonally and parallel, so that $e = b = (b + c)/\sqrt{2} = a$, or $a + 2(b + c) = (1 + 2\sqrt{2})e = 72$, and the square window has each edge equal to $72/(1 + 2\sqrt{2})$ inches or $18.807 -$ inches and an area of 353.703 sq. in.

The problem may be extended by asking for the *rectangular* window of smallest area. One's first thought might be that no answer could be obtained. However, the smaller edge of the window must be $14\frac{2}{5}$ inches since this is only just large enough to admit a cubical box where $a = b = c$ and $a + 2(b + c) = 72$. The other, longer edge satisfies an equation obtained by a discussion analogous to that above.

For convenience denote the shorter edge of the window by e, and suppose the longer edge to be equal to b. The box of cross section $b \times c$ when just fitted in diagonally will divide the side of length b into segments $2bc^2/(b^2 + c^2)$ and $b(b^2 - c^2)/(b^2 + c^2)$ and the side e in segments $2b^2c/(b^2 + c^2)$ and $c(b^2 - c^2)/(b^2 + c^2)$. Further $3b + 2c = 5e$ when $a = b$; also $2b^2c/(b^2 + c^2) + c(b^2 - c^2)/(b^2 + c^2) = e$. Thus $(c/e) = \frac{5}{2} - \frac{3}{2}(b/e)$. Substituting in $(c/e)[3(b/e)^2 - (c/e)^2]/[(b/e)^2 + (c/e)^2] = 1$, we have $9(b/e)^3 + 101(b/e)^2 - 285(b/e) + 175 = 0$, from which the trivial solution $b/e = 1$ may be at once removed, leaving $9(b/e)^2 + 110(b/e) - 175 = 0$. Whence $b = 16\sqrt{46} - 88$, or $20.517 +$ in. for the longer edge of the rectangular window, giving an area of only $295.445 -$ sq. in. for the smallest rectangular window.

NOTE BY OTTO DUNKEL, Washington University, and H. P. MANNING, Providence, R. I.—

We let $e = 72/5$ and $b_1 = (10\sqrt{46} - 55)e/9$. Then a box in which $b = b_1$ will fit diagonally in a window whose dimensions are $b_1 \times e$. Professor Bennett has proved this, but he has not proved that all boxes can be passed through such a window.

Let $u \times e$ be the dimensions of a window in which a box of cross section $b \times c$ (where $3b + 2c = 5e$) will just fit diagonally. u must satisfy the equation $(b - c)^2(u + e)^2 + (b + c)^2(u - e)^2 - 2(b^2 - c^2)^2 = 0$ (see *1920*, 327). Eliminating c we have

$$f(u, b) = 50(b - e)^2(u + e)^2 + 2(5e - b)^2(u - e)^2 - 25(b - e)^2(5e - b)^2 = 0. \tag{1}$$

If $b = e$, $u = e$. If $b > e$ and $< (5 - 2\sqrt{2})e$ this equation has one root less than e and one root greater than e; for $f(e, b) = 25(b - e)^2[8e^2 - (5e - b)^2]$, which is negative. It is the larger value of u that belongs to our problem.

For $u = b$, f becomes $(b - e)^2[50(b + e)^2 - 23(5e - b)^2]$, which is zero when $b = b_1$. Then if $b < b_1$ this is negative and the value of u which is greater than e is greater than b, and if $b > b_1$ this is positive and the value of u which is greater than e is less than b. In the former case the box passes through a smaller window when it is put through in the first way, with its sides parallel to the sides of the window, than when passed through in the second way. In the second case the reverse is true.

It will now be shown that in the second case u is also less than b_1, and therefore that all boxes for which $b > b_1$ will pass diagonally through a window whose dimensions are $b_1 \times e$.

For $b = 5e/3$, f reduces to $\frac{400}{9}e^2(u^2 - \frac{16}{9}e^2)$. That is, $u = 4e/3$, which is less than b_1. Now the coefficient of u^2 in $f(u, b)$ does not vanish for any real value of b, and the two roots of the equation $f(u, b)$ are continuous single-valued functions of b in the interval $e < b < 5e$. When b varies from $5e/3$ to b_1, the value of u which is greater than e will remain less than b_1, or for some particular value of b will become equal to b_1.

When u equals b_1, f becomes

$$\varphi(b) = 50(b - e)^2(b_1 + e)^2 + 2(5e - b)^2(b_1 - e)^2 - 25(b - e)^2(5e - b)^2.$$

One root of $\varphi(b)$ is b_1. We shall prove that $\varphi(b)$ does not become zero for any value of b between b_1 and $5e/3$.

$$\begin{aligned}
\varphi(0) &= 25e^2(4b_1^2 - 21e^2), \text{ which is negative.}\\
\varphi(e) &= 32e^2(b_1 - e)^2, \text{ which is positive.}\\
\varphi(b_1) &= 0.\\
\varphi'(b_1) &= 4(b_1 - e)(b_1^2 + 244b_1e - 345e^2), \text{ which is positive.}
\end{aligned}$$

This shows that when b passes from e to b_1, φ must first be negative in order to increase to zero.

$$\begin{aligned}
&\varphi(5e/3) = \tfrac{400}{9}e^2(b_1^2 - \tfrac{16}{9}e^2), \text{ which is positive.}\\
&\varphi(\infty) \text{ is negative.}
\end{aligned}$$

φ then has one root between 0 and e, one between e and b_1, one at b_1, and one greater than $5e/3$. All of the roots of φ are accounted for, and this proves that it does not become zero between b_1 and $5e/3$.

In other words, for no value of b between b_1 and $5e/3$ can u become equal to b_1, and as u is less than b_1 when $b = 5e/3$, it must remain less than b_1 for all these values of b. All boxes can pass through the window of dimensions $b_1 \times e$, those for which $b \leqq b_1$ by the first way, and those for which $b \geqq b_1$ by the second way.

The shorter side of the window must be at least equal to e. When the shorter side is equal to e the longer side must be equal to b_1. Now if the shorter side is made a little longer, the longer side may be made a little shorter, at least for some boxes. It has not been shown that we cannot in this way make the area of the window smaller. It will be sufficient to show that a window in which a box will fit snugly both ways must have a larger area when its shorter side is of length $v > e$, for with a given v no smaller window will admit this particular box.

Writing $u + v$ and $u - v$ in (1) for $u + e$ and $u - e$, putting $u = b$, and, finally eliminating b by putting $z = bv$, we find that z will be a function of v whose equation may be written

$$\left(\frac{z + v^2}{5ev - z}\right)^2 + \frac{1}{25}\left(\frac{z - v^2}{z - ev}\right)^2 - \frac{1}{2} = 0.$$

Now if we let U and V denote the expressions in the two parentheses we can determine their signs and the signs of their derivatives with respect to z and v, and so prove that

$$\frac{dz}{dv} = -\frac{25U\dfrac{\partial U}{\partial v} + V\dfrac{\partial V}{\partial v}}{25U\dfrac{\partial U}{\partial z} + V\dfrac{\partial U}{\partial z}} > 0.$$

This will complete the proof that the window of dimensions $b_1 \times e$ is the rectangular window of smallest area that will admit all possible boxes.

Also solved by T. M. Blakslee and F. L. Wilmer.

2958 [1922, 82]. Proposed by R. P. BAKER, University of Iowa.

Over a frictionless pulley a weightless cord sustains at one end a mass M, while the other end is wound on the axle of a wheel of mass M and moment of inertia N. At the zero of time the wheel revolves with angular velocity ω and tends to wind up the cord. Describe the motion neglecting friction.

Solution by J. B. Reynolds, Lehigh University.

Let the vertical distances from some fixed horizontal line in the plane of motion *down* to the mass M and to the wheel be x and x_1, respectively. Let θ be the angle turned through by the wheel in unwinding the cord and let T be the tension of the cord. Then we have the geometric condition that $x + x_1 - r\theta = \text{const.}$, r being the radius of the axle. For the equations of motion we have

$$Mg - T = M\ddot{x}, \qquad Mg - T = M\ddot{x}_1, \qquad Tr = N\ddot{\theta}.$$

These equations give

$$\ddot{\theta} = \frac{2Mgr}{2N + Mr^2}; \qquad \text{whence} \qquad \dot{\theta} = \frac{2Mgr}{2N + Mr^2}t - \omega,$$

showing that the cord will wind up for $\dfrac{(2N + Mr^2)}{2Mgr}\,\omega$ seconds and thereafter unwind. Again

$$\ddot{x} = g - \frac{T}{M} = \frac{gMr^2}{2N + Mr^2} \qquad \text{and} \qquad x = \frac{gMr^2}{2(2N + Mr^2)}t^2.$$

That is, the mass M will descend with constant acceleration. Further,

$$\ddot{x}_1 = \ddot{x} = \frac{gMr^2}{2N + Mr^2} \qquad \text{and hence} \qquad \dot{x}_1 = \frac{gMr^2}{2N + Mr^2}t - r\omega.$$

That is, the wheel will rise until $t = \dfrac{2N + Mr^2}{gMr}\,\omega$. Now if a is the initial value of x_1,

$$x_1 = \frac{gMr^2}{2(2N + Mr^2)}t^2 - r\omega t + a \qquad \text{and} \qquad a - x_1 = \frac{2N + Mr^2}{2Mg}\,\omega^2 \qquad \text{where} \qquad t = \frac{2N + Mr^2}{gMr}\,\omega.$$

That is, the wheel will rise a distance $\dfrac{2N + Mr^2}{gMr}\,\omega^2$ above its first position and will thereafter descend with constant acceleration.

Also solved by Philip Fitch and William Hoover.

2965 [1922, 130]. Proposed by C. N. MILLS, Tiffin, Ohio.

If a quadrilateral inscribed in a square has the diagonals a and b and the area A, show that the area of the square is $\dfrac{a^2b^2 - 4A^2}{a^2 + b^2 - 4A}$.

Solution by J. B. Reynolds, Lehigh University.

Let φ be the angle between b and a side of the square into which it projects and let θ be the angle between a and b. We will then have, since either equals the side of the square,

$$a \sin(\theta + \varphi) = b \cos\varphi; \qquad \text{whence} \qquad \tan\varphi = (b - a\sin\theta)/a\cos\theta$$

and, if S is the area of the square,

$$S = b^2 \cos^2 \varphi = a^2b^2 \cos^2 \theta/(a^2 + b^2 - 2ab \sin \theta).$$

But $A = \frac{1}{2}ab \sin \theta$; whence $S = (a^2b^2 - 4A^2)/(a^2 + b^2 - 4A)$.

Also solved by J. Q. McNatt, A. Pelletier and A. V. Richardson.

2967 [1922, 179]. Proposed by ELIJAH SWIFT, University of Vermont.

A plane revolves about one of two non-coplanar lines as an axis. Find the locus, *in the plane*, of the intersection of the plane and the other line.

Solution by R. S. Underwood, Alabama Polytechnic Institute.

Let the x-axis be the axis of rotation and the z-axis, the common perpendicular to the two lines. Then the equations of the other line may be written $z = k$, $Y = mx$. If (x, y) are the coördinates in the rotating plane of its intersection with the other line, the origin being the same as for the space coördinates, then $y^2 - Y^2 = z^2$ or $y^2 - m^2x^2 = k^2$. Hence the curve is an hyperbola unless the two lines are perpendicular, and in this case the curve reduces to two coincident straight lines. Both branches of the hyperbola are traced in a complete revolution of the plane.

Note by the Editors: The above results are evident without the use of equations, for we may suppose that the plane is fixed and that the other line rotates about the first line, fixed in the plane. The rotating line describes an hyperboloid of one sheet, and, since the plane passes through the axis of the hyperboloid, it cuts out an hyperbola.

Also solved by A. Bogard, H. C. Bradley, William Hoover, W. J. Patterson, A. Pelletier, F. L. Wilmer and the Proposer.

2978 [1922, 271]. Proposed by C. N. SCHMALL, New York City.

Given the equation of the general cubic, $f(x, y) \equiv ax^3 + 3bx^2y + 3cxy^2 + dy^3 + a_1x^2 + 2b_1xy + c_1y^2 + a_2x + b_2y + c_3 = 0$, show that the three asymptotes of the curve will be concurrent if

$$\begin{vmatrix} a & b & a_1 \\ b & c & b_1 \\ c & d & c_1 \end{vmatrix} = 0.$$

Solution by J. B. Reynolds, Lehigh University.

Let $y = sx + k$ be an asymptote. Substituting this in the equation of the curve we find the equations of condition for infinite roots to be

$$a + 3bs + 3cs^2 + ds^3 = 0, \tag{1}$$

$$3k(b + 2cs + ds^2) + (a_1 + 2b_1s + c_1s^2) = 0. \tag{2}$$

The equation of the asymptote may then be written

$$3(b + 2cs + ds^2)y - 3(bs + 2cs^2 + ds^3)x + (a_1 + 2b_1s + c_1s^2) = 0,$$

and this equation by aid of (1) as

$$3(a + 2bs + cs^2)x + 3(b + 2cs + ds^2)y + (a_1 + 2b_1s + c_1s^2) = 0. \tag{3}$$

If s_1, s_2, s_3 are the roots of (1), then the determinant of the three equations obtained by inserting these roots in (3) is readily seen to be the product

$$18 \begin{vmatrix} 1 & s_1 & s_1^2 \\ 1 & s_2 & s_2^2 \\ 1 & s_3 & s_3^2 \end{vmatrix} \begin{vmatrix} a & b & a_1 \\ b & c & b_1 \\ c & d & c_1 \end{vmatrix}$$

and hence the theorem of the problem follows at once.

Another way of obtaining the result is as follows: If the determinant in the problem is zero, then the three lines

$$3ax + 3by + a_1 = 0,$$
$$3bx + 3cy + b_1 = 0,$$
$$3cx + 3dy + c_1 = 0$$

meet in a point. Now (3) is a linear combination of these three lines and hence passes through the same point for all values of s.

Also solved by A. Bogard, H. A. DoBell, R. E. Gaines, A. M. Harding, William Hoover, A. V. Richardson, Hazel E. Shoemaker, F. L. Wilmer and the Proposer.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor WILLIAM CAIN, of the University of North Carolina, has been awarded the James R. Croes medal of the American Society of Civil Engineers for his paper on *The circular arch under normal loads.*

Assistant Professor G. W. MULLINS, of Barnard College, Columbia University, has been promoted to an associate professorship of mathematics.

Assistant Professor F. N. BRYANT, of Syracuse University, has been promoted to a full professorship of mathematics.

Mr. B. F. KIMBALL has been appointed instructor of mathematics at Tulane University.

Announcement is made that the Council of the American Mathematical Society has decided to proceed to raise an endowment fund of at least one hundred thousand dollars and a Committee on Endowment has been appointed, consisting of Julian L. Coolidge, Harvard University (Chairman); Arnold Dresden, University of Wisconsin; Griffith C. Evans, Rice Institute; Robert Henderson, Equitable Life Assurance Society; and George E. Roosevelt, 30 Pine Street, New York (Treasurer). While recognizing that there are many other worthy objects, it has been voted that income may be used for publishing of material pertaining to research, such as the transactions and other journals, the colloquium lectures, and books and treatises. The release of a portion of the regular income will permit the Society to attend to some of the other pressing matters.

A campaign for a wider circulation and for an extension of its influence is now being conducted by "SCIENTIA," the international review published at Milan, Italy. This journal aims to promote the work of the philosophic synthesis of science and the intellectual fraternization of peoples. Its authoritative but non-technical articles (each published simultaneously in the language of the author and in French) give a summary and evaluation of scientific truth not found elsewhere. As such a journal necessarily makes its appeal to a select group of intellectual people, it can succeed in its aims only if supported by all those who believe in the formation of an international scientific mind infused with the spirit of a philosophic interpretation of scientific truth. American subscribers can purchase "SCIENTIA" through Messrs. Williams & Wilkins, Mount Royal and Guilford Avenues, Baltimore, Md., at ten dollars per year (twelve issues). Among contributors to recent or forthcoming issues are the following Americans: W. D. MacMillan, S. Nearing, L. E. Dickson, W. S. Adams, and R. D. Carmichael.

CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Annual Meeting, University of Cincinnati, December 27–29, 1923

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Knox College, Galesburg, May 4–5

IOWA, Cornell College, Mount Vernon, April 27–28

KANSAS, Topeka, January 20

KENTUCKY, University of Kentucky, Lexington, April

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Baltimore, May 12

MINNESOTA, St. Paul, May 27

MISSOURI, University of Missouri, Columbia, November 30–December 1

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April

SOUTHEASTERN, Agnes Scott College, Decatur, March 10

TEXAS, Houston, December 1–2

THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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LIFE MEMBERSHIP IN THE MATHEMATICAL ASSOCIATION.

In accordance with the action of the Association at its Rochester meeting, members may obtain life membership in the Association by the payment, at the first of any calendar year, of an amount indicated in the accompanying table. In estimating one's age the birthday anniversary nearest to the first of January when payment is made should be taken.

Age	Fee	Age	Fee	Age	Fee
20	$77.84	40	$65.21	60	$43.54
21	77.40	41	64.33	61	42.28
22	76.94	42	63.44	62	41.02
23	76.47	43	62.52	63	39.76
24	75.97	44	61.57	64	38.49
25	75.46	45	60.60	65	37.22
26	74.93	46	59.61	66	35.96
27	74.38	47	58.59	67	34.69
28	73.81	48	57.55	68	33.44
29	73.22	49	56.48	69	32.20
30	72.60	50	55.40	70	30.96
31	71.97	51	54.29	71	29.74
32	71.31	52	53.16	72	28.54
33	70.63	53	52.01	73	27.35
34	69.93	54	50.85	74	26.18
35	69.20	55	49.66	75	25.04
36	68.45	56	48.47	76	23.92
37	67.68	57	47.25	77	22.82
38	66.88	58	46.03	78	21.76
39	66.05	59	44.78	79	20.72

W. D. CAIRNS, *Secretary-Treasurer.*

THE APRIL MEETING OF THE IOWA SECTION.

The twelfth regular meeting of the Iowa Section of the Mathematical Association of America was held, in conjunction with the thirty-seventh annual meeting of the Iowa Academy of Science, at Cornell College, Mount Vernon, Iowa, on April 27 and 28, 1923. The Chairman of the Section, Professor C. W. Emmons, presided at both the Friday afternoon and Saturday morning sessions.

There were thirty-seven in attendance, including the following twenty-five members of the Association: O. W. Albert, E. W. Chittenden, L. M. Coffin, Marian E. Daniells, R. M. Deming, C. W. Emmons, Fay Farnum, C. Gouwens, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, E. E. Moots, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, C. W. Strom, John Theobald, J. S. Turner, F. M. Weida, C. W. Wester, W. H. Wilson, Roscoe Woods.

Dinner was enjoyed together Friday evening, followed by a number of brief impromptu speeches. At the business meeting the following officers were elected for 1923–1924: Chairman, F. M. McGAW, Cornell College; Vice-chairman,

J. V. McKelvey, Iowa State College; Secretary-treasurer, J. F. Reilly, University of Iowa. A committee consisting of Professors McClenon, Gouwens and Weida presented the following resolution:

"Resolved that the Iowa section of the Mathematical Association of America hereby expresses its feeling of deep regret at the death of Mr. T. M. Blakslee of Ames. Mr. Blakslee had been for many years a member of this section, and had rendered excellent service to the cause of mathematics, especially through his contributions to the Monthly. His death is a loss to our state and to the nation.

"Resolved further that a copy of this resolution be spread upon the minutes of the Iowa section, and published in the Monthly."

The next fall meeting of the Section will be held at Des Moines in November, and the next spring meeting at Ames in April, 1924.

The following papers were presented:

(1) "On the correction of a common error in the calculation of the mean deviation from a given frequency distribution" by Professor H. L. Rietz;

(2) "On the geodesic in four space" by Professor C. Gouwens;

(3) "A general expression for the scedastic function for the generalized double frequency distribution" by Professor E. R. Smith;

(4) "Leibnitz's contribution to the history of complex numbers" by Professor R. B. McClenon;

(5) "Some curves met with in the conformal representation of integral transcendental functions" by Professor McClenon;

(6) "The definite integral in a first course in calculus" by Professor J. V. McKelvey;

(7) "Certain preliminaries to the calculus" by Professor C. W. Emmons;

(8) "The cochleöide" by Professor Roscoe Woods;

(9) "On the theory of wave filters with an application to the theory of acoustic wave filters" by Professor E. W. Chittenden;

(10) "Some functional equations suggested by the mean value theorem" by Professor W. H. Wilson;

(11) "The differentiation of the trigonometric functions" by Professor Wilson;

(12) "What is mathematics?" by Professor J. S. Turner;

(13) "An application of finite differences" by Professor J. F. Reilly;

(14) "The cycloid and its companion" by Professor E. E. Moots.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Rietz dealt with the source and elimination of an error in certain familiar textbook methods for the calculation of the mean deviation when values are grouped into class frequencies. That the error in question is important was shown by examples occurring in statistical practice, where the per cent. error in the mean deviation would vary from 0 to 6.

2. Professor Gouwens showed that, by making use of the fundamental lemma

of the calculus of variations and making one change of independent variable, we have a direct method for obtaining the geodesic in four-space without making use of the δ-process so common in the textbooks on relativity.

3. By means of a general expansion of the double frequency function into a series of Hermitian polynomials, Professor Smith obtained the equation of the scedastic curve. The equation brings out a simple relation between the scedastic and the regressional functions.

4. The pioneer work done by Leibnitz in the study of complex numbers, particularly in connection with the solution of algebraic equations, does not seem to have been sufficiently emphasized by historians of mathematics. In this paper Professor McClenon gave an analysis of this part of Leibnitz's contribution, and brought out the fact that he was a direct precursor of Tschirnhausen, Jean Bernoulli, and Euler in this field.

5. In this paper Professor McClenon mentioned a set of curves that have interesting properties. The equations were obtained by setting simple rational functions of e^{zi} equal to a constant.

6. Professor McKelvey outlined a method of introducing and defining the definite integral by algebraic methods without any use of differentiation or the inverse process. A number of physical and geometrical examples were given which can be set up as the limit of a sum of products, the sum being obtainable by use of formulas for arithmetical or geometrical series or sum of the squares of the integers, etc.

7. Professor Emmons showed the advantage to the beginning student of calculus of a thorough familiarity with the principles of curve tracing. Textbooks generally leave the subject of curve tracing until after the derivative has been introduced. A more opportune time for gaining facility in curve tracing is found to be when the functional notation is introduced. The effect upon the shape, size and nature of a given curve produced by special cases of the homographic transformation may be investigated by the student and the insight thus derived will have its effect upon his later progress.

8. Professor Woods, after making some remarks of an historical nature, discussed the different ways of defining the cochleöide. The relations between the cochleöide and other curves were brought out as well as some of the properties of the curve. To the theorems concerning the tangents to the curve he added some remarks regarding the loci connected with the normals. Finally the applications of the curve were brought out and its connection with a recent problem in physics was given.

9. The theories of electric and acoustic wave filters composed of a number of equal sections in series are based upon linear difference equations of the second order with constant coefficients. If the sections of the filter are unlike the coefficients are variable. After obtaining the necessary and sufficient conditions that a wave of given frequency be transmitted from section to section of a filter without attenuation, Professor Chittenden applied these conditions in the case of filters with alternate sections equal. It was found that, in general, filters of this type transmit two distinct bands of frequencies without attenuation.

10. Professor Wilson investigated certain functional relationships, some of which are special cases of the mean value theorem of the differential calculus while others are generalizations of this theorem and its extensions. The most general continuous functions which satisfy the various equations were found and some of the relationships existing between these functions noted.

11. The belief that students entering the first course in calculus have little or no knowledge (either theoretical or practical) of circular measure and that they have not a working knowledge of proportion leads Professor Wilson to advocate the evaluation of the limit of the quotient of the sine of an angle divided by the angle (as the angle approaches zero) in the form $2\pi/R$, where R is the number of angular units contained in the complete angle about a point. This limit is approached by a comparison of areas one of which, the area of a sector, is derived from its ratio to the area of the entire circle. The general formulæ thereafter found for the derivatives of the trigonometric and inverse trigonometric functions are used in the classroom for most of the semester, emphasis being placed on the simplification introduced by the use of radian measure (defined through the use of $R = 2\pi$). Exercises in graphing, rate of change and expansions in series fix the ideas for the student.

12. In this paper Professor Turner characterized the following subjects: common sense, categorical logic, inductive logic, logical reasoning, philosophy, science. Finally two definitions of mathematics were given, in which certain terms previously explained were employed.

13. In the paper "An interpolation formula for equidistant frequency distributions" by Langman in the *Quarterly Publications of the American Statistical Association* is solved the following problem: Given the frequency over each of the unit sub-intervals of the interval from a to b, to find the frequency over any arbitrary interval. Professor Reilly showed how Langman's result can be obtained much more easily and briefly by applying some principles of the calculus of finite differences.

14. The companion to the cycloid, $x = a\theta$, $y = a(1 - \cos\theta)$, has the same period and amplitude as the cycloid. The two curves are tangent at their highest point, and coincident at their lowest points. Professor Moots showed that, if the generating circle be divided into two parts by a diameter and placed in the position corresponding to $\theta = \pi$, this circle, the cycloid and its companion divide the circumscribing rectangle into eight equal parts each of area $\pi a^2/2$. The rectification of the companion leads to the elliptic integral $E(\varphi, k)$, where $\varphi = 180°$ and $k = 1/\sqrt{2}$, which when evaluated gives the length as $7.6401a$, or approximately $.36a$ less than that of the cycloid.

J. F. Reilly, *Secretary-Treasurer.*

THE MAY MEETING OF THE KENTUCKY SECTION.

The seventh regular meeting of the Kentucky Section of the Mathematical Association of America was held at the Physics Building, University of Kentucky,

Lexington, on Saturday, May 5, 1923. The meeting consisted of two sessions with Dr. Elizabeth LeStourgeon, the chairman, presiding.

The attendance was seventeen, including the following members of the Association: R. V. Blair, P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, Elizabeth LeStourgeon, C. H. Richardson and G. A. Seubert.

Professor A. R. FEHN, Center College, was elected chairman of the Section, and Mr. G. A. SEUBERT, University of Kentucky, secretary-treasurer. The next meeting will be held at Center College next April.

The following papers were read:

(1) "Anallagmatic curves" by Dean P. P. BOYD;

(2) "Some remarks on statistics" by Professor A. R. FEHN;

(3) "A vectorial treatment of the loci of points of a moving body which are exceptional points of their trajectories" by Professor E. L. REES;

(4) "Unified mathematics for freshmen" by Professor C. H. RICHARDSON;

(5) "A treatment by vector methods of order of contact" by Mr. AUGUSTUS SISK (by invitation);

(6) "Illustrated lecture on astronomy" by Professor H. H. DOWNING.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dean Boyd introduced his subject by recalling the necessary facts concerning circular points at infinity and the foci of higher plane curves. Circular inversion was presented as a specialization of the quadratic transformation. The relations between the inverse, the pedal and the polar reciprocal of a curve were demonstrated. Various special anallagmatic curves, bicircular quartics, especially, were shown. The necessary condition for an anallagmatic curve was derived. The curve was derived as the envelope of a pencil of circles orthogonal to the circle of inversion. The geometric construction of points upon and of the foci of the anallagmatic curve, when the deferent is given, was explained. Finally various theorems concerning circular cubics and bicircular quartics as anallagmatic curves were stated.

3. Professor Rees derived by vector methods the equation of the third-order curve of inflections and the third-order surface locus of points at which the osculating planes of their trajectories are stationary. Methods for obtaining the instantaneous loci of points for which the osculating circles and the osculating spheres of their trajectories are stationary were also outlined. The manner in which these loci are related was pointed out and certain properties of these loci were proved in a simple way by vector methods.

4. Assuming that a course of mathematics for freshmen should prepare the student for a more intelligent analysis, interpretation, and understanding of quantitative phenomena, Professor Richardson stated as his opinion that such a course should introduce the student to the more powerful methods for investigating quantitative phenomena; that in view of the recent applications of mathematics to many fields such as psychology, education, biology, economics, the problems should be selected from a wide range of topics; that, where mathe-

matics is required of freshmen, expert facility in some of the processes should be sacrificed in order that an acquaintance with the many applications may be accomplished. He also expressed the belief that such a course emphasizing utility would stimulate a feeling for the need of a better mathematical training, and retrieve for our field some ground lost during the past decade by reason of misplaced emphasis.

5. Mr. Sisk showed by vector methods that in general the tangent line has contact of the first order, the osculating circle has contact of the second, and the osculating plane has contact of the third order.

6. Professor Downing showed about fifty views of the sun, moon, planets, comets, nebulas, star clusters, and star clouds. The principal features of each slide were pointed out.

C. H. Richardson, *Secretary-Treasurer.*

ORGANIZATION MEETING OF THE MICHIGAN SECTION.

A meeting for the purpose of organizing a Michigan Section of the Association was held at the University of Michigan, Ann Arbor, on March 29, 1923. The Mathematics Section of the Michigan Schoolmasters' Club had arranged to have its program for the session of this day of especial interest to teachers of mathematics in the colleges, this to culminate in the organization of the section. Professor Harold Blair of Kalamazoo Normal presided at the first part of the meeting, and then turned the meeting over to Professor T. H. Hildebrandt. After a brief discussion by Professor W. B. Ford outlining the history and purposes of the Association and the policy of the Monthly, the organization was consummated in the drawing up of a petition to the Board of Trustees for recognition as a section and in the adoption of by-laws. Professor L. C. Emmons acted as Secretary for this part of the meeting. The following signed the petition and were declared charter members of the section:[1]

N. H. Anning,
*W. S. Barlow,
J. F. Barnhill,
*G. C. Bartoo,
Harold Blair,
*P. N. Blessing,
J. W. Bradshaw,
*W. M. Coates,
*C. J. Coe,
*C. C. Craig,
S. E. Crowe,
Albertus Darnell,
W. W. Denton,
*J. M. Earl,
L. C. Emmons,
J. P. Everett,
Florence E. Field,
Peter Field,
*R. C. Huffer,
M. F. Johnson,
L. C. Karpinski,
*D. K. Kazarinoff,
A. E. Lampen,
*C. E. Love,
J. L. Markley,
*Jane L. Matteson,
*Selah W. Mullen,
A. L. Nelson,
*Ada A. Norton,
H. L. Olson,
W. H. Pearce,
*V. C. Poor,
Clair Reid,
R. B. Robbins,
*L. J. Rouse,
T. R. Running,

[1] Those whose names are starred were not members of the Association at the time, but have since applied, and by action of the section are to be considered charter members of the section.

S. E. Field,
W. B. Ford,
J. W. Glover,
*A. G. Hall,
T. H. Hildebrandt,
L. A. Hopkins,
J. M. Howie,
*R. H. Schoonover,
E. R. Sleight,
*W. H. Wentworth,
*A. Marie Whelan.
C. B. Williams,
*Orpha E. Worden,
Alexander Ziwet.

The following officers were elected for the ensuing year: Chairman, T. H. HILDEBRANDT; Secretary-Treasurer, J. P. EVERETT; Member of the Executive Committee, E. R. SLEIGHT.

J. P. EVERETT, *Secretary-Treasurer.*

THE MAY MEETING OF THE ILLINOIS SECTION.

The fourth annual meeting of the Illinois Section of the Mathematical Association of America was held at Knox College, Galesburg, Illinois, on May 4 and 5, 1923, in conjunction with the Illinois State Academy of Science. There were three sessions. On Friday afternoon, Chairman Sellew called the meeting to order at 2:20. During the latter part of the session Vice-Chairman Comstock presided. The Friday evening session was a joint meeting of the Academy and the Illinois Section. Prof. Sellew presided at the Saturday morning session.

The attendance was forty-one, including the following eighteen members of the Association:

W. E. Cederberg, C. E. Comstock, M. W. Coultrap, A. R. Crathorne, D. R. Curtiss, C. F. Green, Mabel M. Heren, Mayme I. Logsdon, E. B. Lytle, E. J. Moulton, Mary W. Newson, C. I. Palmer, G. H. Scott, G. T. Sellew, H. E. Slaught, M. G. Smith, E. J. Townsend and Alice Winbigler.

The following officers were elected: C. E. COMSTOCK, Chairman; E. J. MOULTON, Vice-Chairman; G. H. SCOTT, Secretary-Treasurer. By a unanimous vote it was decided that the time and place of the next meeting of the Association will be decided by the executive committee.

The Illinois Section and the State Academy were entertained at a banquet on Friday evening, at Knox College.

After the reading of the minutes of the 1922 meeting, the following papers were presented:

(1) "Similar Perspective Triangles" by Professor F. E. WOOD, Northwestern University (by invitation);

(2) "Differentials, History and Uses" by Professor C. I. PALMER, Armour Institute of Technology;

(3) "Some Problems in Probability" by Professor E. J. MOULTON, Northwestern University;

(4) "Unified Mathematics for Freshmen" by Dr. C. F. GREEN, University of Illinois;

(5) "The Pearson School of Statistics at the University of London" by Professor G. T. SELLEW, Knox College;

Report of Committee "On Some Recent Changes in Mathematical Requirements";

(6) "In the Universities" by Professor E. J. MOULTON, Northwestern University;

(7) "In the Colleges of Illinois" by Professor C. E. COMSTOCK, Bradley Polytechnic Institute;

(8) "In the High Schools" by Professor E. B. LYTLE, University of Illinois.

Abstracts of papers, numbered as in the above list, follow:

1. Professor Wood proved that if two similar triangles are perspective with a finite axis, the circles circumscribing the triangles meet in two real points, one of which is the center of perspectivity, and the other the common Miquel point to each of the two quadrilaterals formed by either triangle and the axis of perspectivity. A special case of the Desargues configuration was obtained, and from that a configuration in circles and points. Certain extensions to equilateral triangles, to congruent triangles, to n-sides and to space were suggested.

2. Professor Palmer made a plea that instructors in the calculus should give more attention to clarifying the meaning of a differential and should make greater use of the idea in teaching the calculus to students who later are to study texts in applied mathematics where the differential is of frequent occurrence. He gave the history of the growth of the differential idea during the eighteenth century, and quoted various texts showing the great confusion of the ideas current at the present time.

3. Professor Moulton reported on some problems in probability which come up in a discussion of the accuracy of grades. The most important problem is described thus: If a paper were graded on many occasions by an instructor, the average of the grades may be called the true grade of the paper on the instructor's standard of grading. Suppose now that an instructor gives grades X_1 and Y_1 to two students, and that X_1 is less than Y_1. What is the probability that for the true grades X and Y the former is as large as the latter? The answer of course depends upon the probable error of a grade and the magnitude of the difference $Y_1 - X_1$. The solution applies to other measurements as well as to grades. An interesting conclusion, based on experimental determinations of the probable error, is that for fairly typical semester grades if one student is given a grade of 80 and another 89, then the probability that the former merits as high a grade as the latter is less than one in a hundred thousand. Most of the paper is found in the *Mathematics Teacher*, vol. 16, pp. 141–149, March, 1923.

4. Dr. Green's paper was a short survey of the field of unified mathematics from the viewpoint of its orientation with regard to other reform movements towards increasing the value and significance of our freshman courses in mathematics. The aims of such courses in meeting the needs both of the students who will continue into further courses and of those who will not can possibly best be gained by more emphasis on insight and understanding of fundamental conceptions and modes of thought, by covering as broad a range of mathematical concepts and processes as feasible and by stimulating "functional thinking"

and analysis. These principles are embodied to the fullest extent in unified mathematics.

Also there was outlined the plan used in an experiment with a few classes at the University of Illinois during the present year together with the results attained.

5. Professor Sellew gave some results of his experience in investigating the opportunities for the study of statistics in Europe. He gave a detailed account of these opportunities in the Department of Applied Statistics (under the direction of Professor Karl Pearson) at University College, University of London.

On account of the world-wide reputation of this great center of inspiration for the study of statistics and the curiosity concerning it, the department is overwhelmed with *visitors* from all parts of the world and has been obliged to curtail the amount of time that can be given to them. But both Professor Pearson personally and his twelve able assistants are most generous of the time and attention they give to all *students* in the three laboratories.

6. As a member of a committee requested to report on recent changes in mathematical requirements, Professor Moulton reported tendencies shown by answers to a questionnaire sent out to a number of typical universities and large colleges. The questions were not restricted strictly to the field assigned. Some conclusions follow: In Liberal Arts Colleges entrance requirements are usually limited to one unit of algebra and one unit of geometry, which is a reduction from the requirements of ten or fifteen years ago, in many cases. In the Eastern and Southern states the amount of mathematics offered for admission by students has remained sensibly constant, but elsewhere there has been an appreciable decrease, solid geometry especially showing a slump. Advanced algebra, including quadratics, is now taught in most of the central and western universities, as a regular course for college credit, but not in most of the eastern and southern institutions. Most of the universities give a course in solid geometry, but seldom is it a required course. About one out of four universities have an absolute mathematics requirement for graduation in excess of entrance requirements; most of the others have a group system of requirements, one group including mathematics. Some institutions give no college credit for any mathematics taken in a preparatory school before graduation, but most universities outside of the eastern states give college credit under some circumstances for trigonometry and college algebra taken in high school. Many have tried methods of separating students on the basis of ability, and for the most part they are well satisfied with results.

7. Professor Comstock stated that fifteen colleges in Illinois (not including University of Illinois, University of Chicago, and Northwestern University) report that, due to recent changes, the entrance requirements in mathematics are one unit in algebra and one unit in geometry. In the last ten years there has been a marked decrease in the percentage of Freshmen presenting one and a half units of both algebra and geometry, as is shown by the figures below:

Years.	Per Cent. Alg.	Per Cent. Geom.
1912	83	55
1917	72	40
1922	56	31

The treatment of these two classes of students varies. In some colleges they are put into different sections of Freshman mathematics, the missing half unit being made up in classes requiring more hours a week. In three colleges the Freshmen are divided into sections according to ability rather than according to the number of units presented for entrance. Three colleges hold diagnostic examinations near the beginning of the Freshman year. Only four of the fifteen require college courses in mathematics of all candidates for graduation. None require solid geometry and only six provide for instruction in that subject.

8. Professor Lytle discussed four changes going on in high schools. (1) There is a strong movement for less emphasis of technique and greater simplification of subject matter, evidenced by magazine discussions, new texts, experimental schools and the National Committee Report. Conservatism, drill-master type of teachers, poor preparation of teachers, manipulative type of examination questions and texts make progress slow. The movement includes better types of drill on the fundamental technique retained. (2) The increasing number of Junior High Schools is bringing about what is probably the best reorganization of elementary mathematics we have ever had because it eliminates some of the long-felt loss of time in the eighth year; this reorganization increases correlation possibilities and gives a more real and vitalized mathematics. (3) A popular approval of General Mathematics is evident, but there seems to be a lack of proper differentiation between "fused" or "unified," "correlated" and "general" courses. (4) A new appreciation of real demonstrative geometry has appeared, due probably to the more careful statements of belief of educators on "transfer" and "training" values. However, extended real demonstrative geometry courses are tending to come later and are more often made elective.

G. H. Scott, *Secretary-Treasurer.*

VECTORIAL TREATMENT OF THE MOTION OF A RIGID BODY IN A PLANE.[1]

By E. L. REES, University of Kentucky.

1. Introduction. Although the simplicity and brevity as well as the elegance of vector methods in the treatment of kinematics have long been recognized, few writers on this subject have made use of vector analysis, especially in works published in English. The advantages of the vector treatment are well illustrated

[1] The subject of this paper in so far as it relates to velocities and accelerations was treated from a somewhat different point of view by Professors Ziwet and Field in the Monthly, 1914, 105–113. The method there used is that of Burali-Forti and Marcolongo. In the present paper the Gibbs' notation and method have been used exclusively.

in the applications to the theory of the motion of a rigid system of points in a plane briefly sketched in this paper.[1]

2. Instantaneous Center. We are to study the motion of a plane (the points of which constitute a rigid system) which glides upon a fixed plane. This motion is completely determined at any instant by the velocity of a point P of the moving plane and the angular velocity of that plane. Let $\boldsymbol{p}(t)$ be the position vector of this point and let $\boldsymbol{w}(t)$ be the angular velocity vector; also let $\boldsymbol{q}(t)$ be the vector from the point P to any other point Q of the moving plane. The equation of the trajectory of Q is then $\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{q}$ and Q's velocity is $\boldsymbol{v} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{q}} = \dot{\boldsymbol{p}} + \boldsymbol{w} \times \boldsymbol{q}$. Placing $\boldsymbol{v} = 0$ and assuming $\boldsymbol{w} \neq 0$, we find $\boldsymbol{q} = \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$. Hence at any instant there is one and only one point at rest (instantaneous center about which the plane is rotating) and its position vector is

$$\boldsymbol{p} + \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}.$$

3. Centrodes. The locus of the instantaneous center in the fixed plane and its locus in the moving plane are called, respectively, the fixed centrode and the moving centrode. The equation of the fixed centrode is obviously $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$, in which t is the scalar parameter.

We refer the moving centrode to the orthogonal unit vectors $\boldsymbol{i}', \boldsymbol{j}'$ (origin at P) in the moving plane, where $\boldsymbol{i}' = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}$ and $\boldsymbol{j}' = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$, $\theta(= \int w dt)$ being the angle of orientation with reference to the fixed vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. To find the equation of the moving centrode we have merely to express the vector $\boldsymbol{Q} = \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$ in terms of $\boldsymbol{i}'$ and $\boldsymbol{j}'$; thus the required equation is

$$\boldsymbol{Q}' = \boldsymbol{Q}\cdot\boldsymbol{i}'\boldsymbol{i}' + \boldsymbol{Q}\cdot\boldsymbol{j}'\boldsymbol{j}' = [\boldsymbol{w}^{-1}\dot{\boldsymbol{p}}\boldsymbol{i}']\boldsymbol{i}' + [\boldsymbol{w}^{-1}\dot{\boldsymbol{p}}\boldsymbol{j}']\boldsymbol{j}',$$

in which $\boldsymbol{i}'$ and $\boldsymbol{j}'$ of the triple products are to be regarded as the functions of t indicated above.

The velocity of the instantaneous center on the fixed centrode is $\dot{\boldsymbol{P}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{Q}}$, and its velocity on the moving centrode relative to the moving plane is

$$\dot{\boldsymbol{Q}}' = \dot{\boldsymbol{Q}}\cdot\boldsymbol{i}'\boldsymbol{i}' + \boldsymbol{Q}\cdot\boldsymbol{j}'\boldsymbol{j}' + \boldsymbol{Q}\cdot\dot{\boldsymbol{i}}'\boldsymbol{i}' + \boldsymbol{Q}\cdot\dot{\boldsymbol{j}}'\boldsymbol{j}' = \dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}.$$

Hence the centrodes are tangent to each other at the instantaneous center and the corresponding centrodal arcs are equal. Whence the following theorem:

The moving centrode rolls without slipping on the fixed centrode.

4. Roulettes. When the centrodes $\boldsymbol{r} = \boldsymbol{P}$, $\boldsymbol{r}' = \boldsymbol{Q}'$ are given, the trajectories of the points of the moving plane are called point roulettes. If the generating point be taken as origin in the moving plane, we see at once that the equation of the roulette is

$$\boldsymbol{r} = \boldsymbol{P} - \boldsymbol{Q} = \boldsymbol{P} - \boldsymbol{Q}'\cdot\boldsymbol{i}\boldsymbol{i} - \boldsymbol{Q}'\cdot\boldsymbol{j}\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ of the last two coefficients are to be expressed in terms of $\boldsymbol{i}'$ and $\boldsymbol{j}'$.

[1] The reader is referred to such elementary works as those of Gibbs-Wilson and Coffin for certain preliminary definitions and theorems which are omitted here for brevity. We use the notation there explained except that primes are used here to indicate vectors which are expressed in terms of the unit reference vectors $\boldsymbol{i}', \boldsymbol{j}'$ of the moving plane. Vectors not affected by primes are expressed in terms of the unit reference vectors $\boldsymbol{i}, \boldsymbol{j}$ of the fixed plane.

If the generating point were any other point Q' of the moving plane, the equation would be

$$r = P + (q' - Q')\cdot ii + (q' - Q')\cdot jj.$$

For a cusp on the roulette $r = p$ we must have $\dot{r} = \dot{p} = 0$, *i.e.*, the generating point must be an instantaneous center at a cusp. Hence in general only points of the moving centrode generate roulettes with cusps, and the locus of these cusps is the fixed centrode.

The envelope of a line fixed in the moving plane is called a line roulette. Let Q be the point of the enveloping line such that the vector q is normal to the line. The equation of the line is then

$$r\cdot q_1 = (p + q)\cdot q_1 = p\cdot q_1 + q.$$

Differentiate with respect to the parameter t and we get $r\cdot\dot{q}_1 = \dot{p}\cdot q_1 + p\cdot\dot{q}_1$, or $r\cdot w_1 \times q_1 = w^{-1}\dot{p}\cdot q_1 + p\cdot w_1 \times q_1$. Solving for r, we have

$$r = r\cdot q_1 q_1 + r\cdot(w_1 \times q_1)w_1 \times q_1 = (p\cdot q_1 + q)q_1 + (w^{-1}\dot{p}\cdot q_1 + p\cdot w_1 \times q_1)w_1 \times q_1.$$

Therefore the equation of the envelope is

$$r = p + q + \dot{p}\cdot q_1 w^{-1} \times q_1.$$

When P is at the instantaneous center, $r = p + q$, which shows that the envelope of the line is the locus of the foot of the perpendicular from the instantaneous center to the line.

To find the equation of a line roulette, given the centrodes (if P be a point of the line), we have simply to replace p in the equation $r = p + \dot{p}\cdot q_1 w^{-1} \times q_1$ by $P - Q$, which gives

$$r = P - Q + (\dot{P} - \dot{Q})\cdot q_1 w^{-1} \times q_1,$$

where $Q = Q'\cdot ii + Q'\cdot jj$.

5. Center of Acceleration. The acceleration of any point Q is

$$\ddot{r} = \ddot{p} + \ddot{q} = \ddot{p} + \dot{w} \times q + w \times \dot{q} = \ddot{p} + \dot{w} \times q - w^2 q.$$

For a point of zero acceleration (acceleration center) we have $\ddot{p} + \dot{w} \times q - w^2 q = 0$. Multiply this equation by $q^{-1}\cdot$ and $q^{-1}\times$, and we obtain respectively $\ddot{p}\cdot q^{-1} = w^2$ and $\ddot{p} \times q^{-1} = \dot{w}$, which give the angular speed and the angular acceleration in terms of the acceleration of any point P of the moving plane and the vector from P to the acceleration center. Multiply the second equation by $\ddot{p}\times$, substitute w^2 for $\ddot{p}\cdot q^{-1}$ and we get $w^2\ddot{p} - \ddot{p}^2 q^{-1} = \ddot{p} \times \dot{w}$. Whence $q^{-1} = w^2\ddot{p}^{-1} + \dot{w} \times \ddot{p}^{-1}$, so that $q = (w^2\ddot{p}^{-1} + \dot{w} \times \ddot{p}^{-1})^{-1} = \ddot{p}^2(w^2 p + \dot{w} \times \ddot{p})^{-1}$, and the center of acceleration is $A = p + \ddot{p}^2(w^2\ddot{p} + \dot{w} \times \ddot{p})^{-1}$.

6. Circle of Inflections. We shall now find the locus of points for which the normal acceleration is zero at a given instant. This is equivalent to the problem of finding the locus of the generating points which are inflection points of their trajectories at the instant considered. The condition to be satisfied by the points is

$$(\dot{p} + q) \times (\ddot{p} + \ddot{q}) = 0.$$

Let P be the instantaneous center at that instant. Then $\dot{p} = 0$ and our condition reduces to

$$(w \times q) \times (\ddot{p} + \dot{w} \times q - w^2 q) = - \ddot{p} \cdot q w + w^2 q^2 w = 0,$$

and therefore $\ddot{p} \cdot q^{-1} = w^2$. From this equation we see that the locus of the tip of q^{-1} is a line perpendicular to the acceleration $\ddot{p}$ of the point P. It follows then that the locus of the tip of q, being the inverse of this line with respect to the instantaneous center, is a circle through the instantaneous center with diameter on $\ddot{p}$. This circle is called the "circle of inflections." If we let k equal the diameter $|\ddot{p}|/w^2$, and φ equal the angle $\widehat{\dot{P}q}$, positive in the direction of minimum rotation from $\dot{P}$ to $\ddot{p}$, the equation in the scalar polar form is $q = k \sin \varphi$.

The circle of inflections is tangent to the centrodes at the instantaneous center; for, differentiating $P = p + w^{-1} \times \dot{p}$ we find for the centrodal tangent vector $\dot{P} = \dot{p} + \dot{w}^{-1} \times \dot{p} + w^{-1} \times \ddot{p}$, and since $\dot{p} = 0$ we have $\dot{P} = w^{-1} \times \ddot{p}$, which is perpendicular to the diameter of the circle of inflections. Incidentally we have shown that the acceleration of the point P at the instantaneous center is normal to the centrodes.

Let us now find the locus of the points for which the tangential acceleration is zero. Here the condition is

$$(\dot{p} + \dot{q}) \cdot (\ddot{p} + \ddot{q}) = 0,$$

which, under the assumption that $\dot{p} = 0$, reduces to

$$(w \times q) \cdot (\ddot{p} + \dot{w} \times q - w^2 q) = \ddot{p} \times w \cdot q + q^2 w \cdot \dot{w} = 0,$$

and therefore $w \times \ddot{p} \cdot q^{-1} = w \cdot \dot{w}$. This likewise represents a circle; and this circle passes through the instantaneous center and has a diameter on $w \times \ddot{p}$. Since $w \times \ddot{p}$ is perpendicular to $\ddot{p}$ this circle intersects the circle of inflections orthogonally.

The other intersection of these two circles is the center of acceleration. This of course is obvious, but we may make a formal analytical proof by showing that $q^{-1} = w^2 \ddot{p}^{-1} + \dot{w} \times \ddot{p}^{-1}$ satisfies both equations. In fact the two equations (see under "center of acceleration") from which q was found are identically the vector equations of these two circles.

7. Locus of Points of Undulation. If the generating point is a point of undulation (a point at which the tangent has contact of the third order) of the trajectory it is easy to show by means of Taylor's series for Δr that $\dot{r}$, $\ddot{r}$, $\dddot{r}$ are scalar multiples of each other, *i.e.*, $\dot{r} \times \ddot{r} = \ddot{r} \times \dddot{r} = \dddot{r} \times \dot{r} = 0$.

Obviously at a given instant the generating point, if it be a point of undulation, must be on the circle of inflections. Regarding w as constant (a restriction which does not affect the geometric properties of the trajectories) we find at once that the condition $\dot{r} \times \dddot{r} = 0$ is equivalent to $\dddot{p} \cdot q = 0$, when P is at the instantaneous center. Hence, in general, at a given instant there is one and but one generating point at a point of undulation of its trajectory, and this

point is on the circle of inflections. The position vector of this point (origin at P) is perpendicular to $\dddot{\boldsymbol{p}}$.

We proceed to find the envelope of the circles of inflections. Differentiating the equation of the circle of inflections, $\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = 0$, we obtain $\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}} = 0$, which, as we have just seen, is equivalent to $\dddot{\boldsymbol{p}} \cdot \boldsymbol{q} = 0$, the locus of which is the line that intersects the circle of inflections in the point of undulation and the instantaneous center. We know from the theory of envelopes that the points of intersection of this line and circle are also the points of intersection of consecutive circles of inflections and that their locus as t varies is the two-branch envelope of the circles of inflections. Hence the following theorem:

The envelope of the circles of inflections consists of two branches, one of which is the locus of the instantaneous centers (fixed centrode) and the other the locus of the points of undulation.

8. Circle of Cusps. We shall now find the locus of the cusps of the line roulettes which are the points of tangency of the enveloping lines at a given instant. Differentiating the equation of the roulette and placing $\dot{\boldsymbol{p}} = 0$, we have $\dot{\boldsymbol{r}} = \boldsymbol{w} \times \boldsymbol{q} + \ddot{\boldsymbol{p}} \cdot \boldsymbol{q}_1 \boldsymbol{w}^{-1} \times \boldsymbol{q}_1$. At a cusp $\dot{\boldsymbol{r}} = 0$. Hence $wq + (\ddot{\boldsymbol{p}} \cdot \boldsymbol{q}_1 / w) = 0$, or $q + k \sin \varphi = 0$. Whence the theorem:

At any instant the locus of the cusps of the line roulettes which correspond to the enveloping lines is a circle symmetric to the circle of inflections with respect to the tangent at the instantaneous center.

It is from this property that the circle receives its name "circle of cusps." Obviously the cuspidal tangents all pass through the opposite extremity of the diameter through the instantaneous center.

9. Formula of Savary. Differentiating $\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{q}$ and multiplying by $\cdot \boldsymbol{q}$ we find $\dot{\boldsymbol{r}} \cdot \boldsymbol{q} = 0$ when P is at the instantaneous center. Hence the center of curvature of the trajectory of Q, for the point Q, is on the line joining Q and the instantaneous center.

The equation of the trajectory may also be written $\boldsymbol{r} = \boldsymbol{P} + \boldsymbol{R}$, where $\boldsymbol{P}$ is the instantaneous center and $\boldsymbol{R}$ the vector from $\boldsymbol{P}$ to Q. Denoting by σ the distance from $\boldsymbol{P}$ to the center of curvature, we have for the equation of the locus of the centers of curvature (evolute of the trajectory of Q)

$$\boldsymbol{r} = \boldsymbol{P} + \sigma \boldsymbol{R}_1.$$

Since the tangent of an evolute is normal to the involute, we may write

$$\left[\frac{d}{dt}(\boldsymbol{P} + \sigma \boldsymbol{R}_1)\right] \times \boldsymbol{R} = 0,$$

which reduces to $\dot{\boldsymbol{P}} \times \boldsymbol{R} + \sigma \dot{\boldsymbol{R}}_1 \times \boldsymbol{R} = 0$, or to

$$\dot{\boldsymbol{P}} \times \boldsymbol{R} + \frac{\sigma}{R} \dot{\boldsymbol{R}} \times \boldsymbol{R} = 0,$$

since $\boldsymbol{R}_1 = \boldsymbol{R}/R$. But $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{P} + \boldsymbol{R}$ and $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$, and therefore $\boldsymbol{R} = \boldsymbol{q} - \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$. When P is at the instantaneous center, $\boldsymbol{R} = \boldsymbol{q}$ and $\dot{\boldsymbol{R}} = \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{w}^{-1} \times \boldsymbol{p}$. Substituting in the equation above, we get $\dot{\boldsymbol{P}} \times \boldsymbol{q}$

$+ \sigma(- qw + \ddot{\boldsymbol{p}}\cdot\boldsymbol{q}_1 w^{-1}) = 0$. Let v be the speed of the instantaneous center. Our vector equation is then equivalent to the scalar equation $vq \sin \varphi + \sigma qw - \sigma wk \sin \varphi = 0$. By differentiating $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$, we easily find

$$v = \frac{|\ddot{\boldsymbol{p}}|}{w} = kw.$$

In view of this relation, if $w \neq 0$, our equation is equivalent to

$$\frac{1}{q} - \frac{1}{\sigma} = \frac{1}{k \sin \varphi}.$$

This is known as the *formula of Savary.*

If $\sigma = \infty$, we get as we should the circle of inflections $q = k \sin \varphi$.

If $q = \infty$, we have $\sigma = - k \sin \varphi$. Hence the locus of the centers of curvature of the trajectories of the points at infinity is the circle of cusps.

If we take the time t equal to the centrodal arc s, the curvatures of the fixed and moving centrodes are respectively

$$\boldsymbol{C}_f = \ddot{\boldsymbol{P}} = \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{Q}}$$

and

$$\boldsymbol{C}_m = \frac{d}{dt}(\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}})' = (\ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}})\cdot\boldsymbol{i}'\boldsymbol{i}' + (\ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}})\cdot\boldsymbol{j}'\boldsymbol{j}' + (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}\cdot\boldsymbol{i}'\boldsymbol{i}'$$
$$+ (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}\cdot\boldsymbol{j}'\boldsymbol{j}' = \ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}} + (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}.$$

Subtracting, $\boldsymbol{C}_m - \boldsymbol{C}_f = (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}$. Since $\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}$ is now a unit vector $C_m \pm C_f = \pm w$, the sign of the second term being plus or minus according as the curvature vectors have opposite or the same directions. The sign of the second member of the equation must be chosen to accord with that of the first. (It should be remembered that C_m, C_f, and w are essentially positive.) This formula states the more or less obvious fact that the rate of turning of the moving plane with respect to the centrodal arc equals the sum or the difference of the curvatures of the centrodes according as their directions have the opposite or the same sense.

Since here $v = 1$, the relation $v = kw$ gives $w = 1/k$ and the formula just found may be written in the equivalent form

$$\frac{1}{\rho_m} \pm \frac{1}{\rho_f} = \pm \frac{1}{k},$$

in which the ρ's are the radii of curvature of the centrodes.

10. Locus of Points of Contact of Stationary Circles of Curvature. For points whose circles of curvature are momentarily stationary, *i.e.*, have contact of the third order (or higher) with the trajectories of those points, we have the following conditions:

$$(\boldsymbol{K} - \boldsymbol{r})\cdot\dot{\ } = 0, \qquad (\boldsymbol{K} - \boldsymbol{r})\cdot\ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}}^2, \qquad (\boldsymbol{K} - \boldsymbol{r})\cdot\dddot{\boldsymbol{r}} = 3\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}},$$

where $\boldsymbol{K}$ is the position vector of the center of the circle of curvature. Eliminating $\boldsymbol{K}$ by solving the first two equations (simultaneously with $(\boldsymbol{K} - \boldsymbol{r})\cdot\boldsymbol{k} = 0$) for

$\mathbf{K} - \mathbf{r}$ and substituting in the third equation, we get $\dot{\mathbf{r}}^2[\mathbf{k}\dot{\mathbf{r}}\dddot{\mathbf{r}}] = 3\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}[\mathbf{k}\dot{\mathbf{r}}\ddot{\mathbf{r}}]$, or $\dot{\mathbf{r}}^2\dot{\mathbf{r}} \times \dddot{\mathbf{r}} = 3\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$. Replacing $\mathbf{r}$ by $\mathbf{p} + \mathbf{q}$ and letting P be at the instantaneous center, we obtain after simplifying

$$3[\mathbf{w}\ddot{\mathbf{p}}\mathbf{q}](\ddot{\mathbf{p}}\cdot\mathbf{q} - w^2q^2) + w^2\dddot{\mathbf{p}}\cdot\mathbf{q}q^2 = 0.$$

Consequently the locus of points of contact of stationary circles of curvature is a third order curve This curve obviously passes through the instantaneous center, and since the degree of the term of lowest degree is the second, the instantaneous center is a double point The equations of the tangents at this point are $\mathbf{w} \times \ddot{\mathbf{p}}\cdot\mathbf{q} = 0$ and $\ddot{\mathbf{p}}\cdot\mathbf{q} = 0$. Consequently these tangents coincide with the common tangent and normal of the centrodes.

Dividing the above equation by w^2q^4, we get

$$3[\mathbf{w}^{-1}\ddot{\mathbf{p}}\mathbf{q}^{-1}](\ddot{\mathbf{p}}\cdot\mathbf{q}^{-1} - w^2) + \dddot{\mathbf{p}}\cdot\mathbf{q}^{-1} = 0,$$

which shows that the locus of points of contact of stationary circles of curvature is the inverse of a conic with the instantaneous center as center of inversion. This conic is an equilateral hyperbola whose asymptotes have the directions of the tangents of the cubic curve at the double point; and since the conic passes through the origin, the cubic extends to infinity in the asymptotic direction given by

$$(\dddot{\mathbf{p}} - 3\mathbf{w} \times \ddot{\mathbf{p}})\cdot\mathbf{q} = 0.$$

11. Locus of Stationary Centers of Curvature. Let $\mathbf{C}$ be the vector from the instantaneous center to the center of a stationary circle of curvature. Savary's formula gives us

$$\mathbf{q}^{-1} = \frac{w^2C + \ddot{\mathbf{p}}\cdot\mathbf{C}_1}{\ddot{\mathbf{p}}\cdot\mathbf{C}}\,\mathbf{C}_1.$$

Substituting this expression for $\mathbf{q}^{-1}$ in $3[\mathbf{w}^{-1}\ddot{\mathbf{p}}\mathbf{q}^{-1}](\ddot{\mathbf{p}}\cdot\mathbf{q}^{-1} - w^2) + \dddot{\mathbf{p}}\cdot\mathbf{q}^{-1} = 0$, we find $3[\mathbf{w}^{-1}\ddot{\mathbf{p}}\mathbf{c}^{-1}]\ddot{\mathbf{p}}\cdot\mathbf{c}^{-1} + \dddot{\mathbf{p}}\cdot\mathbf{c}^{-1} = 0$. Thus the locus of the instantaneously stationary centers of curvature is a curve of the third order having properties similar to those of the cubic found above. There is a double point at the instantaneous center and the tangents of the two cubics at this point coincide with the common tangent and normal of the centrodes.

NOTE ON THE RELIABILITY OF A TEST, WITH SPECIAL REFERENCE TO THE EXAMINATIONS SET BY THE COLLEGE ENTRANCE BOARD.[1]

By W. L. CRUM, Yale University.

Introduction. The occasion for this study was presented by the appearance of the "Ben Wood Report"[2] in early 1922 and the comment thereon appearing on page 2 of the "Advance sheets" of the Board's *Annual Report* for 1921.

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[1] Presented before the American Mathematical Society, 28 October, 1922. Since the presentation of the paper, valuable suggestions have been received from Professors E. L. Dodd and C. N. Haskins.

[2] Ben D. Wood, *The reliability and difficulty of the College Entrance Examination Board examinations in Algebra and in Geometry*, published by the Board, 1922.

Mr. Wood defines [1] reliability as the correlation between the results obtained by administering two forms of the same test in succession to the same group of candidates. No such successive applications of the same test having been made, he states that the custom is to use the correlation between the results for the half of the test consisting of the odd-numbered questions and the results for the other half. To secure the reliability from this half-with-half correlation, he uses Brown's [2] formula to "get the reliability of the whole examination from that of the half of it":

$$\mathrm{r}_{\mathrm{II}} = 2\mathrm{r}_{\frac{1}{2}\mathrm{I}/\mathrm{II}}/(1 + \mathrm{r}_{\frac{1}{2}\mathrm{I}/\mathrm{II}}) \tag{1}$$

where $\mathrm{r}_{\frac{1}{2}\mathrm{I}/\mathrm{II}}$ is the correlation between the halves and r_{II} is the reliability sought.

The objects of this paper are to call attention to the fact that the use of Brown's formula has not been justified and that the conclusions based upon its use are therefore of doubtful validity, and to examine further the problem of estimating the reliability of a test from the results of a single application of that test.

1. We suppose that two forms of the same test are given in succession to the same group of n candidates, and that the deviations from their respective means of the grades of candidate i on parts A and B and on the whole of each test are:

	Part A.	Part B.	Whole.
Test 1	x_i	y_i	w_i
Test 2	s_i	t_i	z_i

The desired measure of reliability is the correlation between w and z:

$$\begin{aligned}\mathrm{r}_{wz} &= \frac{\sum w_i z_i}{n\sigma_w\sigma_z} \\ &= \frac{\sigma_x\sigma_s\mathrm{r}_{xs} + \sigma_x\sigma_t\mathrm{r}_{xt} + \sigma_y s_s\mathrm{r}_{ys} + \sigma_y\sigma_t\mathrm{r}_{yt}}{\sqrt{\sigma_x^2 + 2\sigma_x\sigma_y\mathrm{r}_{xy} + \sigma_y^2}\,\sqrt{\sigma_s^2 + 2\sigma_s\sigma_t\mathrm{r}_{st} + \sigma_t^2}}.\end{aligned} \tag{2}$$

In this notation, Brown's formula is:

$$R = \mathrm{r}_{wz} = 2\mathrm{r}_{xy}/(1 + \mathrm{r}_{xy}) \tag{3}$$

and we note that one set of conditions [3] which will reduce (2) to (3) is:

$$\begin{gathered}\sigma_x = \sigma_y = \sigma_s = \sigma_t, \\ \mathrm{r}_{xy} = \mathrm{r}_{xs} = \mathrm{r}_{xt} = \mathrm{r}_{yt} = \mathrm{r}_{ys} = \mathrm{r}_{st}.\end{gathered} \tag{4}$$

It is obvious that, if we actually administer only the first test, we can know only σ_x, σ_y, and r_{xy}. Not only do we not know the values of the other coefficients, but we have no way of knowing whether they are related by the equalities in (4). If the Wood Report rested upon the assumption of the equalities in (4), no mention was made of that fact. Moreover, it is not stated whether the two standard deviations which can be calculated, σ_x and σ_y, are equal. Furthermore, the Report does not give sufficient of the original data to enable the reader to compute

[1] *Loc. cit.*, p. 4.

[2] William Brown, *Essentials of Mental Measurement*, Camb. Univ. Press, 1911, p. 101.

[3] These are the conditions suggested by Brown: *loc. cit.*, footnote.

these constants; but, judging from the extreme irregularity in the standard deviations of the grades on individual questions, it is very doubtful if σ_x and σ_y can be equal in this case. In fact, Table I on page 5 of the Report enables us to calculate the following:

Question number........	1	2	3	4	5	6	7	8	9	10
Mean grade.............	8.9	9.3	8.4	8.0	2.9	6.7	7.2	8.9	6.8	2.8
Standard deviation.......	1.9	2.2	3.3	2.9	3.9	4.4	3.2	2.8	1.1	4.3
Coefficient of variation....	.22	.23	.39	.36	1.3	.66	.45	.32	.17	1.5

Now, it may be said that the standard deviations of the two halves may still be equal in spite of the lack of uniformity in the dispersions of the several questions. This is certainly true, but the conditions yielding such equality would be extremely exceptional. In fact, there is a simple algebraic relation connecting σ_x with the standard deviations of the individual questions which constitute x and with the correlations of these individual questions with each other. In order for σ_x and σ_y to be equal in spite of the non-uniform dispersion, we must have so great divergences in the correlations of the individual questions with each other as to lead us to an immediate conclusion that the test must have low reliability.

It is accordingly evident that the use of Brown's formula in this instance is not justified by a conformity to the requirements suggested in (4).

2. The equalities (4) are not, however, necessary conditions in order that (2) may reduce to (3): they are sufficient, but the necessary condition is merely that the right members of (2) and (3) be equal to each other.

It is hardly fruitful to go into an extensive analysis of this necessary condition. If one expresses the various symbols in terms of the x_i, y_i, s_i and t_i, and simplifies the resulting equation of condition, one is led to a relation of the 16th degree. Since, by the very nature of the practical problem under consideration, we know nothing precisely about the s_i and t_i, it is not possible to make exact substitutions leading to a simplification of this condition equation. Moreover, it is suggested that we do not have even approximate knowledge—such knowledge as we should base upon considerations of *a priori* probability—of the s_i and t_i; for, here again the essence of the reliability problem is that we do not know the statistical distribution of the grades in the second test. It seems accurate to say that one can not *demonstrate* that the equation of necessary condition is satisfied without actually giving the second test, and that there are no grounds whatever for *assuming* that it is satisfied.

It is not difficult, on the other hand, to find sufficient conditions which are much less restrictive than those given in (4); and which, therefore, have a greater likelihood of being satisfied in actual practice. Suppose we set g equal to the right member of the exact relation (2), and introduce the following notation:

$$\left.\begin{array}{c} \sum x_i y_i = p, \qquad \sum s_i t_i = (1+a)p, \qquad \sum x_i s_i = (1+b)p, \\ \sum x_i t_i = (1+c)p, \qquad \sum y_i s_i = (1+d)p, \qquad \sum y_i t_i = (1+e)p; \\ 4A = b+c+d+e, \qquad \sum x_i^2 = h, \qquad \sum y_i^2 = (1+k)h, \\ \sum s_i^2 = (1+l)h, \qquad \sum t_i^2 = (1+m)h, \qquad 2K = l+m, \end{array}\right\} \tag{5}$$

and we have:

$$g = \frac{4p(1 + A)}{\sqrt{h^2(4 + 2k + 4K + 2kK) + 2ph(4 + 2a + k + 2K + ka) + 4p^2(1 + a)}}, \tag{6}$$

$$R = \frac{2p}{h\sqrt{1 + k} + p}, \tag{7}$$

and a set of conditions sufficient to reduce (6) to (7) is:

$$a = A = k = K = 0. \tag{8}$$

Although the conditions (8) are rather severe, they are far less so than those imposed by (4). Two of them, $a = 0$ and $k = 0$, are included among the conditions of (4). The other two, $A = 0$ and $K = 0$, are more likely to be fulfilled than the remaining conditions under (4). Indeed, these two conditions may be stated verbally as:. the average of the four product-sums of the halves of the first test with the halves of the second test shall be the product-sum of the halves of the first test with each other; and the average of the standard deviations of the halves of the second test shall equal the standard deviation of a half of the first test.

The examination of various hypotheses will of course lead to other possible sets of sufficient conditions for the reduction of (6) to (7). An interesting example is to assume that k is zero, that p is positive, and that b, c, d, e, l, m are all small. It is then possible to expand the right member of (6) and show that it will reduce to (7), except for terms of second and higher orders, if:

$$2A = K = a. \tag{9}$$

Although the conditions (9) are still less restrictive than (8), their practical significance is less apparent, and it is less easy to see how they might be realized in practice. Indeed, it is doubtful whether we can find a group of sufficient conditions the fulfillment of which we have a better chance of establishing on *a priori* grounds than the set given in (8). If we confess our entire inability to show that the necessary condition mentioned at the beginning of this section is satisfied, we must fall back on reasoning about the properties of our distributions as revealed by the dispersions and inter-correlations; and, for this purpose, it is improbable that we shall find a better set of conditions than (8).

3. The chief practical question is, then, under what circumstances we may expect—from a mere knowledge of the first test—the sufficient conditions we have discussed to be fulfilled. The requirement that k be zero, which enters in each group of conditions suggested above, is one which can actually be observed. In a particular test, we can learn by computation whether k is zero; and, with care and practice, we should be able to construct a test in which k is approximately zero. On the same grounds, we should be able to insure that K is zero approximately.

As for the two remaining conditions in (8), it is suggested that one can scarcely hope to have them satisfied in a test such as the entrance algebra or

any other test the parts of which are specifically designed to measure different intellectual capacities—capacities differing certainly in degree and almost surely, in many practical cases, in kind. The assumption that A reduces to zero in such cases is quite unwarrantable: one would expect a marked lack of uniformity in the four inter-correlations and consequently in the four product-sums. Moreover, even a brief experience with the making of mathematics tests convinces one that the construction of two similar tests covering a range of subjects with a view to having the correlation between the two halves of the one test equal that between the halves of the other—or equal to any previously assigned value whatever—is an extremely difficult feat. We are led to conclude that the condition that a should equal zero is also unlikely to be fulfilled.

The essential point is that we can not hope to establish by *a priori* considerations that such a test, covering a range of capacities, will even approximately fulfill the conditions which we impose. There is indeed a remote possibility that it *may* happen to satisfy those conditions, and a slightly greater possibility that it may satisfy the fundamental necessary condition; but such possibility certainly gives no warrant for assuming that the conditions are met. In other words, it does not seem unfair to say that there can be no justification whatever for using Brown's formula in the study of reliability of a test which covers a range of specific topics. It is on such grounds that the analysis in the Wood Report appears to be unsound; and the conclusions based on such analysis, in particular the conclusion in paragraph 2 on page 14, have not been substantiated.

The case is somewhat different with a test designed to measure a single capacity, such as a section of a standard mental test. Here it does not seem impossible to design a test which shall fulfill the requirements (8). The test really consists of a series of similar observations of the same magnitude: the various questions are the instruments of measurement. Clearly, in the nature of the case, it is desirable to have the individual standard deviations all approximately equal and to have a uniformly high correlation of the individual questions with each other. Care and practice will doubtless enable the examiner to insure to a high degree the realization of this ideal. With such a test, it is not difficult to believe that the conditions (8) might be satisfied at least approximately. All this is, however, robbed of much of its importance, when we remark that such a test is after all but a composite of two similar tests given in immediate succession. If we can make the two halves of such a test have high correlation with each other, we should be able to make an entire new test which should have high correlation with the first. Thus the practical problem of estimating reliability is of relatively little moment in the case of a test intended to measure a single capacity.

For the other case, in which there is a strong practical reason for seeking an estimate of the reliability, we have seen that the use of Brown's formula is open to serious objection. Is there any other way in which we can approximate to the reliability of the test from the evidence yielded by a single application of that test? If we are to proceed by correlating one part of the test with another, we must bear in mind that the selection of these parts can hardly be purely by

chance. The Wood report designates a portion of the test consisting of alternate questions as a random half, but this does not accord with the usual significance of the term *random.* The essential requisite for a random sample, in order that the conclusions of the sampling theory may hold, is that the choice of one object be independent of the other choices. Such a condition can not possibly hold in the case before us, in which we select as a sample the half of a limited group of objects. In addition to the special objections, which apply to the selection of the alternate questions in a test designed as mathematics tests usually are designed, we must raise the general objection that no random sample, in the sense of the theory of sampling, is here possible.

Another available method is to select samples by design: to pick out consciously a sample which we may regard as the best representative of the entire test. It would appear desirable to choose as samples two halves which would be equally good representatives of the whole test, and this condition will be met if the two halves have equal standard deviations. With such a choice, it will be found that the correlation of the whole test with either half is given by

$$\sqrt{(1 + r_{xy})/2} \tag{10}$$

and that the partial correlation of the whole with each half is unity. This does not enable us to proceed to an estimate of reliability, for a knowledge of the correlations of the halves with the whole tells us nothing of the correlations of the halves with the second test. The whole first test can not be taken as a random sample of all the tests given, so far as correlation with the halves is concerned, for the direct dependence of the whole on the halves controls the correlation.

We must conclude then that it is impossible to choose random samples and that, no matter how the samples are chosen, the essential dependence of the whole test upon the halves precludes our using the halves as representatives for the purpose of estimating correlation with a subsequent test. In other words, it seems impossible, on theoretical grounds, to solve the reliability problem: we must conclude that the reliability of a test intended to measure a range of capacities can not be estimated from the results of the single application of that test.

SIMILITUDINOUS AND PSEUDO-SIMILITUDINOUS TRANSFORMATIONS IN A PLANE.

By W. H. ECHOLS, University of Virginia.

1. Introduction. The purpose of the present paper is to notice the transformations of certain plane figures in a plane. In a plane two congruent triangles, or figures, are superposable without leaving the plane when their equality is of the same kind, that is, when they are both right- or both left- handedly equal, or as we say are both of the same generation. Otherwise the two figures must be said to be symmetrically equal, or of opposite generation, and one cannot be

moved in the plane to superposition with the other without changing its shape. In like manner two similar figures, in a plane, possess similarity of the same kind or they are symmetrically similar. When we speak of two similar figures, in a plane, we shall mean that they are not symmetrically similar unless otherwise specified. As is well known a figure can be transformed to superposition with any similar figure, in its plane, by movement of its points along continuous path curves, the paths being the members of a family of logarithmic spirals, and in this movement the size of the figure changes continuously while its shape remains invariant.[1] This is the well-known linear transformation. We propose to notice it in a slightly different form showing that there are infinitely many continuous path curves which effect this same transformation.

2. Transformations of Similar Triangles. Let z_1, z_2, z_3 be any three complex numbers, represented by three points in a z-plane. Then any three points w_{12}, w_{23}, w_{31}, which satisfy

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + (z_3 - z_1)w_{31} = 0, \tag{1}$$

will be the corners of a triangle similar to $z_1z_2z_3$. This equation can be written in the dual or reciprocal form

$$(w_{12} - w_{31})z_1 + (w_{23} - w_{12})z_2 + (w_{31} - w_{23})z_3 = 0. \tag{2}$$

Either of these leads to

$$(w_{12} - w_{31})/(w_{23} - w_{31}) = (z_3 - z_2)/(z_1 - z_2), \tag{3}$$

which proves the similarity of the triangles.

In any given triangle ABC, whose angles are A, B, C and opposite sides a, b, c, the relation $c = be^{iA} + ae^{-iB}$ exists. If z_1, z_2, z_3 are the corners of any triangle similar to ABC, then by (2)

$$z_3c = z_2be^{iA} + z_1ae^{-iB}. \tag{4}$$

If z_1' z_2' z_3' be any other triangle similar to $z_1z_2z_3$, then

$$z_3'c = z_2'be^{iA} + z_1'ae^{-iB}. \tag{5}$$

Let k_1, k_2 be any two numbers subject to $k_1 + k_2 = 1$. Multiply (4) by k_1 and (5) by k_2 and add the equations. The result shows that the points

$$Z_1 = k_1z_1 + k_2z_1', \qquad Z_2 = k_1z_2 + k_2z_2', \qquad Z_3 = k_1z_3 + k_2z_3'$$

are the corners of a triangle similar to ABC. We notice three forms of transformation.

3. Similitudinous Translation. If k_1, k_2 are real numbers the points Z_1, Z_2, Z_3 divide the segments z_1z_1', z_2z_2', z_3z_3' respectively in the same ratio $\lambda = k_2/k_1$, and as λ varies continuously from 0 to 1, the triangle $Z_1Z_2Z_3$ transforms, changing ze continuously without changing shape, along straight lines from $z_1z_2z_3$ to $z_1'z_2'z_3'$. Such a movement may be called a similitudinous translation transformation. This admits of quite a simple proof by elementary geometry.

[1] See Townsend, *Functions of a Complex Variable*, p. 162.

4. Similitudinous Rotation. If k_1, k_2 are complex numbers and $|k_1| = m$, $|k_2| = n$, then

$$\lambda = k_2/k_1 = (n/m)e^{i\alpha}.$$

Now Z_1, Z_2, Z_3 are the vertices of triangles constructed on z_1z_1', z_2z_2', z_3z_3' as bases respectively, and which are similar to ABC. If α is constant the points Z_1, Z_2, Z_3 lie on similar arcs of circles passing from z_1, z_2, z_3 to z_1', z_2', z_3' and the triangle $Z_1Z_2Z_3$ is transformed continuously from $z_1z_2z_3$ to $z_1'z_2'z_3'$ along the arcs of circles; the points turning through the same angular rotation, the moving triangle changes size continuously but keeps its shape invariant. Such a movement may be called a similitudinous rotation transformation.

5. Arbitrary Path-curve Transformation. When k_1, k_2 are complex numbers, $k_1 + k_2 = 1$ defines a triangle whose vertices in orthogonal Cartesian coördinates are $(0, 0)$, $(1, 0)$ and (x, y); m and n are the radial coördinates of (x, y) with respect to $(0, 0)$ and $(1, 0)$. Let (x, y) describe any curve $F(m, n) = 0$ passing through $(0, 0)$ and $(1, 0)$ whose parametric equations can be written

$$x = \tfrac{1}{2}(1 + m^2 - n^2),$$
$$y = \tfrac{1}{2}\sqrt{2m^2n^2 + 2m^2 + 2n^2 - m^4 - n^4 - 1}.$$

If $\varphi(x)$ is an arbitrary function having $\varphi(0) = 0$, and $f(x)$ any arbitrary function, such a path curve will be

$$y = \varphi(x)[f(x) - f(1)]. \tag{6}$$

As k_1 varies from 0 to 1 the points Z_1, Z_2, Z_3 describe path curves similar to (6), the triangle varying continuously in size, but invariant in shape, from $z_1z_2z_3$ to $z_1'z_2'z_3'$. Such a path curve could be more briefly defined by $w = k_1e^{k_2}$.

6. Generalization for Similar Figures. Any two similar figures can be decomposed into triangles, each of which in the one is similar and similarly situated to a corresponding triangle in the other. The transformation of the one figure into the other is effected by the transformation of any one of these triangles into the corresponding one.

7. Similar Figures in Parallel Planes. If there be two similar figures in two parallel planes and a surface be generated by a straight line which intersects their boundaries in two corresponding points, then all sections by planes parallel to the planes of the two figures are similar figures, as is obvious by their orthogonal projection on one of the planes.

8. We may notice that, in a plane, if n masses $m_1, \cdots, m_n$ are respectively placed at the vertices of n similar triangles $u_1v_1w_1, \cdots, u_nv_nw_n$, then the triangle whose vertices are $\Sigma m_ru_r/\Sigma m_r$, $\Sigma m_rv_r/\Sigma m_r$, $\Sigma m_rw_r/\Sigma m_r$ is also a similar triangle.

9. Elementary Applications. We note a few consequent geometrical problems which may be of interest.

(*a*) The process of section 3 serves to construct the double infinity of triangles, similar to a given triangle, inscribed on three given straight lines in a plane. Or, the quadrilaterals similar to a given quadrilateral inscribed on four given straight lines. More generally, the construction of the double infinity of

polygons of n sides, similar to a given polygon, inscribed on n given straight lines in a plane.

(*b*) The same process establishes the following: Given three skew lines in space, there are two systems of parallel planes any plane of which cuts the three lines in three points which are the corners of a triangle of invariant shape, the one system cuts triangles of one generation, the second those of contrary similarity.

(*c*) The process of section 4 serves to construct the triangles of given shape whose vertices are respectively on three given circles in a plane.

(*d*) If there be any closed boundary in a plane, whose area is A, and a similar boundary be placed anywhere in a parallel plane distant h, the volume of the solid bounded by the two planes and the ruled surface whose generators are the lines joining corresponding points on the two boundaries is

$$V = \tfrac{1}{3}hA(1 + \rho^2 + \rho \cos \alpha),$$

where α is the angular turn and ρ the coefficient of stretch of the similar figure.

10. Pseudo-Similarity. We come now to examine two quadrilaterals which possess a certain relation to each other which we may, for lack of better designation, express by saying they are pseudo-similar. Thus, suppose $ABCD$ is any given quadrilateral and there is another one $A'B'C'D'$ which has the following property: there is a point E' such that $\triangle\, E'A'B'$ is similar to $\triangle\, CAB$ and $\triangle\, E'C'D'$ is similar to $\triangle\, ACD$. Then there will also be a point F' such that $\triangle\, F'B'C'$ is similar to $\triangle\, DBC$ and $\triangle\, F'D'A'$ is similar to $\triangle\, BDA$. All such quadrilaterals $A'B'C'D'$ are said to be pseudo-similar to each other. The four triangles mentioned above are not independent of each other, when any two are known the other two are determined and the quadrilateral constructed. That there are infinitely many such quadrilaterals is easily shown, for if z_1, z_2, z_3, z_4 represent any quadrilateral $ABCD$, then the w-numbers satisfying

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + \cdots + (z_4 - z_1)w_{41} = 0 \tag{7}$$

are the corners of such a quadrilateral. Equation (7) is therefore the general specification of the group of all pseudo-similar quadrilaterals based on z_1, z_2, z_3, z_4. Moreover we can interchange the z's and w's and rearrange (7), writing it

$$(w_{12} - w_{41})z_1 + (w_{23} - w_{12})z_2 + \cdots + (w_{41} - w_{34})z_4 = 0, \tag{8}$$

so that the relation between the z-figure and the w-figure is a reciprocal one and (8) is also the specification of all pseudo-similar quadrilaterals based on $w_{12}w_{23}w_{34}w_{41}$. The proof by means of (7) of the stated properties is quite simple, thus

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + (z_3 - z_1)w' = 0$$

constructs, by (1), a triangle $w_{12}w_{23}w'$ similar to $z_1z_2z_3$.

$$(z_3 - z_4)w_{34} + (z_4 - z_1)w_{41} + (z_1 - z_3)w'' = 0,$$

by (1), constructs a triangle $w_{34}w_{41}w''$ similar to $z_3z_4z_1$. These relations put in (7) give $(z_3 - z_1)w' + (z_1 - z_3)w'' = 0$. Therefore $w' = w''$ $(= w_{13}$ say) iden-

tifies E'. In like manner we construct F' ($= w_{42}$). In particular, if in the quadrilateral $z_1z_2z_3z_4$ we consider the parallelogram whose corners are the midpoints of its sides, this parallelogram is quite obviously pseudo-similar to the w-quadrilateral, its E and F points being the midpoints of the diagonals of $z_1z_2z_3z_4$.

11. Transformations of Pseudo-Similar Figures. Let there be a second quadrilateral $w_{12}'w_{23}'w_{34}'w_{41}'$ pseudo-similar to $w_{12}w_{23}w_{34}w_{41}$, then the transformations of sections 3, 4, 5 are, by the same process carried out there, true for these quadrilaterals, if we change the word similar into pseudo-similar. Any quadrilateral can be transformed into a pseudo-similar parallelogram. Quadrilaterals based on parallelograms, lozenges or on squares have a number of interesting properties, some of these based on squares, called skewsquares, have been published in the MONTHLY (*1923*, 120).

12. Generalization. If $f(z)$ is any analytic function throughout the region of its use, then its mean value between two points z_i, z_j is

$$w_{ij} = \frac{1}{z_i - z_j}\int_{z_j}^{z_i} f(t)dt,$$

and $w_{ii} = f(z_i)$, when $z_j = z_i$. Corresponding to each number pair z_i, z_j, or segment z_iz_j, in the z-plane there is a point w_{ij} in a w-plane, or if one prefers in the z-plane. If we choose n points $z_1, \cdots, z_n$ as the corners of a closed polygon, then

$$\left.\begin{array}{r}(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + \cdots + (z_n - z_1)w_{n1} = 0,\\ (w_{12} - w_{n1})z_1 + (w_{23} - w_{12})z_2 + \cdots + (w_{n1} - w_{n-1,\,n})z_n = 0.\end{array}\right\} \qquad (9)$$

For every different form of the function $f(z)$ we derive a set of n numbers w_{ij} satisfying (9) and which may be taken as the corners of a w-polygon. We may speak of all such polygons as forming a group of pseudo-similar polygons and, for the purposes of designation, call the w-points corresponding to the diagonals of the z-polygon the foci. If we take $f(z) \equiv z$, then the midpoints of the sides of the z-polygon will be the corners of a w-polygon pseudo-similar to the w-group. The same relation holds between the z-points satisfying (9) and the w-points, the midpoints of the sides of the w-polygon are the corners of a polygon belonging to the group of pseudo-similar z-polygons. It is also clear that any polygon belonging to a pseudo-similar group can be transformed into any other polygon of that group by the processes in sections 3, 4, 5.

13. If in the equations of section 1 we perform the proper elimination we arrive at the standard form of two similar triangles of the same generation

$$\begin{vmatrix} 1, & z_1, & z_1' \\ 1, & z_2, & z_2' \\ 1, & z_3, & z_3' \end{vmatrix} = 0,$$

thus connecting up with the familiar linear transformation in complex variables. In general, if $u, v, w, \cdots, t$ are the n corners of a polygon satisfying (9), and $u_r, v_r, w_r, \cdots, t_r$ $(r = 1, 2, \cdots, n - 1)$ represent $n - 1$ pseudo-similar polygons,

then on elimination from the resultant equations (9),

$$\begin{vmatrix} 1, & u, & u_1, & \cdots, & u_{n-1} \\ 1, & v, & v_1, & \cdots, & v_{n-1} \\ 1, & w, & w_1, & \cdots, & w_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & t, & t_1, & \cdots, & t_{n-1} \end{vmatrix} = 0,$$

which justifies somewhat the term pseudo-similar.

14. We comment on the fact that we are dealing with similarity and homogeneous linear equations and that we are concerned with ratios only. In section 10, when constructing a polygon $ABCD$, we may choose arbitrarily a triangle EAB, then ECD has two degrees of freedom, size and position. If we choose we may make C converge to B as a limit, and when C coincides with B the polygon reduces to a triangle as a special case, but now the point $B \equiv C$ must be counted as two coincident points or as a double point. This being understood, any quadrilateral can be transformed, as in section 3, into a pseudo-similar triangle. It is useless to attempt a like reduction in the case of the triangle. But the degrees of freedom in the construction of the kaleidoscope pattern of a polygon of a pseudo-similar group of n corners permits the transformation to one of $n - 1$ corners one of which however is a double corner. This can be continued until such an n-polygon may be transformed into a triangle having multiple vertices, the sum of their multiplicities being $n - 3$.

15. The following consequences are easily established.

(*a*) If two isosceles triangles have their vertical angles equal to α and this vertex common, their bases are opposite sides of a quadrilateral whose diagonals are equal and intersect at the angle α. The other two sides are the bases of isosceles triangles which have a common vertex and vertex angle $\pi - \alpha$.

(*b*) Reading triangles counter-clockwise in a plane, if ABC, AA_1A_2, BB_1B_2, CC_1C_2 are equilateral, then the midpoints of A_2B_1, B_2C_1, C_2A_1 are the corners of an equilateral triangle.

(*c*) If two pseudo-similar polygons lie respectively in two parallel planes and straight lines be drawn through their corresponding corners, there will be formed a prismoid in space. All planes parallel to the bases cut the surface in pseudo-similar polygons.

(*d*) Areas of closed boundaries being counted in the positive sense of description, let any pseudo-similar polygon satisfying (9) be transformed by section 3; four of its positions being determined by k_1, k_1', k_1'', k_1''' and the respective areas indicated by P, P', P'', P'''. Then

$$\begin{vmatrix} P & 1 & k_1 & k_1^2 \\ P' & 1 & k_1' & k_1'^2 \\ P'' & 1 & k_1'' & k_1''^2 \\ P''' & 1 & k_1''' & k_1'''^2 \end{vmatrix} = 0.$$

THE METHOD OF MOMENTS.[1]

By DUNHAM JACKSON, University of Minnesota.

1. Statement of the problem. One of the simplest problems relating to the fitting of a frequency curve may be stated as follows. A number of measurements have been made of a quantity x, say N measurements in all. These measurements do not all agree, but give the different values $x_1, x_2, \cdots, x_n$, occurring $y_1, y_2, \cdots, y_n$ times respectively, so that $y_1 + y_2 + \cdots + y_n = N$. It is desired to find a polynomial

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p,$$

of given degree p, so that the relation $y = F(x)$ shall give a good approximate representation of the frequencies y_k corresponding to the various values of x.

Since any polynomial in x, other than a constant, becomes very large numerically as x becomes positively or negatively infinite, $F(x)$ can not be expected to give a serviceable representation of a frequency distribution for more than a restricted range of values of x. But it may happen that an approximation is required for only a limited part of a frequency curve, and in such a case a polynomial formula may be entirely satisfactory. So the problem stated has practical importance, in addition to the interest which it possesses as one of the simplest representatives of a general type.

The mathematical discussion of the following paragraphs is of much wider application than the first statement of the problem would suggest. The numbers x_k, instead of being the quantities actually observed, may be representatives of groups of observations which have been thrown together to simplify a computation which would otherwise be excessively laborious. If border-line cases have been regarded as distributed between adjacent groups, some of the group-frequencies y_k may be fractions, instead of integers. More generally still, the reasoning holds without change if the statistical interpretation is abandoned altogether, and $x_1, \cdots, x_n$ are taken to be any n distinct real numbers, while $y_1, \cdots, y_n$ are any n real numbers whatever, distinct or otherwise, positive, negative, or zero.

Let

$$z_k = F(x_k) = a_0 + a_1 x_k + a_2 x_k^2 + \cdots + a_p x_k^p \qquad (k = 1, 2, \cdots, n). \quad (1)$$

The goodness of fit of the curve $y = F(x)$ is to be judged by the extent to which $z_1, \cdots, z_n$ approach coincidence with $y_1, \cdots, y_n$ respectively. The problem remains indefinite until it is specified just what conditions of approximation the numbers z_k must satisfy.

If the degree p is taken equal to $n - 1$, there will be n coefficients in $F(x)$,

[1] Presented to the Minnesota Section of the Association at Northfield, Minn., May 19, 1923. The first two sections of this paper do not require a knowledge of the calculus or of the content of higher courses in mathematics on the part of the reader; the third section, in the nature of an appendix, is of more advanced character.

and they can be determined so as to satisfy exactly the n equations[1] $z_k = y_k$. Usually, however, it will be undesirable, if not impracticable, to calculate so many terms, and some less exacting criterion of approximation must be used.

One suggestion would be to determine the a's so as to minimize the sum of the squares of the errors,

$$(z_1 - y_1)^2 + (z_2 - y_2)^2 + \cdots + (z_n - y_n)^2.$$

Whatever theoretical justification the method of least squares may or may not have in this connection, the suggestion is one of the first that would occur to anybody attacking the problem. From a purely algebraic point of view, it may appear open to the objection that without the use of the calculus it is not clear at first glance how the coefficients are actually to be found in a numerical case.

Another suggestion is that the a's be found by the *method of moments*,[2] that is, that they be made to satisfy the equations

$$\begin{array}{l} z_1 + z_2 + \cdots + z_n = y_1 + y_2 + \cdots + y_n, \\ z_1x_1 + z_2x_2 + \cdots + z_nx_n = y_1x_1 + y_2x_2 + \cdots + y_nx_n, \\ z_1x_1^2 + z_2x_2^2 + \cdots + z_nx_n^2 = y_1x_1^2 + y_2x_2^2 + \cdots + y_nx_n^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_1x_1^p + z_2x_2^p + \cdots + z_nx_n^p = y_1x_1^p + y_2x_2^p + \cdots + y_nx_n^p. \end{array} \tag{2}$$

If the z's are expressed by means of (1), there will be $p + 1$ equations of the first degree, entirely explicit in form, to determine the $p + 1$ unknowns $a_0, \cdots, a_p$. The first equation requires that the total number of observations, as calculated from the approximation formula, have the true value N. The first two equations together make the arithmetical mean of a set of x's[3] distributed according to the frequencies z_k coincide with that found from the observed frequencies y_k, and the third equation, taken with the first two, means that the calculated and observed standard deviations agree. The subsequent equations are perhaps to be regarded as suggested by analogy. It may be objected that it is not clear just what meaning the later equations have for the closeness of the approximation.

The objections to both suggestions are removed, and their advantages are combined, by the following theorem, which it is the purpose of this paper to prove:

THEOREM: *The method of moments gives the solution of the problem of least squares.*

This important fact, while well known, and indeed immediately apparent to anybody familiar with the ordinary procedure in solving linear equations approximately by the method of least squares, seems not to be so universally recognized as to render superfluous an exposition of it here, with a simple algebraic proof. The method of proof is one of standard utility in discussions of this nature. As

[1] These equations are of the first degree in the unknowns $a_0, a_1, \cdots, a_p$, and their determinant is of a well-known type which is always different from zero; cf. third footnote to section 3 below.

[2] See, for example, W. Palin Elderton, *Frequency-curves and correlation*, London, 1906, Chapter III. In practice, the method of moments is used with certain refinements, which are left out of account here.

[3] This is of course an altogether different thing from the mean of the *frequencies*.

the moment equations are in fact simply the "normal equations" of the problem, given by the ordinary rules, the work may be regarded as an elementary algebraic justification of the normal equations, for the case in hand.

It is to be noticed that the reasoning applies only to the fitting of a *polynomial* approximation curve. The conclusion as stated would not be correct for approximation curves of other forms.

2. Proof of identity. A student interested in the theoretical aspects of the two problems under consideration (the problem of least squares and that of moments) would raise the question whether it is clear in advance that each problem has one and just one solution. It will be shown later, necessarily by somewhat less elementary means, that the answer is in the affirmative.[1] For the present, the fact will be assumed, as something easily believed.

This being granted, let it be assumed that the a's have been determined *so as to solve the problem of least squares.* It is to be proved that the same a's must satisfy the moment equations (2). Suppose, if possible, that this is not true; that there is an exponent r, belonging to the range $0 \leqq r \leqq p$, for which

$$z_1x_1^r + z_2x_2^r + \cdots + z_nx_n^r \neq y_1x_1^r + y_2x_2^r + \cdots + y_nx_n^r.$$

Let

$$\sum_{k=1}^{n} (z_k - y_k)x_k^r = S \neq 0.$$

Let h be a constant, about the determination of which more will be said presently, and let

$$F_1(x) = F(x) - hx^r.$$

Then $F_1(x)$ is a polynomial of exactly the same form as $F(x)$, except that the coefficient a_r is replaced by $a_r - h$. Let $F_1(x_k) = z_k'$. Then

$$\sum_{k=1}^{n} (z_k' - y_k)^2 = \sum_{k=1}^{n} (z_k - hx_k^r - y_k)^2$$

$$= \sum_{k=1}^{n} (z_k - y_k)^2 - 2h \sum_{k=1}^{n} (z_k - y_k)x_k^r + h^2 \sum_{k=1}^{n} x_k^{2r}.$$

If $\sum x_k^{2r}$ is denoted by T,

$$\sum_{k=1}^{n} (z_k' - y_k)^2 = \sum_{k=1}^{n} (z_k - y_k)^2 - h(2S - hT).$$

The constant h has been left undetermined so far; let h now be taken equal to S/T. The division by T is certainly possible, because T is a sum of squares, and hence positive. Then

$$\sum (z_k' - y_k)^2 = \sum (z_k - y_k)^2 - \frac{S^2}{T}.$$

[1] It is understood that $n \geqq 2$, and that $p < n - 1$.

The value of S^2/T being positive, it is found that

$$\sum (z_k' - y_k)^2 < \sum (z_k - y_k)^2.$$

But this is impossible, since the a's were supposed determined in the first place so as to make $\sum (z_k - y_k)^2$ the smallest possible sum of squares. So the theorem is proved.

3. Proof of existence and uniqueness. The assumption stated in the first paragraph of the preceding section may be regarded as made up of four parts, (a) that the problem of least squares has at least one solution,[1] (b) that it can not have more than one solution, (c) that the moment equations have at least one solution, (d) that they can not have more than one solution. Only one of these parts, namely (a), was actually used; the discussion really showed that *if* the problem of least squares has a solution, then the moment equations also have a solution, consisting of the same set of a's. The purpose of this concluding section is to complete the justification of (a), (b), (c), and (d). It will be found necessary here to use something more than the methods of elementary algebra.

Let $G = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$. Then[2] $|y_k| \leqq G$ for $k = 1, 2, \cdots, n$. If any one of the numbers $|z_k|$ were greater than $2G$, the corresponding difference $|z_k - y_k|$ would be greater than G, and $\sum (z_k - y_k)^2$ would be greater than G^2. For a specified set of numbers z_k, let the equations

$$a_0 + a_1 x_k + \cdots + a_p x_k^p = z_k, \qquad k = 1, 2, \cdots, p+1,$$

be regarded as a set of linear equations for determining the a's. The corresponding equations for $k = p + 2, \cdots, n$ may be left out of account. (It will be remembered that $p + 1 < n$.) The determinant of the $p + 1$ equations is different from zero;[3] let it be denoted by D. Let the cofactor of x_k^r in this determinant be represented by X_{kr}. By Cramer's rule,

$$a_r = \frac{1}{D} \sum_{k=1}^{p+1} z_k X_{kr}.$$

If H is the largest of the $(p + 1)^2$ numbers $|X_{kr}|$, if the z's are subjected to the restriction that $|z_k| \leqq 2G$, $k = 1, 2, \cdots, p + 1$, and if the number $2(p + 1) \times GH/|D|$ is denoted by K, then[4] $|a_r| \leqq K$, $r = 0, 1, \cdots, p$.

Since $\sum (z_k - y_k)^2$ is a continuous function of the a's, it follows from a fundamental proposition in the theory of functions of real variables that among all sets of a's *subject to the restriction that* $|a_r| \leqq K$, $r = 0, 1, \cdots, p$, there will be

[1] By a "solution" is meant a set of numbers $a_0, a_1, \cdots, a_p$, satisfying the conditions of the problem; and two solutions $(a_0, \cdots, a_p)$, $(a_0', \cdots, a_p')$ are to be regarded as distinct, unless every a_r is equal to the corresponding a_r'.

[2] The discussion is put in such form as to apply even when the y's are not regarded as statistical frequencies, but are allowed to take on arbitrary values, negative as well as positive.

[3] If the determinant were zero, there would exist a set of a's, not all zero, satisfying the equations

$$a_0 + a_1 x_k + \cdots + a_p x_k^p = 0, \qquad k = 1, 2, \cdots, p+1,$$

and the polynomial $a_0 + a_1 x + \cdots + a_p x^p$ would have $p + 1$ distinct roots.

[4] For the method of proof, cf., *e.g.*, L. Tonelli, I polinomi d'approssimazione di Tchebychev, *Annali di matematica pura ed applicata*, series 3, vol. 15 (1908), pp. 47–119; pp. 61–62.

at least one set $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_p$ which reduces the sum of squares to a minimum. Furthermore, this minimum, which may be denoted by g^2, is not greater than G^2, since the sum of squares can be brought down to G^2 simply by taking $a_0 = \cdots = a_p = 0$. On the other hand, if any a_r is taken greater than K in absolute value, it follows from the preceding paragraph that at least one of the (first $p + 1$ of the) z's must violate the condition $|z_k| \leqq 2G$, and then $\sum(z_k - y_k)^2$ will be *greater* than G^2. So the value of $\sum(z_k - y_k)^2$ can not be reduced below g^2 by any choice of the a's whatever; that is, the numbers $\bar{a}_0, \cdots, \bar{a}_p$ give at least one solution of the problem of least squares, and (a) of the first paragraph of the section is proved. The truth of (c) then follows from the work of the preceding section, as has already been pointed out. It remains to consider (b) and (d).

One aspect of the result obtained so far is, that if the x's are held fast, the moment equations (2) will possess at least one solution for arbitrary values of the numbers y_k. But this is equivalent to saying that if the right-hand members are denoted by $Y_0, \cdots, Y_p$, the equations will possess at least one solution for completely arbitrary values of the Y's. For if the Y's are regarded as given, it will always be possible to set $y_{p+2} = \cdots = y_n = 0$, and then to determine $y_1, \cdots, y_{p+1}$ so as to satisfy the $p + 1$ equations

$$y_1x_1^r + y_2x_2^r + \cdots + y_{p+1}x_{p+1}^r = Y_r, \qquad r = 0, 1, \cdots, p,$$

the determinant of these equations being the conjugate of one which has already been seen to be different from zero. Consequently the determinant of the equations (2), regarded as a set of linear equations for the unknowns $a_0, \cdots, a_p$, must be different from zero,[1] and the equations will have *just one* solution for any given values of the y's. This establishes (d).

Finally, if the problem of least squares had two distinct solutions, each of these would give a solution of the moment equations, by section 2, and therefore (d) carries with it the truth of (b).

It would not be difficult to prove (a) and (b) on the one hand, or (c) and (d) on the other hand, without the intervention of section 2, but the present treatment seems simpler when the work of section 2 has been carried through.

ANOTHER DEFINITION OF AMICABLE NUMBERS AND SOME OF THEIR RELATIONS TO DICKSON'S AMICABLES.[2]

By BENJAMIN FRANKLIN YANNEY, Chicago, Illinois.

1. Professor L. E. Dickson (*1913*, 84–92) defines an amicable triple of numbers to be three numbers such that the sum of the proper divisors of each number equals the sum of the remaining two numbers. In a similar manner he defines

[1] If the determinant were zero, there would be values of the Y's for which the equations could not be solved. Suppose the rank of the matrix of the coefficients were $q < p + 1$. Let D_q be a particular non-vanishing q-rowed determinant of the matrix. Let the rth equation be one whose coefficients do not enter into D_q, and let $Y_r = 1$, while all the other Y's are zero. Then the

an amicable k-tuple of numbers, which he expresses in symbols thus:

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k, \tag{A}$$

where $\sigma(n)$ denotes the sum of all the divisors of n. He then proceeds to derive nine sets of amicable triples, closing his paper with a list of ten sets of numbers r, s, t with an appropriate equation in a for each set, from which a value for a relatively prime to each number r, s, t is to be found such that ar, as, at shall be an amicable triple. The last one of the set he himself solves, thus giving a tenth set of amicable triples. He proposes (*1913*, 196, *1919*, 214) Number Theory Problem 191 (erroneously numbered 187), which requires the solution of any one of the remaining nine unsolved equations of the list given in the paper referred to.

The purpose of the present paper is to give another definition of an amicable k-tuple of numbers, and to show some relations existing between amicable numbers as thus defined and amicable numbers as defined by Professor Dickson. An application will be made in an attempt to solve one of the ten equations listed by him. Incidentally also, a still more general definition will be suggested, of which the two compared herein are special cases.

2. Accordingly we shall say that three numbers form an amicable triple if each number equals the sum of the proper divisors of the other two. Thus n_1, n_2, n_3 will form an amicable triple if

$$\begin{aligned} n_1 &= \{\sigma(n_2) - n_2\} + \{\sigma(n_3) - n_3\}, \\ n_2 &= \{\sigma(n_3) - n_3\} + \{\sigma(n_1) - n_1\}, \\ n_3 &= \{\sigma(n_1) - n_1\} + \{\sigma(n_2) - n_2\}; \end{aligned} \tag{1}$$

or, more briefly, if

$$n_1 + n_2 + n_3 = \sigma(n_3) + \sigma(n_1) = \sigma(n_1) + \sigma(n_2) = \sigma(n_2) + \sigma(n_3). \tag{2}$$

Similarly, $n_1, n_2, \cdots, n_k$ form an amicable k-tuple if

$$\sum_{i=1}^{k} n_i = \sum_{1} \sigma(n_i) = \sum_{2} \sigma(n_i) = \cdots = \sum_{k} \sigma(n_i), \tag{B}$$

where the indices under $\sum$ indicate which n is to be omitted in the respective summations.

From (B) we get

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k), \tag{3}$$

which expresses the same relation between the sums of all the divisors of the respective numbers as in (A). But instead of these being equal to the sum of all

$(q + 1)$-rowed determinant of the augmented matrix which contains the minor D_q and the element Y_r is different from zero, the augmented matrix is of rank $q + 1$, and the equations are inconsistent.

[2] Professor Dickson, to whom the author acknowledges indebtedness for criticisms and suggestions given during the preparation of this paper, must not in any way be held responsible for any infelicities or inaccuracies that may have found their way into the final selection and arrangement of the material employed. It is only fair to state also that definition (B) had occurred to Professor Dickson when he made his investigations, but that he chose (A) because of its greater simplicity.

the numbers, as in (A), we have, from (B) and (3),

$$\sum_{i=1}^{k} n_i = (k-1)\sigma(n_j), \qquad j = 1, 2, \cdots, \text{or } k; \tag{4}$$

so that (2) becomes

$$n_1 + n_2 + n_3 = 2\sigma(n_j), \qquad j = 1, 2, \text{or } 3. \tag{5}$$

From (4) it easily follows, as indeed it does from the definition directly, the same as in the case of Dickson's amicables, that if $k = 2$, we have the usual case of two amicable numbers. There is, however, one marked difference between (A) and (B). It is the simplicity with which examples may be furnished of amicable k-tuples of numbers as defined in (B). If we let n be any prime number, then $k\ (= n + 1)$ such equal prime numbers constitute a k-tuple of numbers amicable by (B). Thus, we have as amicable sets[1] of numbers the following: 2, 2, 2; 3, 3, 3, 3; 5, 5, 5, 5, 5, 5; and so on for every prime. Of a different type is the amicable triple[1]

$$2^2\cdot 7\cdot 11,\ 5\cdot 7\cdot 13,\ 7\cdot 83, \tag{6}$$

which set of numbers fulfills (2). The numbers $2^2\cdot 11$, $5\cdot 13$, 83 are the three numbers involved in one of Dickson's problems already referred to, and considered later in this paper.

We may state as follows the problem of which (6) is the solution:

Problem. From the three numbers $2^2\cdot 11$, $5\cdot 13$, 83 to derive a triple of numbers amicable by (B).

To solve the problem, we note first that the given numbers satisfy (3). What we desire, then, is a number a, not divisible by 2, 5, 11, 13, or 83, such that $(2^2\cdot 11 + 5\cdot 13 + 83)a = 2\sigma(83a)$, or such that $2^3a = 7\sigma(a)$. This latter equation is evidently satisfied if $a = 7$.

3. It will now be shown how we may derive from an amicable set of numbers fulfilling (B), another set fulfilling (A). From (B) we see that

$$k\left(\sum_{i=1}^{k} n_i\right) = (k-1)\left[\sum_{i=1}^{k}\sigma(n_i)\right]. \tag{7}$$

Now let a be a number such that $an_1, an_2, \cdots, an_k$ form an amicable k-tuple in accordance with (A). Then it follows that

$$k\left(\sum_{i=1}^{k} an_i\right) = \sum_{i=1}^{k}\sigma(an_i). \tag{8}$$

From (7) and (8) we obtain

$$\sum_{i=1}^{k}\sigma(an_i) = a(k-1)\left[\sum_{i=1}^{k}\sigma(n_i)\right], \tag{9}$$

[1] "In 1899, E. B. Escott raised the question of the existence of three or more numbers such that each is equal to the sum of the [aliquot] divisors of the others." Dickson's *History of the Theory of Numbers*, vol. I, p. 50.

which, in view of (3), may be written

$$\sigma(an_i) = a(k-1)\sigma(n_i), \qquad i = 1, 2, \cdots, k. \tag{10}$$

This result is a necessary condition. It is also sufficient. For if (10) is true, we obtain the relation (9) by adding the k equations of (10). Then by means of (7) we obtain (8). But from (10), in view of (3), we get, after dividing out k,

$$\sigma(an_1) = \sigma(an_2) = \cdots = \sigma(an_k) = \sum_{i=1}^{k} an_i, \tag{11}$$

which fulfills the conditions in (A). We have thus completed the proof of the

Theorem. If $n_1, n_2, \cdots, n_k$ form an amicable k-tuple of numbers as defined in (B), then will $an_1, an_2, \cdots, an_k$ form an amicable k-tuple of numbers as defined in (A), if and only if $\sigma(an_i) = a(k-1)\sigma(n_i)$, $i = 1, 2, \cdots$, or k.

There follow the corollaries:

Corollary 1. If a is relatively prime to each n_i, then the necessary and sufficient condition of the theorem may be replaced by $\sigma(a) = a(k-1)$.

For if a is relatively prime to n, $\sigma(an) = \sigma(a)\sigma(n)$.

If, further, $k = 3$, a must be a perfect number. In general, if $k > 2$, a must be a multiply perfect number, the order of multiplicity being $k - 1$.

Corollary 2. If a is relatively prime to each n_i, it can not be a prime number.

For then $a + 1 = a(k-1)$, and, hence, $a(k-2) = 1$, which is absurd.

Corollary 3. If n_1, n_2 are amicable numbers, an_1, an_2 can not be amicable for $a > 1$.

For then surely $\sigma(an_i) > a\sigma(n_i)$, and, hence, $\sigma(an_i) > an_j$, $i, j = 1, 2, i \neq j$.

4. We now apply some of the foregoing principles to the solution of the first of the list of ten problems proposed by Professor Dickson.

Problem. Given $2^2 \cdot 11$, $5 \cdot 13$, 83, and the equation $7\sigma(a) = 2^4 a$; to find a value for a, not divisible by 2, 5, 11, 13, or 83, such that $2^2 \cdot 11a$, $5 \cdot 13a$, $83a$ shall form an amicable triple [as defined by Dickson].

As already found in the problem previously considered, (6) is an amicable triple in accordance with (B). Hence, we let $a = 7b$. We have, then, from (10) the equation $\sigma(7b \cdot 83) = 2b\sigma(7 \cdot 83)$, from which we get $\sigma(7b) = 2^4 b$. Of this latter equation, every known perfect number except 28 is a solution. But since every known perfect number is even, while the conditions of the problem require that b shall be an odd number, a solution of the problem seems doubtful. Of course, an odd perfect number not divisible by 7, if such exists, would be a solution provided it met also the condition of non-divisibility stated in the problem. There are, however, other solutions than perfect numbers for b in the foregoing equation, as $3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 19$. But this solution of the equation is not a solution of the problem, since it contains two prohibited factors.

5. As examples of sets of triples related as in (10), are 2, 2, 2 and 120, 120, 120; 2, 2, 2 and 672, 672, 672; or, 2, 2, 2 and $2a$, $2a$, $2a$ such that $2a$ is any of the known multiply perfect numbers of multiplicity two. No example has yet been found in which precisely two of the numbers in each set are equal, or

in which the numbers in each set are all distinct. It may be of interest also to note that it can easily be shown that in none of the cases of amicable triples found by Professor Dickson and listed by him in the article referred to does there exist an amicable triple as defined in (B), which is related to Dickson's as in (10). Furthermore, in the cases of at least two of the ten problems proposed by him for solution, viz., the second and fifth, it can be shown that there does not exist for either case a corresponding triple of amicable numbers as defined in (B). For example, in problem 2 of the list, the corresponding equation in a, necessary for an amicable triple as defined in (10), is $2.3^2\sigma(a) = 19a$. This necessitates a to have some power of 2 as a factor, while at the same time one of the three numbers given in the fundamental set prohibits 2 from being a divisor of a.

6. A More General Definition of Amicable Numbers. The following is suggested as a more general definition of an amicable k-tuple of numbers, which includes definitions (A) and (B) as special cases:

A set of k numbers $n_1, n_2, \cdots, n_k$ shall be said to form an amicable k-tuple of numbers if

$$\sum_{i=1}^{k} n_i = \sum \sigma(n_{j1}) = \sum \sigma(n_{j2}) = \cdots = \sum \sigma(n_{js}), \qquad s = {}_kC_r, \tag{K}$$

where each j ranges over r of the k values of i, s being the number of combinations of k things taken r at a time.

From (K) again follows (3). Hence, instead of (K) we have the simpler expression

$$\sum_{i=1}^{k} n_i = r\sigma(n_j), \qquad j = 1, 2, \cdots, \text{or } k. \tag{12}$$

We shall refer to r as designating the multiplicity of the amicable set of k numbers. Thus, the multiplicity of the amicable set as defined in (A) is 1; and as defined in (B), $k - 1$.

As a simple example of (12), we may take the amicable sextuple 2, 2, 2, 2, 2, 2, which is of multiplicity 4. Of a different type is the amicable quadruple $3 \cdot 7 \cdot 41$, $7 \cdot 11 \cdot 13$, $7 \cdot 11 \cdot 13$, $7 \cdot 167$, of multiplicity 3. If each of the four numbers is multiplied by 2, the resulting numbers form an amicable quadruple of multiplicity 2.

In a manner analogous to that used in proving (10), it can be shown that if the numbers $n_1, n_2, \cdots, n_k$ form an amicable k-tuple of multiplicity r, which is to be not less than one half k, then will $an_1, an_2, \cdots, an_k$ form an amicable k-tuple of numbers of multiplicity $k - r$, if and only if

$$(k - r)\sigma(an_i) = ar\sigma(n_i), \qquad i = 1, 2, \cdots, \text{or } k. \tag{13}$$

ACTUAL SOLUTION OF SIMULTANEOUS LINEAR NUMERICAL EQUATIONS.

By WILLIAM R. RANSOM, Tufts College.

1. Introduction. The theory of the solution of simultaneous linear equations is well known, and can be very briefly and neatly disposed of, as in the fourth chapter of Bôcher's *Introduction to Higher Algebra*. But an actual method of procedure for any case where the coefficients are numerical has not been described in detail in print so far as the writer is aware. It will be the purpose of this paper to state a set of directions in this connection.

2. Notation. We may employ the notation

$$E_k \equiv a_{k0} + a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n$$

and represent a set of linear equations by

$$E_1 = E_2 = \cdots = E_m = 0.$$

From these equations write the n-columned, m-rowed matrix

$$M \equiv \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{matrix} \right\|$$

from which certain determinants are to be chosen by striking out rows and columns. A set of a's (from the equations, not from M) whose second suffix is zero will be referred to as the "absolute column." A set of a's whose second suffix is p will be called the x_p-column.

When $(m - p)$ rows and $(n - p)$ columns of M are struck out, the remaining a's form a determinant, of order p, which will be called Δ_p. The rows and columns of M which are represented in Δ_p correspond to certain E's and x's, and it may be supposed, without loss of generality, that the E's and x's are renumbered so that they may be referred to as $E_1 \cdots E_p$, and $x_1 \cdots x_p$. Let them be thus renumbered.

Begin by finding a Δ_p whose value is different from zero. This can always be done[1] since a single non-vanishing coefficient, a_{11}, is a Δ_1 and may be used if one does not easily discover a non-vanishing determinant of higher order.

3. Directions. If $p = m$, proceed at once to (V) below, and solve.

If $p = n$, change the notation, using one of the non-vanishing minors as Δ_p, and proceed as in (I) below.

(I) If p is less than m or n, border Δ_p by means of terms from the $(p + 1)$st row and x_{p+1} column of M, and call the resulting determinant Δ_{p+1}. From Δ_{p+1} take the cofactors of the elements $a_{1,p+1} \cdots a_{p+1,p+1}$, calling them $A_{1,p+1} \cdots A_{p+1,p+1}$, and note that $A_{p+1,p+1} \equiv \Delta_p \neq 0$. Evaluate Δ_{p+1} by means of the expansion formula

$$\Delta_{p+1} = a_{1,p+1}A_{1,p+1} + \cdots + a_{p+1,p+1}\Delta_p.$$

At this point two cases appear: IIa and IIb.

[1] The trivial case in which the a's are all zero is excluded.

(IIa) If $\Delta_{p+1} \neq 0$: begin again at (I), using this higher order determinant as the Δ_p and noting that it will be the last one of the next set of cofactors. (Continue to go back to (I) in this way until a case of (IIb) is reached. When $p = m$ a row of zeros, or when $p = n$ a column of zeros, is adjoined to M in order that a case of (IIb) may be reached.)

(IIb) If $\Delta_{p+1} = 0$: multiply $E_1, E_2, \cdots E_{p+1}$ by $A_{1,p+1} \cdots A_{p+1,p+1}$, respectively, and add the results. As a check on the work thus far, note that the x_{p+1} column adds up to $\Delta_{p+1}x_{p+1}$, and therefore vanishes, and that the $x_1, \cdots x_p$ columns give zero in each case. It is possible that some other columns may give zero: in fact, at this point three cases appear, IIIa, IIIb, IIIc, as follows.

(IIIa) If every x column gives zero, but the absolute column gives

$$a_{10}A_{1,p+1} + \cdots a_{0,p+1}A_{p+1,p+1} \equiv Q \neq 0,$$

the equations are inconsistent, and in this case the work terminates by stating this conclusion and supporting it by the formula just found

$$A_{1,p+1}E_1 + \cdots + \Delta_p E_{p+1} \equiv Q \neq 0$$

which derives from the $(p + 1)$ equations, $E_1 = \cdots = E_{p+1} = 0$, a result inconsistent with them.

(IIIb) If some x column, say the x_{p+2} column, does not give zero (regardless of what happens to the absolute column): the determinant found by bordering the original Δ_p with coefficients from the x_{p+2} column and the E_{p+1} row will be a non-vanishing determinant of order $(p + 1)$, and having Δ_p as minor, just as in (IIa), and we go back to I as directed in (IIa), renumbering the columns.

(IIIc) If each column (including the absolute column) gives zero: the equation $E_{p+1} = 0$ is satisfied by any set of values that satisfies $E_1 = \cdots E_p = 0$. In this case the dependence of E_{p+1} upon the first p equations should be exhibited by recording the formula just found

$$E_{p+1} \equiv (A_{1,p+1}E_1 + \cdots A_{p,p+1}E_p) \div (-\Delta_p).$$

Then proceed to (IV).

(IV) Take each remaining equation and treat it as the E_{p+1} in (I). What will happen will be that either:

IIIa: an inconsistency will be found.

IIIb: a higher order non-vanishing determinant will be found.

IIIc: another formula indicating dependence will be found. (IIIc) is continued until each one of the last $(m - p)$ E's is expressed in terms of the first p of them. Then proceed to (V).

(V) The highest order non-vanishing determinant found in the last (IIIb) is now denoted Δ_p and the corresponding p equations are multiplied each by the cofactor of the corresponding element in the last column of Δ_p, and then the equations are added and the result divided by Δ_p. In this addition, the columns containing $x_1 \cdots x_{p-1}$ give zeros, and the result gives x_p, possibly in terms of $x_{p+1} \cdots x_n$ if $n > p$, or as an absolute number if $n = p$.[1]

[1] Since $\Delta_p \neq 0$, n cannot be less than p.

(VI) Take from the Δ_p just used the cofactor of $a_{p,p}$ as a new Δ_p of order lower by one. Substitute the value of x_p just found in (V) into the first $(p - 1)$ equations, and proceeding as in (V), find the value of another x, and so on until the value of x_1 is reached.

4. Conclusion. The complete solution will consist of either:

A: a formula showing that certain equations are inconsistent, (III*a*),[1] or

B: A set of formulas for $x_1 \cdots x_p$ found from p of the equations (V), together with $(m - p)$ formulas, if $m > p$, showing how the last $(m - p)$ equations depend upon those which precede them, (III*c*).

Although these directions give no clue as to the theory on which they are based, one who follows them carefully will find that the process explains and justifies itself as it is carried out.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. THE TEN MOST IMPORTANT MATHEMATICAL BOOKS IN THE WORLD.

By WALTER C. EELLS, Whitman College, Walla Walla, Wash.

This title is suggested by H. G. Wells's article *The Ten Most Important Books in the World* in the *American Magazine* for April, 1923. "What are the ten most important books in the world?" an interviewer asked Mr. Wells, and in his reply he says, "Absurd questions sometimes make the most interesting discussions. . . . Following the precedents, I will first show how unreasonable a question it is, and then give myself up to its insidious fascination."

The question, "*What are the ten most important mathematical books in the world?*", is equally unreasonable, but to the mathematician it may prove equally or even more fascinating. I suggested a similar question to my class in the History of Mathematics last term, with rather interesting results. I suggest this question now for discussion on the part of the readers of the MONTHLY who may be tempted to yield to its "insidious fascination." To be sure, no two lists will agree, but this very fact will give the discussion its interest and value. What books, if any, will be common to all lists suggested?[2]

The result of such a discussion should be similar to that stated by Mr. Wells in discussing the answer to the question of the six greatest men in the world,

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[1] The geometrical and algebraic problems in their conventional form are distinguished at this stage. For most algebraic purposes a system which is inconsistent may be dismissed as such, while the study of a system of linear spreads does not terminate when two of the spreads are found to be parallel.

[2] It is hoped that a number of readers will follow up the suggestion made by the author of this discussion, either by contributing short papers, as in this instance, or by supplying lists of ten which may be of use when the time comes for a final summing up.—EDITOR.

as a result of which he says "endless people were set thinking, very profitably, and sent to their encyclopedias and histories and biographies for refreshing and stimulating reading."

A list of the ten most important mathematical books is not necessarily synonymous with a list of the ten greatest mathematicians. Archimedes or Leibnitz, for instance, should doubtless be included in the latter class, but it is difficult to pick out a single outstanding work of either which nearly approaches the importance and influence of Euclid's *Elements*, or Newton's *Principia*. Much, too, of the important and influential work in mathematics of the modern period has appeared in scattered articles in the journals, not in books.

Neither is a list of the ten most important mathematical books necessarily the ten most important ones for present-day study, any more than is Mr. Wells's list suitable for a similar purpose. In fact less than half of his list does he recommend as valuable reading at the present time.

It is interesting to note that four of Mr. Wells's ten books are scientific, but none of them are mathematical. The nearest he comes is when he considers Newton's *Principia*, "which brought the whole material universe under the domain of natural law," but reluctantly he rejects it as one of his ten.

As a first approximation toward a mathematical list, and as a basis for discussion and suggestion of other lists, I venture to offer the following as my choice, arranged in chronological order, accompanied, in some cases, by noteworthy characterizations of the contents of these books or of their influence which have been made by others.

EUCLID'S "*Elements*" (Alexandria, *c.* 325 B.C.), which "has been for nearly twenty-two centuries the encouragement and guide of scientific thought" (Clifford), which has passed through more than two thousand editions and has exercised such profound influence on the teaching and knowledge of geometry for more than two thousand years, and which is "still regarded by some as the best introduction to the mathematical sciences" (Cajori).

APOLLONIUS'S "*Conic Sections*" (Alexandria ? *c.* 210 B.C.), the great systematic treatise which developed the geometrical "theory of conic sections, and was the prelude to the theory of geometrical curves of all degrees—and of the geometry of form and position" (Cajori) as distinguished from the geometry of measurement.

LEONARDO OF PISA'S "*Liber Abaci*" (Pisa, 1202), which marked the first renaissance of mathematics on Christian soil, introduced Arabian algebra, and brought into general use in Europe the labor-saving Hindu-Arabic numerals, and for centuries was a storehouse of material for later writers on arithmetic and algebra; among others forming the basis for the first printed work on arithmetic, algebra, and geometry, Pacioli's, which was printed at Venice in 1494.

NAPIER'S "*Mirifici Logarithmorum Canonis Descriptio*" (Edinburgh, 1614), which gave to the world Napier's great invention of logarithms with their miraculous power in modern computation, than which "with the exception

of the *Principia* of Newton there is no mathematical work published in the country which has produced such important consequences" (Glaisher, in *Encyclopædia Britannica*).

DESCARTES's "*Geometrie*" (Leyden, 1637), which in spite of its obscure style was of epoch-making importance in giving to the world the powerful method of analytic geometry "which far transcended everything that ever could have been reached upon the path pursued by the ancients" (Hankel), and than which "there cannot be a language more universal and more simple, . . . and better adapted to express the invariable relations of nature" (Fourier), and which contains in addition the modern exponential and literal notation of algebra.

NEWTON's "*Principia*" (Full Title: *Philosophiæ Naturalis Principia Mathematica*) (London, 1687), which established the mathematical foundation of the universe, "the greatest production of the human mind" (Lagrange); "the brightest page in the records of human wisdom—and preëminent above all the productions of human intellect" (Brewster's Life of Newton); and which, Laplace says, will always be assured "a preëminence above all the other productions of human genius."

LAGRANGE's "*Mécanique Analytique*" (Paris, 1788), "an epoch-making work . . . a most consummate example of analytic generality" (Cajori), "a kind of scientific poem" (Hamilton), the foundation of all later work on analytic mechanics, in which Lagrange "impressed on mechanics, as a branch of pure mathematics, that generality and completeness toward which his labours invariably tended" (Ball).

LAPLACE's "*Mécanique Céleste*" (Paris, 5 vols., 1799–1825), "the translation of the *Principia* into the language of the differential calculus" (Ball), which according to the author was intended to "offer a complete solution of the great mechanical problem presented by the solar system."

BOLYAI's "*Science Absolute of Space*" (Hungary, 1833), which, although only the appendix to a two-volume work by his father, is characterized by Halsted as "the most extraordinary two dozen pages in the history of human thought," and which, together with Lobachevski's work, opened up the whole fascinating and broadening field of non-Euclidean geometries.

HAMILTON's "*Lectures on Quaternions*" (Dublin, 1852), "the great discovery of our nineteenth century . . . (in which) there is as much real promise of benefit to mankind as in any event of Victoria's reign" (Thomas Hill), which is the foundation of all modern developments in the field of vector analysis, with its important applications in mathematical physics, including electromagnetic theory and Einstein's generalizations.

It is with much regret that the arbitrary limit of ten forbids the inclusion of such works as Diophantus's "*Arithmetic,*" Alkowarezmi's "*Algebra,*" Cardan's "*Ars Magna,*" Euler's "*Analysin Infinitorum,*" Legendre's "*Fonctions elliptiques*" and "*Théorie des Nombres,*" Gauss's "*Disquisitiones Arithmeticæ,*" Cantor's "*Geschichte der Mathematik*" and others which could easily be mentioned.

This list of ten important books is well distributed among the great fields of mathematics, as well as in time and in nationality. It ranges over twenty-two centuries. In it are represented two Greeks, two (or one) Italians (depending upon whether Lagrange is considered Italian or French), one Scotchman, two (or three) Frenchmen, an Englishman, a Hungarian and an Irishman—a very cosmopolitan group.

II. Note on the Algebraic Solution of the Cubic.

By E. J. Oglesby, New York University.

Take the general cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{1}$$

and assume that the roots x_1, x_2, x_3 have the form [1]

$$x_1 = a + b + c,$$
$$x_2 = a + \omega b + \omega^2 c,$$
$$x_3 = a + \omega^2 b + \omega c,$$

where $\omega^2 + \omega + 1 = 0$ and a, b, and c are to be determined.

We then have the following identities:

$$(a + b + c) + (a + \omega b + \omega^2 c) + (a + \omega^2 b + \omega c) = 3a, \tag{2}$$

$$\begin{aligned}(a + b + c)(a + \omega b + \omega^2 c) + (a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)& \\ + (a + \omega^2 b + \omega c)(a + b + c) &= 3(a^2 - bc),\end{aligned} \tag{3}$$

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^3 + b^3 + c^3 - 3abc. \tag{4}$$

From (2), (3), and (4)

$$x_1 + x_2 + x_3 = 3a,$$
$$x_1x_2 + x_2x_3 + x_3x_1 = 3(a^2 - bc),$$
$$x_1x_2x_3 = a^3 + b^3 + c^3 - 3abc.$$

From (1)

$$x_1 + x_2 + x_3 = -\frac{3a_1}{a_0},$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{3a_2}{a_0},$$

$$x_1x_2x_3 = -\frac{a_3}{a_0};$$

[1] This assumption may be justified in advance, since a, b, and c can always be found to satisfy these equations, for any x_1, x_2, x_3.—Editor.

whence

$$bc = -\frac{5}{9};$$

$$a^3 + b^3 + c^3 - 3abc = -\frac{a_3}{a_0} = -4,$$

whence

$$b^3 + c^3 = -\frac{70}{27}.$$

Therefore

$$b^3 - \frac{125}{729b^3} = -\frac{70}{27},$$

so that

$$b^3 = \frac{-35 + 15\sqrt{6}}{27}, \qquad c^3 = \frac{-35 - 15\sqrt{6}}{27},$$

$$b = -\tfrac{1}{3}\sqrt[3]{35 - 15\sqrt{6}}, \qquad c = -\tfrac{1}{3}\sqrt[3]{35 + 15\sqrt{6}},$$

whence

$$x_1 = a + b + c = -\tfrac{1}{3}[2 + \sqrt[3]{35 - 15\sqrt{6}} + \sqrt[3]{35 + 15\sqrt{6}}],$$

$$x_2 = a + \omega b + \omega^2 c = -\tfrac{1}{3}[2 + \omega\sqrt[3]{35 - 15\sqrt{6}} + \omega^2\sqrt[3]{35 + 15\sqrt{6}}],$$

$$x_3 = a + \omega^2 b + \omega c = -\tfrac{1}{3}[2 + \omega^2\sqrt[3]{35 - 15\sqrt{6}} + \omega\sqrt[3]{35 + 15\sqrt{6}}].$$

III. The "Limiting Case" in Enumerative Geometry.

By Albert A. Bennett, University of Texas.

H. G. Zeuthen, in his work on Enumerative Geometry, entitled *Lehrbuch der Abzählenden Methoden der Geometrie,* points out many significant dangers in enumerative methods. One of the most interesting cautions urged upon the reader is to approach all *special cases* as *limiting cases.* He remarks that in making the count of geometrical constants, usually evidenced in the order of the algebraic equation involved, there is a temptation to treat the special cases by special methods which may introduce new constants or omit, implicitly, significant elements of the general problem which become trivial in the special case. It often happens that a special case is contained in each of two general classes of the same number of parameters. This special case is special, but in a different way for the two classes, and the count must be made differently from the two points of view. As a precaution Zeuthen urges that one treat all such problems as *limiting* cases. Thus in the trivial case of the intersections of a straight line with a curve, a tangent line is to be regarded as the limit of a secant.

When algebraic data are given and algebraic results are desired, is it too much to ask that algebraic methods be employed? Now passing to a limit is *not*

an algebraic method, if one restricts this term severely. One has merely to examine modular geometries to find examples in which secants and tangents exist, for which no limiting processes may be applied. One may indeed replace x by *zero*, but not by letting x *approach* zero. Enumerative results hold in finite geometries with suitable restrictions, but Zeuthen's methods do not apply. What shall we do?

RECENT PUBLICATIONS.

REVIEWS.

Theorie der Gruppen von endlicher Ordnung. By A. SPEISER. Berlin, Julius Springer. 1923. 8vo. viii + 194 pages.

About two years ago there appeared a small volume of 120 pages in the Sammlung Göschen, entitled *Gruppentheorie,* by Ludwig Baumgartner. The fact that the present volume on the same general subject appeared so soon thereafter seems to indicate that there is now a considerable demand in Germany for new introductory books along this line. This is the more remarkable since the German student was already supplied with various useful expository works, including the two books by Netto, published in 1882 and 1908, and Weber's classic *Lehrbuch der Algebra.* While the present volume has much in common with these earlier works its main object seems to be to lead the reader rapidly into some of the more modern developments, especially into those inaugurated by Frobenius about 1896.

The book is divided into 15 chapters, the longest of which is devoted to group characteristics and covers only 20 pages. The applications to crystallography are especially emphasized in view of the simplicity and the precision of these applications. Hence Chapter 6 is devoted to the groups of crystallography, the subheadings being plane lattices, space lattices, and classes of crystals. The first five chapters are devoted mainly to the fundamental theorems relating to finite abstract groups. The reference near the bottom of page 2 should be to volume 1 of Kronecker's *Werke,* not to volume 2, and on page 38 the name L. C. Dickson appears instead of L. E. Dickson.

On page 17 the commutator of A and B is defined as $B^{-1}A^{-1}BA$ instead of $A^{-1}B^{-1}AB$ as is done by A. Loewy on page 189 of volume 1 of the second edition of Pascal's *Repertorium der höheren Mathematik,* 1910, and by H. Vogt on page 540 of tome 1, volume 1, *Encyclopédie des Sciences Mathématiques.* Our author's definition is, however, in accord with the one given by R. Dedekind, who introduced this term, *Mathematische Annalen,* volume 48 (1897), page 553. Although this is not a very important matter it would clearly be desirable to secure uniformity of usage so as to smooth the path of the student who desires to enter this field of study. The author of the present work does not seem to have adopted the custom followed by the majority in this respect. Similar remarks apply to the notation used sometimes to represent the type of a prime power group. On

page 38, for instance, it is noted that the group of isomorphisms of the cyclic group of order 2^m is the abelian group of type $(2^{m-2}, 2)$. This group is more commonly said to be of type $(m - 2, 1)$.

The brevity of the work made it necessary to select the theorems from a large body of known results. Hence our author found it desirable to say only little as regards different possible sets of independent generators of an abstract group. In particular, the theorem that the number of the independent generators of every prime power group is fixed by the group itself and is not dependent upon how the generating operators are selected is not treated. In the small amount of space at his command to lay the foundation for a study of linear substitution groups the author did, however, succeed in giving in a clear and attractive manner a considerable number of the most important theorems relating to abstract groups and to the so-called permutation groups. Chapter 7 is explicitly devoted to the latter.

The proofs of theorems 48 and 49 are vitiated by a slight mistake in the first line of page 45, and the closing developments of Chapter 5 are marred by the use of r in place of s and vice versa, as well as by several statements which are likely to be misunderstood at first by the beginner. For instance, near the middle of page 51 the author states that it is now easy to determine all the p-groups with a cyclic invariant subgroup of index p or of index p^2. In the consideration of the groups of index p it is not assumed that this cyclic invariant subgroup involves an operator of highest order but in the following paragraph this assumption is implicitly made since otherwise the number of the groups coming under the first case would be 6 instead of 3 as here stated. These slight oversights are, however, easily detected by the careful reader and hence are not apt to cause him much annoyance.

Chapters 8 and 9 deal with automorphisms and monomial groups respectively. In the former of these the concepts of automorphism and group of automorphisms are clearly explained and illustrated. This is followed by the ordinary definition of characteristic subgroups to which attention was called recently in this MONTHLY, volume 29, page 320. The automorphisms of abelian groups are treated more fully than those of other groups, and at the close of the section on this subject reference is made to a practical method of establishing all such automorphisms, which seems to be due to the author of the present review: see *Bulletin of the American Mathematical Society*, volume 6 (1900), page 337. The statement that when the multiplying group is the same as the group itself each element of the group corresponds to its square is clearly not generally true as may be seen by considering such automorphisms of the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$.

Chapters 10 to 14 are devoted mainly to linear substitution groups and occupy more than one third of the space of the book, while chapter 15 is devoted to the theory of equations. The final chapter begins with a brief account of the Lagrange theory of equations. This is followed by the Galois theory and by applications of general group theory and substitution groups. In view of the

advanced character of the book, readers may be somewhat surprised to find so much emphasis on the literature in the German language. For instance, in the preface it is stated that the readers who found the presentation too brief might consult the books by Weber and Netto for more complete treatments.

On the whole, the volume under review furnishes a very attractive introduction into some of the most modern developments of the theory of groups of finite order, with emphasis on its applications, and we can only wish it success along with the other volumes of the interesting series to which it belongs now being published under the general editorship of R. Courant of the University of Göttingen.

G. A. MILLER.

A Treatise on the Theory of Bessel Functions. By G. N. WATSON. Cambridge University Press, 1922. ii + 804 pages. Price $22.50.

"This book has been designed with two objects in view. The first is the development of applications of the fundamental processes of the theory of functions of complex variables. For this purpose Bessel functions are admirably adapted; while they offer at the same time a rather wider scope for the application of parts of the theory of functions of a real variable than is provided by trigonometrical functions in the theory of Fourier series.

"The second object is the compilation of a collection of results which would be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The existence of such a collection seems to be demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics."

Bessel functions are perhaps from the point of view of applied mathematics the most important transcendants after the exponential group and their inverses.

It is this importance that justifies this great treatise on their theoretic properties and while the applied mathematician might have wished for illustrations of their use in physical problems, such additions would have vastly increased the size of the book and proved disappointing to those largely interested in the theoretic aspects of these functions. That Professor Watson has examined a large amount of material is shown by the index of titles of more than seven hundred papers and an author index of more than three hundred names of which among the sixteen American mathematicians the late Professor Bôcher's stands out preëminent. The book does ample justice to the large contributions of Sturm, Lommel, Carl Neumann, Sonine and Kapteyn who have created so much of this theory.

It is unfortunate that so many different Bessel functions of the second kind have been used by different mathematicians and still more unfortunate that the often elegant identities involving these functions cannot be expressed in terms of one type without loss of simplicity.

It suffices to say that this book is a rich mine of methods and results and is full of interesting problems of which we might mention the transcendence of the non-zero roots of $J_n(x) = 0$ when n is rational.

Professor Watson deserves the gratitude of his colleagues for the able and careful manner in which he has brought together this large amount of material

as well as for the well-balanced presentation of a subject which owes much to his own personal investigations.

It is unnecessary to say that the book work of the Cambridge Press is beyond all praise and is absolutely unrivalled elsewhere, fine paper, beautiful printing and wide margins only marred by a somewhat flimsy binding.

The book should certainly find a place, despite the high price, in every good mathematical library, but the cost will doubtless prevent its purchase by many persons desiring to own a copy of it.

M. B. PORTER.

Algebraic Numbers. Report of the Committee on Algebraic Numbers (National Research Council), by L. E. DICKSON, H. H. MITCHELL, H. S. VANDIVER, and G. E. WAHLIN. The National Academy of Sciences, Washington, D. C., February, 1923. Paper, 8vo. 96 pages. Price $1.50.

The first object of this paper is to bring up to date the extensive report on the theory of algebraic number fields by D. Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung,* vol. 4, 1894–5, pp. 175–546; French translation in *Annales de la Faculté des Sciences de Toulouse,* series 3, vol. I, 1909, 257–328; II, 1910, 226–456; III, 1911, 1–62 (notes by Humbert and Got, and errata). This supplementary report deals with the articles subsequent to Hilbert's report and such earlier papers as were not cited by him.

The second object of the paper is to deal with the literature, not in Hilbert's report, on fields of functions and related topics, such as Hensel's p-adic numbers and modular systems. Some of these topics have already been treated in the French *Encyclopédie.* In two special cases mentioned explicitly the present report leaves it to the reader to get the requisite information from the French article; otherwise, it is entirely independent of that article. Hence, with these two exceptions, the present report (taken with that of Hilbert) is intended to exhaust the literature of the subjects named.

Great care has been taken to make the list of references wholly complete; and a most painstaking search of the literature has been made with this end in view. In this respect this paper and the report of Hilbert give a full direction to the literature of that part of the theory of numbers which is not being covered in Dickson's monumental *History of the Theory of Numbers.* The importance of this report is greatly enhanced by its having this relation to that comprehensive work.

In the treatment of the forty-five topics which form the subject matter of as many articles into which the four chapters are divided there is a great variety of procedure ranging from a mere listing of references to articles at one extreme to an outline of the theory at the other extreme. In the chapter on Hensel's p-adic numbers there is a careful outline of the leading ideas with a survey of the literature and a statement of the more important theorems. In such topics as cubic fields, Galois fields, Abelian fields, and units of a general field there is but little more than a list of references.

From the supplementary character of the report it is inevitable that it should be rather fragmentary in character in many places; for it contains merely the necessary additions to a previous well-made report of great importance. But this does not decrease the value of the paper to a man of research who wishes to get access to the whole literature of the subject treated. The devotees of the theory of numbers owe a distinct debt of gratitude to the authors of this report for their difficult and careful labor in bringing together and systematizing the literature of their subject not already cited in the report of Hilbert.

R. D. CARMICHAEL.

Frequency Arrays. By H. E. SOPER. Cambridge University Press, 1922. 48 pages.

"Since in other branches of science symbols bearing an objective or logical significance have been usefully employed, conjoined with symbols of number, to express quantity, it may be expected that in the science of statistics, which of its essence is the enumeration of logical classes, such symbols will find serviceable application.

The aim, then, has been to recommend the use of logical symbols in the enumeration of logical classes. It has been no material part of the purpose to establish new formulæ and results in the mathematical theory of statistics and if new conclusions have been reached these will serve chiefly to help point the precept, since no difficult analysis has been undertaken or is anywhere involved.

It is possible that, with wider familiarity in their use, the applications of the symbols of denomination may bear extension and that they may be found of assistance in the development of the higher theory both of statistical and other distributions."

United States Life Tables—1890, 1901, 1910, and 1901–1910. By JAMES W. GLOVER, expert special agent of the Bureau of the Census. Washington, Government Printing Office, 1921. 4vo. 496 pages. Price $1.25.

Mr. Sam L. Rogers, director of the Census, writes in the "Letter of Transmittal": "The Bureau has had the advice and coöperation of a special census committee representing the Actuarial Society of America, composed of John K. Gore, chairman, Robert Henderson, Arthur Hunter, William A. Hutcheson, and Henry Moir. The tables have been prepared along lines approved by this committee."

It is difficult to describe in a few words a book that contains so much valuable information, a book valuable not only to actuaries and statisticians but to the general public—which must have some interest in vital statistics, in life insurance, in old-age pensions, in the settlement of estates involving life interests, and in the settlement of claims for injury or death. Indeed, Part I, consisting of 25 large pages, is a "non-technical description and explanation of life table functions, graphs, and other parts of text and tables": it gives in a particularly lucid manner, by the question-answer method, with continual reference to specific tables, just the information sought by the general public. For the United States Tables (Part II), the classifications, in addition to dates, are: Male and female, white and negro, native and foreign-born white, urban and rural. There are tables for the "selected registration states," Indiana, Massachusetts, Michigan, New Jersey, and New York; and for the cities of Boston, Chicago, New York, and

NOTES ON RECENT PUBLICATIONS.

In Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, mathematisch-physische Klasse, 1922, volume 44, no. 3, pages 157–160, there is a "Nachruf" by G. Herglotz, "Zum Gedächtnis Johannes Thomae 1840–1921" (*1921*, 336).

A French translation of WOODS and BAILEY's *Analytic Geometry and Calculus* (Boston, Ginn, 1917) is now in the press and permission has been asked to translate *A Course in Mathematics for Students of Engineering and Applied Science* (2 volumes, Ginn, 1907–1909) by the same authors.

Under the auspices of the Society of Sciences in Göttingen, volumes I–X_1 of Gauss' Werke were published 1863–1917. The second part of volume X (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik), the first part of volume XI (Nachlese und Briefwechsel zur Physik, Astronomie und Chronologie) and the second part of volume XI (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik) are in the press. Volumes X_2 and XI_2 are to be composed of single monographs separately paged. We have already referred (*1921*, 79) to the first edition of eight of these as they appeared in the Göttinger Nachrichten, 1911–1920. Two parts of volume X_2 have been published. The first (1922) contained Bachmann's "Ueber Gauss' zahlentheoretische Arbeiten" (1911) and Oskar Bolza's "Gauss und die Variationsrechnung" (95 pages not previously printed). The second part (1923) contains (123 pages) an almost unchanged reprint of Stäckel's essay on "Gauss als Geometer" (1917) which the late author had intended should be much extended. The next essay published is to be Schlesinger's "Ueber Gauss' Arbeiten zur Funktionentheorie" (1912) completely rewritten. Volume XI_2 is to contain the essays in Astronomy by Brendel (1919), on Physics by Schäfer, on Geodesy by Galle and on "Gauss als Zahlenrechner" by Männchen (1918).

The first number of *The Quarterly Journal of Pure and Applied Mathematics*, volume 49, appeared in October, 1920; the fourth number issued in March, 1923, contained a paper "Singly infinite class number relations" (pages 322–327) by E. T. BELL, of the University of Washington. We have already referred (*1921*, 137; *1922*, 224) to Professor Bell's other papers in the volume. Number 3 (October, 1922) contains "Abstract definitions of the symmetric and alternating groups and certain other permutation groups" (pages 226–283) by R. D. CARMICHAEL, of the University of Illinois.

We have already referred (*1921*, 317–318; *1922*, 222) to *Fundamenta Mathematicæ*, the remarkable annual periodical dealing wholly with the theory of aggregates, and published in Warsaw, Poland. Volume 4 (4 + 372 pages) appeared in April, 1923, and contains two papers by Americans: "On the generation of a simple surface by means of a set of equicontinuous curves" by R. L. MOORE, 106–117; "Note on a paper of M. Banach" by N. WEINER, 136–143. The extremely low subscription price (20 French francs, less than $1.40) is in happy contrast to the extortionate sums demanded by the Germans

for their periodicals. American libraries should encourage this notable publication by their subscriptions.

On the two hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international coöperation for the publication of his collected works. Academies and individuals subscribed for about 300 sets, and about one hundred thousand (100,000) Swiss francs were collected, for the most part in Switzerland; the American Mathematical Society subscribed five thousand (5,000) francs. Eighteen (18) of the estimated seventy (70) volumes have been published.

By reason of the European War, nearly one half of the subscribers have been unable to meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. There are three plans for subscribing. Those libraries or individuals wishing for information with a view to joining in promoting this great international undertaking should communicate with the Official Representative of the Euler Committee for the United States and Canada, Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

ARTICLES IN CURRENT PERIODICALS.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, January, 1923: "Les modules des zéros des polynomes" by M. P. Montel, 1–32.

ANNALI DI MATEMATICA, series 3, volume 31, March, 1922: "Nuove ricerche sulle corrispondenze algebraiche fra i punti di una curva algebraica" by C. Rosati, 1–49; "Determinazione delle ipersuperficie che ammettono rappresentazioni geodetiche" by E. Bompiani, 51–80; "Sulle equazioni integrali non lineari" by A. Vergerio, 81–119; "Sopra due teoremi di Dirichlet" by A. M. Bedarida, 121–125; "Intorno alle involuzioni situate sopra le superficie iperellittiche con due fasci di curve ellitiche" by N. Spampinato, 127–148; "I gobbo-circolanti e i divisori di zero di un particolare sistema di numeri complessi ad n unità" by V. Amato, 149–164. November, 1922: "Riavvicinamento di geometrie differenziali delle superficie: metriche, affine, proiettiva" by G. Sannia, 165–189; "L'intorno d'un punto d'una superficie considerato dal punto di vista projettivo" by E. Cech, 191–206; "Sulle serie di polinomi di una variabile complessa. Le serie di Darboux" by N. Abramescu, 207–249; "I fondamenti della geometria proiettivo- differenziale secondo il metodo di Fubini" by E. Cech, 252–278; "Sull' indipendenza di un integrale daparametri nel caso più generale" by G. Usai, 279–294.

BULLETIN DES SCIENCES MATHÉMATIQUES, series 2, volume 47, February, 1923: "Réduction des systèmes algébriques de points appartenant à une même courbe algébrique. Théorème d'Abel" by B. Gambier, 76–96.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, January, 1923: "The October meeting of the Society" by R. G. D. Richardson, 1–10; "The October meeting of the San Francisco Section" by B. A. Bernstein, 10–13; "The Frank Nelson Cole prize in algebra" by R. G. D. Richardson, 14; "Periodic solutions in the problem of three bodies" by F. H. Murray, 15–16; "Note on quartiles and allied measures" by D. Jackson, 17–20; "Ruled surfaces with director planes" by J. K. Whittemore, 21–25; "On transformable systems and covariants of algebraic forms" by C. C. Macduffee, 26–33; Reviews: by A. J. Kempner of O. Perron, *Irrationalzahlen* (Berlin and Leipzig, 1921), 34–36; and of P. Bachmann, *Grundlehren der neueren Zahlentheorie* (Berlin and Leipzig, 1921), 36–37; by A. B. Coble of H. Malet, *Étude Géométrique des Transformations Birationelles et des Courbes Planes* (Paris, 1921), 38; by J. W. Young of E. H. Neville, *The Fourth Dimension* (Cambridge, 1921), 38; and of G. Kowalewski, *Mathematica Delectans* (Leipzig, 1921), 40; by P. J. Daniell of L. Page, *An Introduction to Electrodynamics* (Boston, 1922), 39, and of R. Gramel, *Der Kreisel* (Braunschweig, 1920), 40; and by H. L. Rietz of E. Czuber, *Wahrscheinlichkeitsrechnung* (Vol. II, 3d ed., Leipzig and Berlin, 1921) and *Die statistischen Forschungsmethoden* (Wien, 1921), 39; Notes, 41–42; New publications, 43–48—

February: "An uncountable, closed and non-dense point set each of whose complementary intervals abuts on another one at each of its ends" by R. L. Moore, 49–50; "Total geodesic curvature and geodesic torsion" by J. K. Whittemore, 51–54; "The name 'divergent' series" by F. Cajori, 55; "A qualitative definition of the trigonometric and hyperbolic functions" by P. Franklin, 56–64; "Groups in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup" by G. A. Miller, 64–70; "On curves kinematically related to a given curve" by H. Poritzky, 71–78; Reviews: by D. E. Smith of T. L. Heath, *A History of Greek Mathematics* (2 vols., Oxford, 1921), 79–84; by L. W. Dowling of F. Amodeo, *Lezioni di Geometria Proiettiva* (3d ed., 2d reprint, Naples, 1920), 85–86; by H. L. Rietz of J. W. Glover, *United States Life Tables 1890, 1901, 1910 and 1901–10* (Washington, 1921), 86; and of F. Insolera, *Lezioni di Statistica Metodologica* (Turin, 1921), 90; by K. W. Lamson of A. S. Eddington, *Espace, Temps et Gravitation*, French translation by J. Rossignol (Paris, 1921), 87; by W. C. Graustein of W. de Tannenberg, *Conférences sur les Transformations en Géométrie Plane* (Paris, 1921), 87; by J. B. Shaw of G. Juvet, *Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu* (Paris, 1922), 88–89; by E. W. Brown of *Annuaire du Bureau des Longitudes pour l'An 1922* (Paris, 1922), 89; by A. Emch of M. Groll, *Kartenkunde* (Berlin and Leipzig, 1922), 89–90; and by E. B. Lytle of B. Branford, *A Study of Mathematical Education* (2d ed., Oxford, 1921), 90; Notes, 91–92; New publications, 93–96.

Journal de Mathématiques Pures et Appliquées, series 9, volume 2, no. 1, 1923: "Differential properties of functions of a complex variable which are invariant under linear transformations" by E. J. Wilczynski, 1–51. The first part appeared in this journal, series 9, volume 1, part 4, 1922. "Sur un ensemble non mensurable B" by N. Lusin and W. Sierpinski, 53–72; "Recherche des systèmes cycliques de triples de Steiner différents pour N premier (ou puissance de nombre premier) de la forme $6n + 1$" by S. Bays, 73–98.

Mathematics Teacher, volume 15, October, 1922: "The strength of the mental connections formed in algebra" by E. L. Thorndike, 317–331; "Fundamental principles of algebra" by R. L. Modesitt, 332–346; "Rabbi Ben Ezra on permutations and combinations" by J. Ginsburg, 347–356; "Introducing Mechalus to geometry" by Mary A. Potter, 357–360; "Note on Mr. Evans's paper in the March Teacher" by H. F. Hart, 360–361; "Original solution in plane geometry" by R. A. Laird, 361–364; "Professor Hedrick's report on the function concept in elementary mathematics" by H. E. Webb, 364–368; "News and notes," 369–374; "Research department," 375–376; "New books," 377–379—November: "The case for general mathematics" by W. D. Reeve, 381–391; "Errors in computations and the rounded number" by H. Rice, 392–404; "The constitution of algebraic abilities" by E. L. Thorndike, 405–415; "Romance in science" by Bessie I. Miller, 416–422; "Some mathematics of the calculating machine" by L. L. Locke, 423–428; "Mathematical clubs in the high school" by Sophia Refior, 434–435; "Recent articles of interest to mathematics teachers" by N. R. Howell, 435–439; "News and notes," 440–443—December: "Non-Euclidean geometry" by W. H. Bussey, 445–459; "The study of mathematics under the individual system" by Mary M. Reese, 460–466; "Problems concerning the teaching of secondary mathematics" by A. Davis, 467–477; "The future development of mathematical education" by C. N. Moore, 478–483; [Quotation: "In addition to bringing home to the student the wide use of mathematical knowledge in the activities of the modern world, we must also give him some notion of its origin and growth and its important rôle in the development of our civilization. We must not let him rest under the impression that mathematics was invented in order to provide intricate and vexatious puzzles for the adolescent mind. We must demonstrate to him that man was led to the pursuit of mathematical knowledge by his eager desire to understand the universe and to control the forces of nature, that he found the knowledge essential for the higher developments of trade and commerce and all the other varied developments that have had a place in the creation of our present-day civilization, in short that the progress of the world is now and always has been bound up with the development of our knowledge of mathematics."] "The function concept in high school mathematics" by J. M. Kinney, 484–495; "An historic theorem in plane geometry" by W. H. Carnahan, 496; "Geometry speaks" by Eva M. Palmer, 496–500; Review by A. S. Otis of The Thurstone Vocational Guidance Tests: Arithmetic, Algebra, Geometry (World Book Company, Yonkers, N. Y.), 506–507.

Messenger of Mathematics, volume 52, no. 4, August, 1922: "Notes on some points in the integral calculus" by G. H. Hardy, 49–53; "The dissection of rectilinear figures" by W. H. Macaulay, 53–56; "Relations between the numbers of Bernoulli, Euler, Genocchi, and Lucas" by E. T. Bell, 56–64.

Nouvelles Annales de Mathématiques, volume 80, October, 1922: "Sur l'étude

algébrique des problèmes de division des arcs" by E. Vessiot, 1–4; "Sur la trisection d'un angle" by B. Niewenglowski, 4–8; "Sur une notion d'equivalence locale apte à préciser certains points de la théorie des enveloppes" by G. Bouligand, 8–21; "Sur les ombilics" by P. Montel, 21–23—November, 1922: "Sur une manière simple d'obtenir géométriquement les formules de Lorentz" by M. Morand, 41–49; "Introduction à l'étude de la mécanique et de ses principes" by G. Bouligand, 50–58; "Sur la conservation de la courbure geodésique dans la déformation d'une surface" by R. Brichard, 58–61; "Sur les coniques focales" by C. Bioche, 62–63; "Remarques sur les trièdres" by C. Bioche, 63–65—December, 1922: "Monographie des polynomes de Kummer" by P. Humbert, 81–92; "Introduction à l'étude de la Mécanique et de ses principes" (continued) by G. Bouligand, 93–109; "Notes sur les podaires" by M. F. Egan, 109–112—January, 1923: "Système harmonique de trois coniques" by J. Lemaire, 121–135; "Introduction à l'étude de la mécanique et de ses principes" (continued) by G. Bouligand, 135–147; "Démonstration de la formule de l'accélération dans le mouvement relatif" by B. Niewenglowski, 147–150; "Remarques sur les Jacobiens" by C. Bioche, 150–153; "Developpables formées avec les normales d'une quadrique" by L. de la Roëre, 153–159—February, 1923: "Un théorème sur les équations algébriques entières" by G. Casabonne, 161–165; "Remarques au sujet d'un des problèmes de mécanique donnés au concours d'agrégation en 1921" by R. Thiry, 165–172; "Sur deux familles de courbes orthogonales" by G. Fontené, 173–180; "Introduction à l'étude de la mécanique et de ses principes" (concluded) by G. Bouligand, 181–188.

SCIENCE, volume 56, December 22, 1922: "The Frank Nelson Cole prize in algebra" by R. G. D. Richardson, 710–711—December 29: Review by R. Pearl of *United States Life Tables 1890, 1901, 1910 and 1901-1910* prepared by J. W. Glover (Washington, Bureau of the Census, 1921), 756–757—Volume 57, January 26, 1923: Reports of the Boston meetings of Section A (Mathematics) of the American Association for the Advancement of Science (by W. H. Roever), of the American Mathematical Society (by R. G. D. Richardson) and of the Mathematical Association of America (by W. D. Cairns), 107–108—February 2: "Geometry and physics" by O. Veblen [address as retiring vice-president and chairman of Section A (Mathematics) of the American Association for the Advancement of Science, Boston, December, 1922], 129–139—March 2: "The American Mathematical Society" [report of the Chicago Section meeting at Evanston, December 29, 1922] by A. Dresden, 276—March 23: "The American Mathematical Society" [report of the meeting at New York, February 24, 1923] by R. G. D. Richardson, 364.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 23, April, 1922 [published April, 1923]: "Some generalizations of geodesics" by E. J. Wilczynski, 223–239; "On the gyroscope" by W. F. Osgood, 240–264; "The relative distribution of the real roots of a system of polynomials" by C. F. Gummer, 265–282; "A general theory of conjugate nets" by E. P. Lane, 283–297; "Parallel maps of surfaces" by W. C. Graustein, 298–332.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 53, nos. 11–12, published December 5, 1922: "Die zeichnerische Ermittlung der Konstanten eines Luftlichtbildes, sowie Herstellung des Grund- und Aufrisses von Geländepunkten bei unebenem Gelände" by M. Brettar, 249–257; "Über den Anfangsunterricht in der Differentialrechnung" by K. Bögel, 257–258; "Zur Gruppierung der Vierecke" by A. Gottschalk, 259–260; "Einige Anwendungen der Spiegelung" by W. Weber, 264–265; "Die Mathematikertagung in Leipzig" [report of the annual meeting of the Deutsche Mathematiker-Vereinigung, September 17–22, 1922] by M. Zacharias, 270–272; "Bücherbesprechungen," 274–279; "Zeitschriftenschau," 279–281—Volume 54, no. 1, published January 30, 1923: "Ergänzungen zur Elementarmathematik" by W. F. Meyer, 1–10; "Die Lehre von den Wurzeln im elementaren Unterricht" by K. Kommerell, 10–21; "Graphische Darstellung von Sätzen der elementaren Algebra" by E. Dintzel, 21–26; "Elementares Verfahren zur Bestimmung der Elektricitätsverteilung auf dem Ellipsoid und zur Ermittlung der Kapazität einer Scheibe und eines Stabes" by H. Dörrie, 26–29; "Physik und Mathematik auf der Oberstufe von Vollanstalten" by K. Hahn, 30–34; "Zur Einführung der komplexen Zahlen im Mittelschulunterricht" by S. Weich, 34–36; "Die artilleristische Aufgabe im Unterricht der analytischen Geometrie" by H. Semiller, 37; "Zur Empfindlichkeit der Wage" by Janss, 38–39; "Aufgabenrepertorium," 39–45; "Bücherbesprechungen," 53–61; "Zeitschriftenschau," 61–62.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE CORNELL PARABOLA, CORNELL UNIVERSITY, ITHACA, N. Y.
[*1920*, 479.]

For membership in the Cornell Parabola, the mathematics faculty and students who have taken elementary calculus are eligible. As officers for the year 1922–1923, the following were elected: President, John Wood '24; secretary-treasurer, Violet Holloway '23; member of the executive committee, Joe Nobile '25; faculty adviser, Professor H. M. Morse. The club met at 7:30 p.m., as follows:

November 17, 1922: "Theory of numbers" by Professor W. L. G. Williams.
December 7: "The Parabola" by Ida Itzkowitz, Gr.
December 16: Christmas party at the home of Professor and Mrs. F. B. Owens.
February 15, 1923: "The mathematics of navigation" by John Wood '24.
March 30: "Scales of notation" by H. I. Lane, Gr.
April 25: "Einstein's theory of relativity" by Professor Morse. An open meeting with an attendance of nearly 200.
May 24: "Modern mathematics in Italy" by Professor Virgil Snyder.

(Reported by Miss Holloway.)

THE EUCLIDEAN CIRCLE OF THE UNIVERSITY OF INDIANA, BLOOMINGTON, IND.
[*1918*, 228.]

As officers for 1922–1923, the following were elected: President, Baker Hindman '23; vice-president-treasurer, Joseph Sheedy '23; secretary, Trula Sidwell '24. Meetings were held as follows:

December 11, 1922: "Mathematical recreation" by Professor Cora B. Hennel.
December 15: "Life of Isaac Newton" by Edna Ellis '23; "The famous problems of antiquity" by George Murphy '23.
February 6, 1923: "A paper on π" by Josephine Rich '23.
February 19: "Unified mathematics" by Joseph Sheedy '23.
March 12, 1923: "Probability" by Marie Sangennebo '24 and Jessie McAtee '23.
March 26: "Mathematics of the calendar" by Avie Burkett '23; "Mathematics of astronomy" by Henry Yarbrough, instructor.
April 17: "Women and mathematics" by Agnes McLeaster '24.
April 30: "Concurrent lines and collinear points" by Lionel Martin '24; "Proof that the angle cannot be trisected" by Mary Winget '23.
May 14: "Systems of notation" by Earl Klinger '24; "Mathematical prodigies" by Dorothy Munns '24.

(Reported by Miss Sidwell.)

THE MATHEMATICS CLUB OF IOWA STATE TEACHERS COLLEGE, CEDAR FALLS, IA.
[*1920*, 29.]

With an average attendance of about thirty-five members, meetings were held in 1922–1923, as follows:

October 18, 1922: "New solutions for equations of higher order" by Professor C. W. Wester.
November 8: "The Burroughs calculating machine and its inventor" by Professor C. A. Speer,—with a demonstration of the machine.
November 22: "Construction of a parabolic arch" by B. A. Jenson '24; "Velocity of the earth in its orbit" by Harrietta Shrimp '25; "The golden section and the use of conic sections" by Merrill Muzzey '23; "The area and the construction of an ellipse" by Mildred Chatterton '23; "Reflectors and whispering galleries" by Harold White '24.
January 17, 1923: "A four step solution of the general form of the quadratic equation" by Myrtle Frederickson '25; "Introduction of logarithms" by Bessie Hutchinson '25, Edward Bark '25, and Vale Dunn '26; "The W. A. G. plane" by Professor R. G. Dougherty.
February 7: "Radio-activity" by Professor Kadish.

February 28: "Mathematical drills and recreations" by Professor E. E. Watson.
April 4: "Correlated mathematics in Iowa" by Professor I. S. Condit.
May 9: "The content of a course in freshman mathematics" by Professor Watson.

(Reported by Professor Watson.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF MAINE, ORONO, ME.
[*1918*, 453.]

The following officers were elected for the year 1922–1923: President, Helen Shorey '23; vice-president, Edwin Hadlock '24; secretary and treasurer, Vera Savage '24; faculty adviser, Mr. Frank S. Beale, instructor. The following papers were read:

October 4, 1922: "History of the Club" by Dean J. N. Hart; "Mathematics clubs" by Professor N. R. Bryan.
October 25: "Rithmomachia" by Chester Austin '23.
November 8: "Reorganization of mathematics in secondary schools" by Professor Bryan.
November 22: "Another world" by Edwin Hadlock '24.
December 13: "The use of the sine function expansion in determining the area of an egg-shaped figure" by Mr. Warren Loving, instructor.
January 10, 1923: "A class of seven elements as applied to algebra" by Helen Shorey '23, retiring president. Chester Austin '23 was elected president for the second semester.
February 14: "Recreations in mathematics." General discussion.
February 28: "The vector method used in proving some propositions in geometry" by Mr. Warren Lucas, instructor.
March 14: Repetition of the talk given by Mr. Lucas at the last meeting.
April 12: "The fourth dimension and the Bible" by Ethelyn Percival '24.
April 26: "Interpolation" by Professor H. R. Willard. The following were elected as officers for the coming year: President, Vera Savage '24; vice-president, Ethelyn Percival '24; secretary and treasurer, Donald Trouant '25; faculty adviser, Professor Bryan.
May 15: A short play on the fourth dimension was presented by Misses Harkness, Percival, Savage, and Shorey. Professor Bryan reviewed the work of the club for the current year, and gave suggestions for the coming year.

(Reported by Miss Savage.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, EUGENE, ORE.
[*1918*, 134.]

As officers for the year 1922–1923, the following were elected: President, R. M. Elliot, Gr.; vice-president, Virl Bennehoff '23; secretary, Wave Lesley '23; treasurer, Ted McAlister '23; members of the executive committee, Gertrude Tolle '23, Don Wilkinson '23, and Willa Loomis '24. Meetings were held as follows:

March 15, 1922: "The theory of numbers" by Professor W. E. Milne.
April 19: "Flat Land" by Laura Hammer, Gr.; "The fourth dimension" by R. M. Elliot, Gr. Election of officers.
May 18: "Nomograms" by Professor McAlister, of the Department of Mechanics and Astronomy. Problem solving.
June: Picnic.
October 31: "Japanese mathematics" by Professor E. E. DeCou.
November 15: Mystery program of puzzles, stunts, and charades by Gertrude Tolle '23 and Wave Lesley '23.
December 8: "Einstein's theory of relativity" by Professor L. L. Smail, of the University of Washington.
January 31, 1923: Models for physics and mathematics displayed and explained by Professor W. P. Boynton, of the Department of Physics.
February 22: Social evening at the home of Professor and Mrs. DeCou.
April 18: "Gamma functions" by Ted McAlister '23.
May 8: "Concrete bridges" by Professor McAlister. Election of officers for 1923–1924: President, Wave Lesley '23; vice-president, Sylvia Veatch '25; secretary, Mary Search '25; treasurer, Vera Hughes '25.
May 29: Picnic and installation of officers.

(Reported by Miss Lesley.)

THE TULANE MATHEMATICAL SOCIETY OF TULANE UNIVERSITY, NEW ORLEANS, LA.

The membership of this society includes many of the mathematics teachers of New Orleans. As officers for 1922–1923, the following were elected: President, Professor H. E. Buchanan; secretary-treasurer, Professor Anna M. Howe, of Newcomb College. Papers were presented as follows:

Fall meeting: "Applications of the binomial theorem" by Professor Howe.

Winter meeting: "Trade mathematics" by Mr. J. P. McGuire, teacher in mathematics at Delgado Trade School, New Orleans.

Spring meeting: "The motion of a heavy particle on the surface of a torus" by Mr. F. G. Eberle, teacher in mathematics at the Warren Easton Boys High School, New Orleans.

The Newcomb Science Club of Newcomb College, New Orleans, La., was organized in the fall of 1922 to stimulate interest in science and to correlate the work of the departments of science. As officers were elected: President, Odessa Lastrapes '23; vice-president, Beatrice Cosgrove '23; secretary-treasurer, Vivia de Milt '23. Papers on general science are read at the monthly meetings. At the December meeting, Carlotta Kraft '24 read a paper: "A report of the Einstein expedition to Christmas Island."

(Reported by Miss Lastrapes.)

THE WHITE MATHEMATICS CLUB, UNIVERSITY OF KENTUCKY, LEXINGTON, KY.
[*1922*, 417.]

October 4, 1922: Election of officers: President, Professor H. H. Downing; secretary, Professor Elizabeth Le Stourgeon.

October 19: "Some ancient methods of multiplication" by Dean P. P. Boyd; "Equivalence of equations" by Professor J. M. Davis.

November 2: "Fourier series" by Professor Le Stourgeon.

November 16: "Descartes' rule of signs" by Minnie Peterson '24; "Sturm's theorem" by H. Mobley '24.

November 29: "Vector treatment of the motion of a rigid body about a point" by Professor E. L. Rees.

January 10, 1923: "Force at a point due to a circular charge of electricity, the point being in the plane of the charge" by Professor M. N. States, of the Department of Physics.

February 14: "Solution of cubic equations" by Mary Gordon '25; "Solution of quartic equations" by Marion Brown '24.

February 28: "Foci" by Dean Boyd.

March 15: "Interference methods applied to astronomy" by Norman Beese '23.

April 5: "The analytical polygon" by Mr. R. V. Blair, instructor.

April 19: "The cissoid of Diocles" by Tomie Bronsbon '23; "The conchoid of Sluse" by W. Hutcherson, Gr.

May 3: "Sylvester's method of elimination" by Mr. George Seubert, instructor.

(Reported by Professor Le Stourgeon.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3035. Proposed by R. M. MATHEWS, Wesleyan University.

Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the first class.

3036. Proposed by F. M. GARNETT, Savannah, Georgia.

The inside dimensions of a chest are $l \times w \times d$; find the greatest length of a rectangular piece of timber with the cross-section $a \times b$ which fits in the closed chest. Numerical application: $l = 6$ feet. $w = 3$ feet, $d = 2$ feet, $a = b = 2$ inches.

3037. Proposed by E. O. BROWER, Chicago, Illinois.

The angles A, B, C of a triangle are given. A logarithmic spiral is tangent to AB at B and to AC at C; at what angle does the curve cut each radius vector?

3038. Proposed by B. F. FINKEL, Drury College.

A fly starts from a point on the periphery of a right elliptic cone, altitude h and equation of the periphery of the base $b^2x^2 + a^2y^2 = a^2b^2$, and walks obliquely along the surface of the cone, crossing the elements of the cone at a constant angle, α. How far has the fly travelled when coming, for the first time, to the element of the cone passing through the point of departure?

3039. Proposed by J. K. WHITTEMORE, Yale University.

Given a conic S, a point A, and a line l. Through A is drawn a variable line cutting S in P and Q. Find the envelope of a conic which is tangent to S at P and Q and which is tangent to l.

SOLUTIONS.

2877 [1921, 89]. Proposed by J. B. REYNOLDS, Lehigh University.

A particle slides down the rough arc of a cardioid, $r = a(1 - \cos\theta)$, which lies in a vertical plane, the initial line being horizontal. If the coefficient of friction, μ, equals 1/3, find θ for the point where the particle leaves the curve, if it starts at $\theta = 90°$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio, AND OTTO DUNKEL, Washington University.

Let v be the velocity of the particle at any point of the curve, φ the angle which the tangent at this point makes with the initial line, ρ the radius of curvature, R the reaction of the curve, and mg the weight of the particle. Resolving the forces along the tangent and the normal, we obtain for the equations of motion

$$mg \sin\varphi - \mu R = m\frac{dv}{dt}, \qquad mg\cos\varphi + R = -\frac{mv^2}{\rho}; \tag{1}$$

and, eliminating R from these two equations, we have

$$g(\sin\varphi + \mu\cos\varphi) = \frac{dv}{dt} - \mu\frac{v^2}{\rho}. \tag{2}$$

If s is the length of the curve measured from the cusp ($\varphi = 0$), then

$$-v = \frac{ds}{dt} = \frac{ds}{d\varphi}\frac{d\varphi}{dt} = \rho\frac{d\varphi}{dt}, \qquad \frac{dv}{dt} = \frac{dv}{d\varphi}\frac{d\varphi}{dt} = -\frac{v}{\rho}\frac{dv}{d\varphi}. \tag{3}$$

By use of (3) the equation (2) becomes

$$\frac{d(v^2)}{d\varphi} + 2\mu v^2 = -2g\rho(\sin\varphi + \mu\cos\varphi). \tag{4}$$

Now for the cardioid $\varphi = 3\theta/2$ and $\rho = (4a/3)\sin(\varphi/3)$, and after this value of ρ is inserted equation (4) may be written

$$d(v^2e^{2\mu\varphi}) = -\frac{8ag}{3}e^{2\mu\varphi}\sin\frac{\varphi}{3}(\sin\varphi + \mu\cos\varphi)d\varphi. \tag{5}$$

The integration of this equation gives

$$v^2 = ke^{-2\mu\varphi} + 2ag\left[\frac{5\mu\cos\frac{4\varphi}{3} + (2 - 3\mu^2)\sin\frac{4\varphi}{3}}{4 + 9\mu^2} + \frac{(3\mu^2 - 1)\sin\frac{2\varphi}{3} - 4\mu\cos\frac{2\varphi}{3}}{1 + 9\mu^2}\right]. \tag{6}$$

Inserting the value of φ above and $\mu = 1/3$, this equation becomes

$$v^2 = \frac{2ag}{3}[\cos 2\theta + \sin 2\theta - \sin\theta - 2\cos\theta + 2\,e^{(\pi/2)-\theta}], \tag{7}$$

where the constant k has been determined so that $v = 0$ when $\theta = \pi/2$. The particle leaves the curve at the moment when $R = 0$, and this value of R and v^2 above inserted in the second equation of (1) gives

$$\cos 2\theta + 2\sin 2\theta - 2\sin\theta - 2\cos\theta + 2e^{(\pi/2)-\theta} = 0. \tag{8}$$

If we set $f(\theta) = e^{\theta-(\pi/2)}(\cos 2\theta + 2\sin 2\theta - 2\sin\theta - 2\cos\theta) + 2$, then $f'(\theta) = e^{\theta-(\pi/2)}(5\cos 2\theta - 4\cos\theta)$. When $\theta = 0$, both $f(\theta)$ and $f'(\theta)$ are positive, and when $\theta = \pi/2$, both are negative. Now $f'(\theta)$ vanishes only once in this interval, *i.e.*, when $\cos\theta = (2 + \sqrt{54})/10$. Hence $f(\theta)$ vanishes once and only once in this same interval. This root is found to be $\theta = 1.36019$ or, in degrees, $\theta = 77° 56'$.

Also solved by F. L. WILMER.

2904 [1921, 277]. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Pairs of tangents to a conic intersect on a fixed straight line; find the locus of the middle points of the chords of contact.

SOLUTION BY OTTO DUNKEL, Washington University.

Let P be the pole of the fixed line and suppose that the conic has a center C and a diameter $A'CA$ passing through P. Let M be the middle point of a chord through P. Draw the chord AN parallel to PM. Then CM produced bisects AN in M'. It follows then from similar triangles that

$$PM = \frac{PC}{AC}\cdot AM' = \frac{PC}{AC}\cdot\frac{AN}{2} = \frac{PC}{AA'}\cdot AN.$$

This shows that, as N describes the given conic, M describes a similar conic with the ratio of similitude PC/AA'.

In the case of a parabola $PC/AA' = \frac{1}{2}$ and the locus of M is also a parabola.

Also solved by T. M. BLAKSLEE, A. M. HARDING, WILLIAM HOOVER and A. PELLETIER.

2914 [1921, 326]. Proposed by HARRIS HANCOCK, University of Cincinnati.

If $a_1, a_2, \cdots, a_n$ are n positive integers, a_{ij} the greatest common divisor of a_i and a_j, d_m the greatest common divisor of all products of every m of these numbers ($m = 1, 2, \cdots, n-1$), then is

$$\prod_{ij} a_{ij} = d_1 d_2 \cdots d_{n-1}; \qquad (j > i;\ i = 1, 2, \cdots, n;\ j = 2, 3, \cdots, n).$$

In general, show that this theorem is true if $A_1, A_2, \cdots, A_n$ are any functions integral in any number of variables, with rational integral coefficients, or with algebraic coefficients.

Remark: This is a generalized statement in positive rational integers of the following theorem of much importance in the theory of algebraic numbers and due to Dedekind (Dirichlet, *Zahlentheorie*, Supplement XI): Let A, B, C be three moduls (Dedekind). Denote the greatest common divisor of A and B by $A + B$ and their product by $A \cdot B$. Dedekind proves that

$$(A + B)(B + C)(C + A) = (A + B + C)(AB + BC + CA).$$

Denote the greatest common divisor of two moduls A_i and A_j, that is, $A_i + A_j$, by A_{ij}; and write

$$\begin{aligned} D_1 &= A_1 + A_2 + \cdots + A_n, \qquad D_2 = A_1A_2 + A_1A_3 + \cdots + A_{n-1}A_n, \\ D_3 &= A_1A_2A_3 + A_1A_2A_4 + \cdots + A_{n-2}A_{n-1}A_n, \cdots, \\ D_{n-1} &= A_2A_3\cdots A_n + A_1A_3\cdots A_n \cdots + A_1A_2\cdots A_{n-1}. \end{aligned}$$

Show that

$$A_{12}A_{13}\cdots A_{n-1}A_n = D_1D_2\cdots D_{n-1}.$$

SOLUTION BY THE PROPOSER.

The following proof is found in one of my note books entitled "Miscellaneous." The proof is probably due to or suggested by *Frobenius*.

Let p be any prime integer that enters a_λ to the k_λ power and arrange the integers $a_1, a_2, \cdots, a_n$ so that

$$k_1 \leqq k_2 \leqq k_3 \leqq \cdots \leqq k_n.$$

Denote the greatest common divisor of a_i and a_j by a_{ij}. It is clear that p appears to the k_i power in a_{ij} and consequently to the power $(n-1)k_1 + (n-2)k_2 + \cdots + 1\cdot k_{n-1}$ in the product $\prod_{ij} a_{ij}$, this product being defined in the statement of the problem. On the other hand p appears in d_1 to the k_1 power; in d_2 to the $(k_1 + k_2)$ power; $\cdots$ in d_{n-1} to the $(k_1 + k_2 + \cdots + k_{n-1})$ power.

With this the correctness of the formula as stated in the problem is established.

2916 [1921, 327]. Proposed by HARRIS HANCOCK, University of Cincinnati.

If p is any rational prime integer, and if $\alpha\ (\neq 1)$ is any root of $x^p - 1 = 0$, show that $p = P_1 \cdot P_2 \cdots P_{p-1}$, where $P_i\ (i = 1, 2, 3, \cdots p-1)$ are the ideals $(p, 1 - \alpha^i)$, which in turn may be reduced to principal ideals. [Remark: This is rather a good elementary example to show that an integer prime in one realm is factorable in a more extended realm.]

Solution by the Proposer.

Let $\alpha_0 = 1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ be the roots of $x^p - 1 = 0$, so that

$$x^p - 1 = (x-1)(x-\alpha_1)\cdots(x-\alpha_{p-1}).$$

Differentiate with regard to x, and then put $x = 1$, with the result

$$p = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{p-1}).$$

The proof of the problem is then immediate.

Note by the Editors: The final result is obtained by observing that, p being a prime integer > 2, $\alpha_i = \alpha^j$, where α is any root not equal to 1, $j \leqq p - 1$, and no two j's are the same for different i's. Hence

$$p = (1-\alpha)(1-\alpha^2)(1-\alpha^3)\cdots(1-\alpha^{p-1}).$$

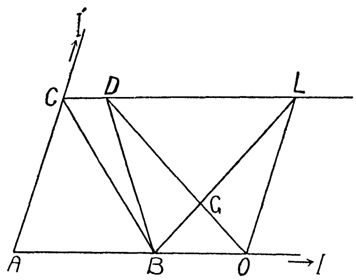
2921 [1921, 392]. Proposed by J. W. CLAWSON, Ursinus College, Pa.

ABC is a triangle cut by a transversal *PQR*, so that *A*, *P*; *B*, *Q*; and *C*, *R* are opposite vertices of a complete quadrilateral. Draw *CD*, *PF*, *QE*, chords in the circles circumscribing, respectively, triangles *ABC*, *BRP*, *AQR*, all these chords being parallel to *AB*.

Prove that (i) *D*, *E*, *F* are collinear; (ii) the line *DEF* passes through the Wallace point of the quadrilateral (the point of concurrency of the circles mentioned above); (iii) the line *DEF* intersects *AB* at the point of tangency to *AB* of the parabola which touches the four sides of the quadrilateral.

Solution by A. Pelletier, Montreal, Canada.

Let O be the point of contact of AB with the parabola which touches the sides of the quadrilateral and let us consider the triangle ABC. Apply Brianchon's theorem to the hexagon CB, BO, OI, IT, TI', $I'C$ where I and I' are the points at infinity on AB and AC, respectively, and T is the point of tangency of the line at infinity. Then CI, OI' and BT meet in L and $ACLO$ is a parallelogram: also BL is parallel to the axis. Draw OG cutting BL in G so that $\angle GOB = \angle GBO$. We shall show that the point D where OG meets CL is the point D of the theorem. For $OD = BL$ and hence the triangles OBL and BOD are congruent: thus $\angle DBA = \angle CAB$ and the circle through A, B and C passes through D. From the properties of the parabola, W, the focus, must lie on OD since triangle GOB is isosceles. It is also known that the above circle passes through W. In the same manner it follows that E and F lie on OW.

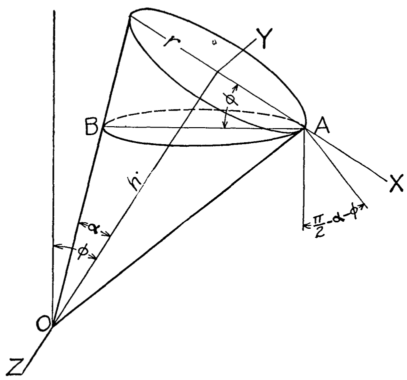


2936 [1921, 467]. Proposed by J. P. BALLANTINE, Chicago, Ill.

A person in drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a maximum?

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the height of the cone be h, its radius r and vertical angle 2α. Assume three mutually perpendicular axes, the x-axis in the plane of rotation of the axis of the cone, the y-axis perpendicular to this plane, the z-axis being the axis of the cone and the origin the center of the base.



We have then for the equation of the conical surface

$$r^2(h - z)^2 = h^2(x^2 + y^2), \tag{1}$$

and for the equation of the surface of the water when the axis of the cone makes an angle φ with the vertical

$$z = (r - x) \tan \varphi. \tag{2}$$

Now (1) and (2) give for the projection of the intersection of these two surfaces

$$r^2(h - r \tan \varphi + x \tan \varphi)^2 = h^2(x^2 + y^2),$$

which gives as a maximum width of this section

$$2y = 2r\sqrt{\frac{h - r \tan \varphi}{h + r \tan \varphi}} = 2r\sqrt{\frac{\cos(\varphi + \alpha)}{\cos(\varphi - \alpha)}}.$$

If AB is the length of this elliptic section, we have by the law of sines $AB\cos(\varphi - \alpha) = 2r \cos \alpha$, giving for the area

$$A = \tfrac{1}{2}\pi \overline{AB} \cdot y = \frac{\pi r^2 \cos \alpha \cos^{1/2}(\varphi + \alpha)}{\cos^{3/2}(\varphi - \alpha)}$$

which is a maximum when $\tan(\varphi + \alpha) = 3 \tan(\varphi - \alpha)$ or $\varphi = \frac{1}{2} \arcsin(2 \sin 2\alpha)$. This indicates that the first surface is a maximum unless the vertical angle of the cone is less than 30° or greater than 150°. Now $\sin 2\varphi = 2 \sin 2\alpha$; hence 2φ is between 2α and $180° - 2\alpha$ in value. Since physical considerations require that $0° < \varphi < 90° - \alpha$, it follows that $\alpha < \varphi$, and therefore $\alpha < 15°$.

Let the two values of φ obtained from $\sin 2\varphi = 2 \sin 2\alpha$ be φ_1 and φ_2, φ_2 being the greater; then φ_1 makes A a minimum and φ_2 makes A a maximum. For $\alpha = 15°$ the values of φ_1 and φ_2 are each 45° so that for this limiting value we have A neither a maximum nor a minimum. As α decreases to 0°, φ_1 decreases to 0° and φ_2 increases to 90°. It remains to determine for what values of α the maximum given by φ_2 is greater than the first surface. For this condition we must have

$$\frac{\pi r^2 \cos \alpha \cos^{1/2}(\varphi_2 + \alpha)}{\cos^{3/2}(\varphi_2 - \alpha)} > \pi r^2$$

which may be written $\cos^3 \alpha \sin \alpha > \cos^3(\varphi_2 - \alpha) \sin(\varphi_2 - \alpha)$ since $\tan(\varphi_2 + \alpha) = 3 \tan(\varphi_2 - \alpha)$. If now $x = \cos 2\alpha$, we have

$$\sin^2(\varphi_2 - \alpha) = \tfrac{1}{2}\{1 - \cos 2(\varphi_2 - \alpha)\} = \tfrac{1}{2}\{2x^2 - 1 + x\sqrt{4x^2 - 3}\},$$
$$\cos^2(\varphi_2 - \alpha) = \tfrac{1}{2}\{1 + \cos 2(\varphi_2 - \alpha)\} = \tfrac{1}{2}\{3 - 2x^2 - x\sqrt{4x^2 - 3}\}.$$

After extended reduction the above inequality becomes

$$(1 - x)^2(1 + x)^3(16x^5 + 16x^4 - 20x^3 - 20x^2 - 42x + 49) < 0.$$

In the range between 0 and 1, this inequality is satisfied by values of $x > 0.91401$ whence 2α or the vertical angle of the cone must be less than 23° 56′ for this maximum to exceed the area of the first surface.

To find the angle at which the delivery will be a maximum, assuming that the surface of the water remains level we must first find when the product of the area of the surface and the vertical velocity of the lip relative to the vertex is a maximum. If ω is the angular velocity of the cone, the velocity of the point A (relative to O) is $\omega\sqrt{h^2 + r^2}$ perpendicular to OA. Then there must be a maximum of

$$\frac{\pi r^2 \cos \alpha \cos^{1/2}(\varphi + \alpha)\sqrt{h^2 + r^2} \sin(\varphi + \alpha)\omega}{\cos^{3/2}(\varphi - \alpha)}$$

or of $\{\cos(\varphi + \alpha) \sin^2(\varphi + \alpha)\}/\cos^3(\varphi - \alpha)$. This is a maximum when $3 \cos(\varphi + \alpha) \cos 2\alpha = \cos(\varphi - \alpha)$ or

$$\tan \varphi = \cot \alpha(3 \cos 2\alpha - 1)/(3 \cos 2\alpha + 1).$$

This maximum will be greater than the initial rate of flow provided

$$\frac{\pi r^2 \cos\alpha \cos^{1/2}(\varphi+\alpha)\sqrt{h^2+r^2}\sin(\varphi+\alpha)}{\cos^{3/2}(\varphi-\alpha)} > \pi r^2\sqrt{h^2+r^2}\sin\alpha$$

which reduces for the above value of φ to

$$\left(\frac{3\cos 2\alpha - 1}{3\cos 2\alpha + 1}\right)^2 \cos^2 2\alpha(4 - 3\cos 2\alpha) > 0,$$

a condition that is true for all values of α, so that the maximum delivery is given in any case by

$$\varphi = \arctan\left(\frac{3\cos 2\alpha - 1}{3\cos 2\alpha + 1}\cot\alpha\right).$$

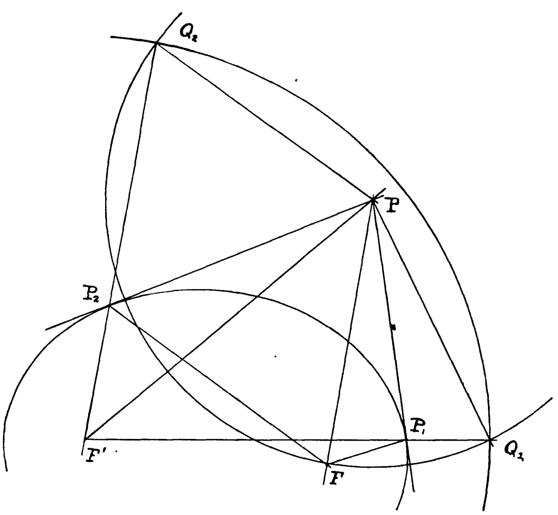
Also solved by E. M. BERRY, PHILIP FITCH and WILLIAM HOOVER.

2971 [1922, 179]. Proposed by C. E. HORNE, University of Porto Rico, Mayaguez, P. R.

Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at a focus. (This problem is found in some text-books, but the proposer is anxious to see an analytic solution of it.)

I. SOLUTION BY THE LATE T. M. BLAKSLEE, Ames, Iowa.

The theorem will be proved by a construction for the two tangents from an external point P to the ellipse whose two foci are F' and F. Describe a circle with center P and radius PF, then a second circle with center F' and radius $2a$, the length of the major axis, cutting the first circle in Q_1 and Q_2. Let the straight lines Q_1F' and Q_2F' cut the ellipse in P_1 and P_2, respectively. Then $FP_2 = P_2Q_2$, since the sum of the focal radii, $F'P_2 + FP_2$, is equal to $2a$, the radius of the F' circle. Also $PF = PQ_2$ and hence $\angle PP_2Q_2 = \angle PP_2F$. It now follows from the theory of conic sections that PP_2 is the tangent at P_2. Similarly, PP_1 is the tangent at P_1.



Since the two triangles $Q_2F'P$ and $Q_1F'P$ are equal, the angles $\angle P_2F'P$ and $\angle P_1F'P$ are equal. A similar proof applies to the focus F.

NOTE BY THE EDITORS: The two circles always intersect in two points, for, since P is an external point, we have

$$|F'P - FP| \leqq F'F < 2a < F'P + FP.$$

An examination of the three pairs of congruent triangles arising in this construction shows that $\angle P_2PP_1$ and $\angle F'PF$ have a common bisector, observing that in the case of the ellipse the first angle encloses the second. This is a known theorem from which may be easily deduced the results given in C. Smith's *Conic Sections*, §§ 227, 228, where the above two theorems and others are proved.

Grace and Rosenberg (*The Conic*, p. 251) arrive at a neat solution by the use of polar coordinates. The tangents to the conic, $l = r + re\cos\theta$, at the points whose vectorial angles are α and β are

$$l = re\cos\theta + r\cos(\theta - \alpha)$$

and

$$l = re\cos\theta + r\cos(\theta - \beta).$$

Solving these for the r and θ of the point of intersection of the tangents, we find readily that $2\theta = \alpha + \beta$. The theorem follows.

II. Solution by A. M. Harding, University of Arkansas.

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be any two points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. The tangents at the two points have the equations

$$b^2x_1x + a^2y_1y = a^2b^2, \qquad b^2x_2x + a^2y_2y = a^2b^2 \tag{1}$$

and they intersect in the point P_3, whose coördinates are obtained by solving the pair of equations (1),

$$x_3 = \frac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1}, \qquad y_3 = \frac{b^2(x_1 - x_2)}{x_1y_2 - x_2y_1}. \tag{2}$$

The equation of the straight line through P_1 and the focus $F(ae, 0)$ is

$$y_1x - (x_1 - ae)y - aey_1 = 0; \tag{3}$$

and if this equation be divided by $\sqrt{y_1^2 + (x_1 - ae)^2} = a - ex_1$ we obtain the normal form. The distance, D, of P_3 from the line FP_1 is obtained by inserting in the equation (3), in the normal form, the coördinates of P_3 given in (2). Making this substitution and noting that $b^2x_1^2 + a^2y_1^2 = a^2b^2$, $a^2e^2 = a^2 - b^2$, the factor $a - ex_1$ divides out from numerator and denominator, and there results

$$D = \frac{b^2x_1x_2 + a^2y_1y_2 - a^2b^2}{\pm a(x_1y_2 - x_2y_1)}.$$

The distance of P_3 from FP_2 may now be obtained by interchanging the subscripts 1 and 2; but this does not alter the numerical value of D. Hence the angle $\angle P_2FP_1$ is bisected by FP_3.

Also solved by R. E. Gaines, William Hoover, J. B. Reynolds, W. J. Patterson, Hazel E. Schoonmaker and Mabel M. Young. Professor Blakslee sent in three other solutions.

2973 [1922, 225]. Proposed by N. K. CHAFFEE, Rutland, Vt.

A straight brass bar 800 feet long expands 8 inches. The ends are fixed, so that it is distorted. If the new shape is that of an arc of a circle of which the original bar is the chord, how far above the center of the bar in its original position will the center be in the new position?

It is instructive to guess at this distance before attempting to solve the problem.

Solution by A. Pelletier, Montreal, Canada.

Let 2θ be the central angle in radians subtended by the arc of $800\frac{2}{3}$ ft. and r the radius of the circle. From the equations $r \sin \theta = 400$ and $r\theta = 400\frac{1}{3}$ follow

$$\frac{\sin \theta}{\theta} = \frac{1200}{1201},$$

and this equation is to be solved for the positive root different from zero. Replacing $\sin \theta$ by its approximate value $\theta - \theta^3/6$, we obtain $\theta = \sqrt{6/1201} = .07068$. The required height is then $400 \tan \theta/2$, or with sufficient accuracy, $400\ \theta/2 = 14.14$ ft.

Also solved by A. Bogard, A. M. Harding and G. R. Livingston.

2974 [1922, 225]. Proposed by the late L. G. WELD.

A man standing on a straight railway track watches a train starting from a station half a mile distant and notices that it approaches at such speed that each puff of the exhaust is heard at the same instant that the next succeeding puff is seen. How long will it be before the train reaches him, the drive wheels of the locomotive being sixteen feet in circumference?

Solution by B. F. Finkel, Drury College.

Let c be the circumference of the drive wheel; n the number of puffs of the exhaust to each revolution of the drive wheel; d the distance from the station to the observer; and v_s the velocity of sound.

Neglecting the velocity of light and the personal equation of the observer, we have the time, t_1, between the first and second puffs $= \dfrac{d}{v_s}$; t_2, the time between the second and third puffs $= \dfrac{d - \frac{c}{n}}{v_s}$; t_3, the time between the third and fourth puffs $= \dfrac{d - 2\frac{c}{n}}{v_s}$; $\cdots$; and, the time between the last puff and the last puff but one $= \dfrac{d - \left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}$, $c \div n$ being the distance traveled by the train between two consecutive puffs of the exhaust and $d/(c/n) = dn/c$ being the number of (c/n) intervals between the station and the observer.

Hence,

$$t = \sum_{i=1}^{nd/c} t_i = \frac{d}{v_s} + \frac{d - \frac{c}{n}}{v_s} + \frac{d - 2\frac{c}{n}}{v_s} + \cdots + \frac{d - \left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}$$

$$= \frac{d}{v_s} \cdot \frac{nd}{c} - \left[\frac{c}{nv_s} + \frac{2c}{nv_s} + \cdots + \frac{\left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}\right] = \frac{nd^2}{cv_s} - \frac{1}{2}\left[\frac{c}{nv_s} + \frac{\left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}\right]\left(\frac{nd}{c} - 1\right)$$

$$= \frac{nd^2}{2cv_s} + \frac{d}{2v_s} = \frac{d}{2v_s}\left(\frac{nd}{c} + 1\right).$$

In the modern locomotive engine, there are two cylinders, one on either side of the boiler, the difference of phase between the two being $\frac{1}{4}$. Hence, there are four puffs to each revolution of the drive-wheel.

Hence, substituting for c, 16; for n, 4; for d, 2640; and v_s, 1100, the velocity of sound at about 4.7° C., we find t = 13 min. 13.2 sec.

Also solved by H. A. BENDER, MOE BUCHMAN, S. A. COREY, and A. PELLETIER.

2979 [1922, 271]. Proposed by V. M. SPUNAR, Chicago, Illinois.

If a tangent be drawn from a variable point of an ellipse of length equal to n times the semi-conjugate diameter at the point, the locus of its extremity will be a concentric ellipse with semi-axes equal to $a\sqrt{n^2 + 1}$, $b\sqrt{n^2 + 1}$.

I. SOLUTION BY R. E. GAINES, University of Richmond.

Let $P_1(a \cos \varphi, b \sin \varphi)$ be the variable point on the ellipse; then the coördinates of an extremity P_2 of the diameter conjugate to OP_1 are $(a \sin \varphi, -b \cos \varphi)$. If OQ_2 is the prolongation of OP_2 to n times its length, the coördinates of Q_2 are $(na \sin \varphi, -nb \cos \varphi)$; and, if Q is the corresponding point on the locus, OQ_2QP_1 is a parallelogram and hence the coördinates of Q (α, β) are the sums of the coördinates of P_1 and Q_2. Hence

$$\frac{\alpha}{a} = \cos \varphi + n \sin \varphi, \qquad \frac{\beta}{b} = \sin \varphi - n \cos \varphi,$$

and, after squaring and adding, we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1 + n^2.$$

II. SOLUTION BY A. M. HARDING, University of Arkansas.

Let $P_1(x_1, y_1)$ be the variable point on the ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$. The coördinates of the extremity of the conjugate diameter are $(ay_1/b, -bx_1/a)$. We then have from the conditions of the problem

$$(1) \qquad b^2x_1^2 + a^2y_1^2 = a^2b^2,$$

$$(2) \qquad b^2x_1\alpha + a^2y_1\beta = a^2b^2,$$

$$(3) \qquad (\alpha - x_1)^2 + (\beta - y_1)^2 = n^2\left[\frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2}\right],$$

where (α, β) is the extremity of the tangent.

From (1) and (2) we find

$$b^2x_1(\alpha - x_1) + a^2y_1(\beta - y_1) = 0,$$

or

$$\frac{\alpha - x_1}{\beta - y_1} = -\frac{a^2y_1}{b^2x_1}. \tag{4}$$

Then from (3) and (4),

$$b(\alpha - x_1) = nay_1,$$
$$a(\beta - y_1) = -nbx_1.$$

Whence,

$$bx_1(1 + n^2) = b\alpha - na\beta,$$
$$ay_1(1 + n^2) = a\beta + nb\alpha.$$

Square and add,

$$a^2b^2(1 + n^2)^2 = (b^2\alpha^2 + a^2\beta^2)(1 + n^2),$$

or

$$b^2\alpha^2 + a^2\beta^2 = (1 + n^2)a^2b^2.$$

Whence,

$$\frac{\alpha^2}{a^2(1 + n^2)} + \frac{\beta^2}{b^2(1 + n^2)} = 1.$$

The locus is, therefore, a concentric ellipse whose semi-axes are $a\sqrt{1 + n^2}$, $b\sqrt{1 + n^2}$. It may be easily shown that a similar theorem holds true for the hyperbola.

Also solved by T. M. BLAKSLEE, E. J. OGLESBY, and J. B. REYNOLDS.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

The principal features of the annual meeting of the Inland Empire Council of Teachers of Mathematics were addresses by Professor W. A. BRATTON, of Whitman College, on "What a high school teacher of mathematics ought to know about Einstein," and by Professor F. L. GRIFFIN, of Reed College, on "The rôle of mathematics in human progress." Professor W. C. Eells, of Whitman College, has retired as president of the Council after three years' service. Professor C. A. Isaacs, of Washington State College, is a member of the executive committee.

The Ohio Section of the Association has definitely interested itself in the high school situation through the formulation by a committee of a letter to high school seniors presenting in a telling form reasons for continuing the study of mathematics in college, and of a letter to high school freshmen setting forth in a form particularly well suited to pupils of that age the importance of following up the study of arithmetic by that of algebra and geometry. The State Department of Education of Ohio has agreed to send copies of these letters to every superintendent of schools in the state. This substantial piece of committee activity will commend itself to other sections of the Association as well as to other organizations interested in setting forth the value of mathematics.

Professor A. A. MICHELSON of the University of Chicago has been elected president of the National Academy of Sciences. Professors E. W. BROWN of Yale and D. L. WEBSTER of Stanford have been elected to membership in the Academy.

Dr. H. M. DADOURIAN, associate professor of physics at Trinity College, has been appointed Seabury professor of mathematics.

Dr. LOUIS WEISNER, of Columbia University, has been appointed instructor of mathematics at the University of Rochester.

Mr. J. B. ROSENBACH, instructor in mathematics at the Carnegie Institute of Technology, has been promoted to an assistant professorship of mathematics.

Mr. O. H. RECHARD, Jr., instructor in mathematics at the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Wyoming.

Professor C. N. MILLS, professor of mathematics at Heidelberg University, Tiffin, O., and one of the associate editors of the MONTHLY, has accepted a professorship at the South Dakota State Normal. He taught for six years at South Dakota State College before moving to Ohio.

Associate Professor R. C. ARCHIBALD of Brown University, past president of the Association and for several years editor-in-chief of the MONTHLY, has been promoted to a full professorship.

Chancellor A. B. CHACE of Brown University, as a vice-president, represented the Mathematical Association of America at the inauguration of Dr. S. W. Stratton as president of the Massachusetts Institute of Technology on June 11, 1923.

Assistant Professor W. L. CRUM of Yale University has been appointed assistant professor of statistics in the department of economics of Harvard University.

Dr. C. H. YEATON and Dr. F. E. CARR have been given permanent appointments as assistant professors of mathematics at Oberlin College.

Mr. C. A. GARABEDIAN has received from Harvard University the degree of doctor of philosophy and an appointment as Parker Travelling Fellow for the next academic year.

Mr. NORMAN ANNING, instructor in mathematics at the University of Michigan, has been promoted to an assistant professorship.

On commencement day, June 20, at Brown University the honorary degree of Sc.D. was conferred upon Professor G. D. BIRKHOFF of Harvard University.

THE CINCINNATI MEETING OF THE ASSOCIATION

The Mathematical Association of America will meet at the University of Cincinnati, in affiliation with the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, on Thursday and Friday, December 27–28, 1923. The first session of the Mathematical Association will be held on Thursday afternoon, a joint session of the Association with Section A of the American Association and the Chicago Section will be held on Friday afternoon, and another session of the Association simultaneous with that of the Chicago Section will be held on Friday morning. Other sessions of the Chicago Section will occur on Saturday morning and probably Saturday afternoon. This plan will enable our members to attend the mathematical meetings during these three days and then have free opportunity to visit the meetings of the American Association during the first three days of the next week.

NOTABLE GIFTS TO THE ASSOCIATION

A notable bequest in the form of a last will and testament has just been executed in favor of the Mathematical Association of America which is designated as sole legatee. The name of the donor is withheld at present by request, but the person through whose influence this bequest was secured is Mr. C. C. Carter of Bluffs, Illinois, who has been a member of the Association since 1918. He deserves great credit for the service which he has thus rendered to the Association, a kind of service which may lie within reach of any member who is on the alert for such an opportunity. To quote his own modest words: "I am not much of a mathematician, but I have all my life studied mathematics as a pastime and I have always had a responsive chord in my cardiac impulses for the fellow who sits up nights to pore over the cabalistic characters of this great subject. While this is only a will, I have perfect confidence that its provisions will be faithfully carried out. I am thoroughly familiar with the property, which is mostly in land, and of course at low ebb in the market just now, but I conservatively estimate its potential value at $100,000."

The will provides for the creation of a Memorial Fund whose income "shall be used for the promotion of mathematical science and its teaching in America in such manner as the officers of the above named corporation may deem fit." The Trustees of the Association are thus given a free hand in determining the objects to which the funds should be applied, the only condition being that "each and every thing done by the financial aid of the above mentioned fund shall bear some distinguishing mark, print, character, or notification, showing the source from which the financial aid to do the said thing or act was derived."

Another notable service to the Association has been very unostentatiously rendered by Chancellor A. B. Chace, of Brown University, who is a Vice-President of the Association for 1923. For each of the two years, 1922, 1923, he has contributed the sum of four hundred dollars in cash to the current funds of the Association. To these generous gifts is largely due the fact that the Association has been able to proceed without drawing upon its slender reserve funds inherited from the former management of the MONTHLY. Chancellor Chace is not engaged professionally in the field of mathematics, but he evidently finds great delight and satisfaction in certain phases of mathematical activity. For instance, he has been occupied for some years in making a translation of the Ahmes Papyrus which he expects soon to publish. The Association is honored by the presence of such a person on its membership roll and among its officers.

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BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

Eighth Annual Meeting, University of Cincinnati, December 27–28, 1923

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NOTES FOR THE ASSOCIATION.

The Trustees of the Association have approved the establishment of a Michigan Section.

They have elected to membership in the Association the following one hundred applicants, on applications duly certified:

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To institutional membership.

The University of Oklahoma, Professor S. W. Reaves, Official representative.

W. D. Cairns, *Secretary-Treasurer.*

THE MAY MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The thirteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Association was held at Johns Hopkins University, Baltimore, Maryland, on Saturday, May 12, 1923. The Chairman of the Section, Professor Frank Morley, presided at both morning and afternoon sessions.

There were forty-four in attendance, including the following twenty-eight members of the Association: O. S. Adams, H. G. Avers, Clara L. Bacon, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, Harry English, H. W. Ficken, W. M. Hamilton, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, L. T. Moore, Frank Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, R. E. Root, W. F. Shenton, C. A. Shook, Miss A. M. Whelan, E. W. Woolard. Those in attendance at the meeting were guests of the Johns Hopkins University for lunch.

The officers elected for the year 1923–1924 are: G. R. Clements, Chairman; Harry English, Secretary-Treasurer; Florence P. Lewis and J. J. Luck, additional members of the executive committee. The fourteenth regular meeting of the Section will be held at Annapolis, Maryland, on Saturday, December 8, 1923.

The following papers were presented:

(1) "An application of tensor analysis to dynamics" by Professor F. D. Murnaghan;

(2) "Periodic functions in machine design" by Professor R. E. ROOT;

(3) "Invariants of the binary octavic" by Miss A. M. WHELAN (introduced by Professor Morley);

(4) "The volume bounded by Steiner's quartic surface" by Professor J. A. BULLARD;

(5) "The lemniscate functions as developed by Gauss" by Mr. O. S. ADAMS;

(6) "The motion of a particle on a rotating sphere; MONTHLY, Problem No. 2956 (*1922*, 81)" by Mr. W. D. LAMBERT;

(7) "An example of conformal mapping" by Professor FRANK MORLEY.

Abstracts of six of these papers follow:

1. Professor Murnaghan discussed the tensor form of the canonical equations of classical dynamics. Adopting the usual notation, the state of a dynamical system is indicated by a representative point in a *state-space* of $2n + 1$ dimensions (n being the number of degrees of freedom of the system) whose coördinates are (q, p, t). This space must not be confused with the more usual phase-space of Gibbs which separates out the time variable t from the others. Then the history of the dynamical system is represented by a curve in the state-space, which is an extremal for the line integral

$$\int p_1 dq_1 + \cdots + p_n dq_n - H dt,$$

where H is the Hamiltonian Function. The dynamical equations are merely the first Pfaff's system for the linear differential form; hence they state the vanishing of a covariant tensor of rank one. Classical dynamics conflicts with Relativistic geometry at the point where the latter demands that all physical equations be tensor equations in the *physical space-time continuum of four dimensions*, and not merely in the idealistic representative space.

2. Professor Root considered functions arising in problems in machine design that are expressed as algebraic functions of the sine or cosine of an angle. It is profitable to express such functions as Fourier series for convenience in computing their values, and to reveal their character in relation to vibrations. The representation in Fourier series is easily found, in most cases, by expanding the function as a power series in the sine or cosine, then expressing the successive powers in terms of functions of multiple angles and collecting coefficients. Professor Root contrasted this method of expanding in Fourier series with those usually treated in mathematical textbooks.

4. Professor Bullard called attention to the problem found in Granville's "*Elements of Differential and Integral Calculus*, rev. ed.," p. 420: Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. The speaker proposed this problem to the Monthly (*1920*, 326) and an incorrect answer was published (*1922*, 39).

If positive signs are taken before the square roots, no volume is enclosed by the portion of the surface where x, y and z are all less than a; but the rationalized equation is the equation of a Steiner's quartic surface.

When a regular tetrahedron of reference is chosen the Steiner's quartic, as a

model shows, is symmetrical, consists of four lobes of finite extent, and the three double lines intersect at right angles.

For the above equation the three coördinate planes and the plane at infinity form the tetrahedron of reference. Thus the surface lies entirely in the first octant (except that the double lines extend into the octants where just two coördinates are negative), three lobes are infinite in extent and one only encloses a finite volume. The three double lines intersect at (a, a, a) and are respectively perpendicular to the coördinate axes. The required volume lies entirely within a cube of edge a with opposite vertices at $(0, 0, 0)$ and (a, a, a) and may be considered as enclosed by $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$, $x^{1/2} + y^{1/2} - z^{1/2} = a^{1/2}$, $x^{1/2} - y^{1/2} + z^{1/2} = a^{1/2}$, $-x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. Computing the space in the corners and subtracting from a^3, we have for the required volume

$$a^3 - (a^3/90) - 3 \times (19/90)a^3 = (16/45)a^3.$$

Similarly for $(x/a)^{1/2} + (y/b)^{1/2} + (z/c)^{1/2} = 1$ we have $(16/45)abc$.

5. Mr. Adams pointed out that Gauss took the integral $\phi = \int_0^x (dx/\sqrt{1 - x^4})$ and inverted it by setting $x = \sin \operatorname{lemn} \phi$, called the lemniscate sine of ϕ. Gauss noted in his diary "Functiones lemniscaticas considerare coeperamus, 1797, Jan. 8." This is the first example of the inversion of an integral to obtain a single valued function by means of which to study the properties of the many valued function defined by the integral. The paper identified the auxiliary functions of Gauss with the sigma functions of Weierstrass, limited to the special case of the lemniscate functions. It is to be regretted that Gauss never prepared his work for publication. As it is, we have no more than the list of various results, mainly without proof, given in the third volume of his collected works.

6. Problem 2956 (*1922*, 81) was discussed by the proposer, Mr. Lambert. The expression for the poleward force there given, $m\omega^2 \sin \phi \cos \phi$, should be corrected by multiplying it by the radius of the sphere. The problem is one that has long interested meteorologists as tending to throw light on the much more complex problem of atmospheric circulation. The compound centrifugal acceleration, which causes a moving particle to be deflected toward the right in the northern hemisphere, has long been known and its effects studied. For a small range of latitude the path of a particle moving under the action of no forces is approximately a small circle, as may be seen from elementary considerations. The rigorous expressions for the path of the particle involve elliptic functions. The path is not exactly a circle but a series of loops which are nearly coincident with one another for moderate velocities of projection, except near the equator. For high velocities and for regions near the equator the loops separate considerably and the possible paths resemble the various forms of the elastica.

7. In Professor Morley's note the mapping of a quadrangle on a rectangle was considered. It was shown that when the quadrangle is a parallelogram, if two opposite sides be levels and the other two sides stream lines, the flex-loci for all levels and all stream lines are the maps of the lines of symmetry of the rectangle.

G. R. CLEMENTS, *Secretary-Treasurer.*

of movable graduated scales, or movable figures on millimeter and tracing papers, were discussed.

2. The method of generating a parabola by folding a piece of paper suggested to Mr. Rupp a possible generalization, which turned out to be a birational point-line transformation applicable to any curve. The inverse transformation has been called a *Podoid* by Brocard. Five theorems were stated, dealing with the mechanical construction by paper-folding of homothetic transformations, pedals and negative pedals, podoids and negative podoids, and a simple way of constructing an ellipse or hyperbola. Developments of these general ideas were suggested as possible for spaces other than flat, as non-Euclidean or spherical, or simple three-space.

3. Professor Bussey's paper was an elementary exposition of Galois fields designed especially for those not familiar with any finite fields other than the modular fields of prime order.

4. President Reuterdahl's paper contained a warning against too ready an acceptance of certain parts of mathematical theory as a correct interpretation of the physical universe.

5. Professor Jackson's paper was published in the MONTHLY for September-October, pp. 307–311.

6. Professor Tate applied the fundamental theorem of Dimensional Analysis to certain examples in physics, and showed how it may be used as a check and guide to the investigator.

7. Professor Fath showed slides of the various types of instruments used in eclipse observations and of the solar corona as shown at times of spot maximum and spot minimum.

8. Professor Johnson's paper appeared in full in the MONTHLY for July–August, pp. 250–252.

R. W. BRINK, *Secretary.*

THE DEVELOPMENT OF "PARTITIO NUMERORUM," WITH PARTICULAR REFERENCE TO THE WORK OF MESSRS. HARDY, LITTLEWOOD AND RAMANUJAN.[1]

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Part I.

HISTORICAL SKETCH LEADING UP TO THE WORK OF HARDY, LITTLEWOOD AND RAMANUJAN.

Introduction. In the last few years the genius of a small group of mathematicians has enriched the additive theory of numbers by new and powerful methods. At present we merely mention the names of these authors: G. H. Hardy (English) and J. E. Littlewood (English) have developed the bulk of the theory; the first papers were under the joint authorship of Hardy and S. Ramanujan (Hindu, *died,* 1920). A very important step in the theory could only be taken by making use of an independent investigation by H. Weyl (German).

[1] Among the references to the literature of the subject, the following are to be noted:
(*a*) *Sources of reference for "partitio numerorum" in general and for the main divisions of the subject,*

The papers in which the new theory is exposed are, in the main, of highly technical character. It is hoped that the present report, partly historical and partly synoptic, may indicate to readers, who are not primarily interested in the theory of numbers, the character of the new methods. A certain knowledge of the historical background is necessary for any proper appreciation of the new work; and since the problems with which the additive theory of numbers is concerned are not widely known, it seems justifiable to give a condensed sketch of the development of the subject.

1. In a rather vague manner, a distinction is often made in the theory of numbers between the "multiplicative" theory and the "additive" theory. For example, the theorem $a^p - a \equiv 0 \bmod p$ for any integer a and any prime p

(1) Bachman, P., *Niedere Zahlentheorie*, Zweiter Teil: *Additive Zahlentheorie* (1910).

(2) Dickson, L. E., *History of the Theory of Numbers*, vol. II (1920). Ch. III: Partitions; Chs. VI, VII, VIII; Ch. IX: Sums of Squares; Ch. XXV: Waring's problem.

(3) *Encyklopädie der mathematischen Wissenschaften*, I C 3, Bachmann, P., *Analytische Zahlentheorie.*

(4) *Encyclopédie des Sciences mathématiques*, I 17, Bachmann-Hadamard-Maillet, *Propositions transcendantes de la théorie des nombres.* Netto-Vogt, I 2, *Analyse combinatoire et théorie des déterminants.*

(5) MacMahon, P. A., *Combinatory Analysis*, vol. I (1915), II (1916).

(6) Netto, E., *Lehrbuch der Combinatorik* (1901).

(7) Sylvester, J. J., *Mathematical Papers*, vol. II (1908), 120–175. "Outlines of seven lectures on the partition of numbers." (Delivered 1859.)

(*b*) *Papers by* G. H. Hardy, J. E. Littlewood *and* S. Ramanujan *on the new method of dealing with problems of "partitio numerorum."* This list has been completed by taking into account the literature references contained in Siegel's (**28**) list.

(8) Hardy, "Asymptotic formulæ in combinatory analysis." *Comptes rendus du quatrième congrès des mathématiciens Scandinaviens à Stockholm* (1916), pp. 45–53.

(9) Hardy and Ramanujan, "Une formule asymptotique pour le nombre des partitions de *n*." *Comptes rendus de l'Académie des Sciences*, Paris, vol. 164 (1917), pp. 35–38.

(10) Ramanujan, "On certain trigonometric sums and their application in the theory of numbers." *Transactions of the Cambridge Philosophical Society*, vol. 22 (1918), pp. 259–276.

(11) Hardy, "On the coefficients in the expansion of certain modular functions." *Proceedings of the Royal Society*, London, A 95, 1918, pp. 144–155.

(12) Hardy and Ramanujan, "Asymptotic formulæ for the distribution of integers of various types." (Read May, 1916.) *Proceedings of the London Mathematical Society*, ser. 2, vol. 16 (1918), pp. 112–132.

(13) Hardy and Ramanujan, "Asymptotic formulæ in combinatory analysis." (Read Jan. 1917.) *Proceedings of the London Mathematical Society*, ser. 2, vol. 17 (1918), pp. 75–115.

(14) Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five and seven." *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 189–193.

(15) Hardy and Littlewood, "A new solution of Waring's problem." *Quarterly Journal of Mathematics*, vol. 48 (1919), pp. 272–293.

(16) Hardy and Littlewood, Note on Messrs. Shah and Wilson's paper entitled "On an empirical formula connected with Goldbach's Theorem." *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 245–254.

(17) Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five." *Transactions of the American Mathematical Society*, vol. 21 (1920), pp. 255–284.

(18) Hardy, "Some famous problems of the theory of numbers and in particular Waring's problem." An inaugural lecture delivered before the University of Oxford. Oxford, 1920.

represents a "multiplicative" property; the theorem that every positive integer is either a cube, or the sum of two positive cubes, or the sum of three, or four, . . ., or nine positive cubes, will be considered as belonging to the "additive" theory.

Since the time of Legendre and Gauss—particularly since 1801, the year in which appeared Gauss' monumental *Disquisitiones Arithmeticæ*—the systematic development of the theory of numbers has been predominantly along the lines of the multiplicative theory. It must however not be thought that no important results are known in the additive theory. Indeed, there is a profusion of dazzling isolated theorems and groups of theorems.[1] If we merely count interesting theorems, we find that the additive theory is perhaps not much weaker than

A series of papers, under the collective title: Hardy and Littlewood, Some problems of "Partitio Numerorum."

(**19**) I. "A new solution of Waring's problem." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1920, pp. 33–54.

(**20**) II. "Proof that every large number is the sum of at most 21 biquadrates." *Mathematische Zeitschrift*, vol. 9 (1921), pp. 14–27.

(**21**) III. "On the expression of a number as a sum of primes." *Acta Mathematica*, vol. 44 (1922) pp. 1–70.

(**22a**) IV. "The singular series in Waring's problem and the value of the number $G(k)$." *Mathematische Zeitschrift*, vol. XII, 1922, pp. 161–188. (Reprinted from *Hilbert Festschrift*, Jan. 1922, pp. 365–392.)

(**22b**) V. "A further contribution to the study of Goldbach's problem." *Proceedings of the London Mathematical Society*, ser. 2, vol. 22 (1923), pp. 46–56.

(*c*) *Papers on the new method by other mathematicians*,

(**23a**) *Fortschritte der Mathematik*, vol. 46 (covering the years 1916–18), pp. 200, 201.

(**23b**) Carmichael, R. D., *Bulletin of the American Mathematical Society*, vol. 27 (1921), 471–5 (Review of (**18**)).

(**24**) Landau, E., "Über die Hardy-Littlewoodschen Arbeiten zur Additiven Zahlentheorie." *Jahresbericht der Deutschen Mathematiker Vereinigung*, vol. 30 (1921), pp. 179–185.

(**25**) Landau, E., "Zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1921, pp. 88–92.

(**26**) Landau, E., "Zum Waringschen Problem." *Mathematische Zeitschrift*, vol. XII (1922), pp. 219–247. (Reprinted from *Hilbert Festschrift*, Jan. 1922, pp. 422–451.)

(**27**) Ostrowski, A., "Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Mathematische Zeitschrift*, vol. IX (1921), pp. 28–34.

(**28**) Siegel, C. L., "Additive Theorie der Zahlkörper I." *Mathematische Annalen*, vol. 87 (1922), pp. 1–35.

(**29**) Weyl, H., "Über die Gleichverteilung von Zahlen mod. 1." *Mathematische Annalen*, vol. 77 (1916), pp. 313–352.

(**30**) Weyl, H., "Bemerkung über die Hardy-Littlewoodschen Untersuchungen zum Waringschen Problem." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1921, pp. 189–192.

(**31**) Weyl, H., "Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1922.

Numbers (**8**), (**21**), (**22 b**), (**25**), (**30**), (**31**) were not available to the writer.

Reference of the text to the preceding literature is made by corresponding numbers. Other text references are given by footnotes.

As preparation for a detailed study of the new theory, the (partly) synoptic papers (**15**), (**18**), (**23**), (**24**) should be of great value.

In (**26**) Landau derives *ab ovo* the main results of Hardy and Littlewood for Waring's problem, with original contributions and modifications.

It is strongly to be hoped that American mathematicians will contribute their share toward the development of the new field.

[1] (**1**), (**2**).

the multiplicative. But while in the multiplicative theory the theorems all find their proper place in a beautiful system, as do the building stones in an imposing structure, the theorems of the additive theory may be compared to a vast accumulation of building material for a palace which is at present barely planned.

It is also true that a large number of theorems, which by their enunciation should be counted to the additive theory, at present derive their proof from methods belonging to the multiplicative theory.

2. One of the most important branches of the additive theory, and one which contains in itself an inexhaustible supply of large problems and groups of problems, is the "partitio numerorum," or the theory of the partition of positive integers into positive summands. This name is probably due to L. Euler, who called chapter XVI of his famous *Introductio in Analysin Infinitorum*,[1] "De Partitione Numerorum."

As starting point we may consider the problem of determining the total number $p(n)$ of ways in which a given positive integer n may be broken up into positive integral summands, counting as identical two partitions which are distinguished only by the order of the summands; thus $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1, p(5) = 7$.

This problem had already been suggested about a century earlier by Leibniz, to one of the Bernoullis.

The range of problems belonging to partitio numerorum is much wider than might be at first thought. Instead of admitting as summands the range of all positive integers, we may select any particular, finite or infinite, range of positive integers. As of interest for our purposes, we mention the following ranges:

1. All positive integers, $1, 2, 3, \cdots$,
2. A finite set of integers, for example, 1, 2, 3,
3. All odd numbers, $1, 3, 5, \cdots$,
4. All odd primes and the number 1,
5. The "triangular" numbers, $1, 3, 6, 10, \cdots$,
6. The "square" numbers, $1, 4, 9, 16, \cdots$,
7. The "k-gon" numbers for a given integer $k \geqq 3$ (the nth k-gon number $= \frac{1}{2}n(n - 1)(k - 2) + n$),
8. The cubes $1, 8, 27, \cdots$,
9. The kth powers, for a given positive integer k, $1^k, 2^k, 3^k, \cdots$.

The original problem (1 above) would then be modified into the determination of the number of ways in which (for example, for 6. above) the positive integer n may be written as a sum of positive squares (as: $10 = 1^2 + 1^2 + \cdots + 1^2 = 2^2 + 1^2 + \cdots + 1^2 = 2^2 + 2^2 + 1^2 + 1^2 = 3^2 + 1^2, p(10) = 4$).

[1] *Introductio in Analysin Infinitorum*, vol. I, 1748. It may be noted that Ch. XV of Euler's work, which deals with certain types of infinite series, has also been of great influence in the development of the theory of numbers. Many results which were stated by Euler without proof have yielded to treatment only in our generation. As a sample we mention that a certain series whose terms depend on the prime numbers and which starts with $1 + 1/3 - 1/5 + 1/7 + 1/9 + 1/11 - 1/13 - 1/15 \cdots$ converges to the value $\pi/2$. See E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1909, vol. II, pp. 645 ff.

3. A very important group of problems arises when we restrict the number of summands to be used in the representation of the number n; for example, when we ask for the number of ways in which n may be the sum of exactly five positive squares, or the sum of exactly nine positive cubes, etc. Equally important is the question concerning the number of ways in which n can be represented as the sum of *not more* than a given number k of elements of the set; for example, the number of ways in which n may be the sum of *five or fewer* positive squares, of *nine or fewer* positive cubes, etc. It may further be specified that the same summand may be used only once, or may be repeated. Thus, in how many ways can n be the sum of (exactly) five *distinct* positive squares; of five or fewer *distinct* squares? These remarks will serve to indicate the bewildering complexity of the group of problems under consideration.

A very general type of problem then is the following: *Given a set of positive integers $c_1 < c_2 < c_3 < \cdots$, finite or infinite in number, to find, for a given positive integer n, the number of ways in which n may be formed of summands chosen from the c's, be it that the number of summands is limited, or is not limited, be it that the summands may be repeated or must be distinct.*

Every such problem may be expressed in the form of a Diophantine equation: for example, the number of positive solutions of $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ gives the number of ways of expressing n as a sum of five positive squares. In $n = 1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3$ the number of (non-negative) solutions gives the number of ways in which n may be built up from 1, 2, 3, allowing repetitions. Thus, $p(7) = 8$, corresponding to the eight solutions of $1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3 = 7$: (7, 0, 0), (5, 1, 0), (4, 0, 1), (3, 2, 0), (2, 1, 1), (1, 3, 0), (1, 0, 2), (0, 2, 1). However, we do not possess any powerful general method of attacking the problems from this direction.

4. We remember that we did not count as distinct two partitions which differ only in the order of their summands; for example, $1 + 2 + 1 + 2$ and $1 + 1 + 2 + 2$, would not be counted separately. When the order of the summands is to be taken into account, one sometimes speaks of "compositions" instead of "partitions," so that the problem of determining the number $c(n)$ of compositions of n means to find the number of partitions, but to count each partition with a multiplicity measured by the number of ways in which the order of the summands may be chosen. For $n = 3$, $c(n) = 4$, from $3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$.

The problems of composition would seem to be in some respects of less importance than the problems of partition. It does not usually seem to be possible to derive, from the solution of a problem of composition, the solution of the corresponding problem of partition. Quite an elaborate theory of the composition of integers into sums of squares is in existence.

We shall restrict our attention essentially to the problems of partition. Problems of partition and composition can also be expressed in a natural fashion as problems of the determination of the number of "lattice-points" in certain regions—"Gitterpunkt Anzahlen." To mention only two examples: The num-

ber of *compositions* of n into two squares is obviously equal to the number of lattice-points (points whose coördinates are integers or zero) lying on the circumference of a quadrant of the circle $x^2 + y^2 = n$ in the (xy)-plane. The number of *compositions* of n into three or fewer squares is represented by the number of lattice-points lying on the surface of the sphere $x^2 + y^2 + z^2 = n$ within the first octant of the (xyz)-space; while the number of *partitions* of n into three squares would be given by the number of lattice-points lying on a certain smaller portion [1] of the surface of the sphere. Similarly for a composition involving a set of k summands (for example, the representation of a number as the sum of nine non-negative cubes) an illustration in k-dimensional space (nine dimensional, working with the "surface" $x_1^3 + x_2^3 + \cdots + x_9^3 = n$) can be given.

However, the problem of finding such "Gitterpunkt Anzahlen" to the degree of accuracy necessary in order to be useful in partitio numerorum is, in practically all cases, one of desperate difficulty.

5. If $p(n) = 0$, in a given problem of partitions, there is no representation of n in the required form while, when $p(n) > 0$, there is at least one representation. The further questions thereby arise:

Which numbers [2] *n can be and which cannot be represented in the desired form? How must an infinite set of numbers be chosen in order that every n can be represented as a sum of a fixed number of elements of the set?*

We shall assume, throughout, that every element may be used repeatedly, as in $7 = 2^2 + 1^2 + 1^2 + 1^2$.

Concerning the first question, a large number of more or less isolated theorems are known. To mention some classical results concerning the representation of integers as sums of powers:

1. A number is the sum of two positive squares when, and only when, it contains no prime factor of the form $4k + 3$ to an odd power.

2. A number is the sum of three or fewer positive squares when, and only when, it is not of the form $4^\alpha \cdot (8k + 7)$, $\alpha \geqq 0$.

3. Every number is the sum of four or fewer positive squares.

4. Every number is the sum of nine or fewer positive cubes.

Concerning the second question no very deep-lying results seem to be known, in the sense of establishing necessary and sufficient conditions for a set of integers to have the property indicated.[3]

There are important reasons why one should in many cases restrict one's self in problems of partition to the representation of large numbers n: this tends to eliminate what may be called disturbing individual properties of the separate c's. The most clearly outstanding results of the new work of Hardy, Littlewood and Ramanujan are in this direction.

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[1] Of area 1/48 of the total surface of the sphere.

[2] We frequently write "number" for "positive integer," where no misunderstanding is to be feared.

[3] The following *necessary* condition is not deep-lying and is easily established: *If* $0 < c_0 < c_1 < c_2 < \cdots$ *are an infinite set of integers possessing the property that every positive integer can be represented as the sum of a limited number of the set, then the power series* $c_0 + c_1x + c_2x^2 + \cdots$ *has the unit circle for its circle of convergence.*

It is essential, in problems of partition, at least at present, that we consider only addition of the positive elements c_r, not subtraction. Therefore Goldbach's conjectured theorem: *Every even number is a sum of two primes,* is properly a problem of partition, but Fermat's conjectured theorem: $x^n + y^n = z^n$ *is not solvable in positive integers* x, y, z *for* $n > 2$ does not belong to this domain, because the problem would require discussion of the equation $0 = -z^n + x^n + y^n$.

It should be remarked that our most powerful methods of attack on Fermat's problem are of distinctly multiplicative character.

6. Already Euler (*loc. cit.*) connected the most important types of problems of partition with certain infinite series and infinite products, the so-called generating functions of the problem, and since most of the serious attacks on the problems of partition which have been made since Euler's time have used this method of generating functions as a starting point, and since, in particular, the new methods of Hardy, Littlewood, and Ramanujan consist in an extreme sharpening of this weapon, we must indicate in some examples this method.

I. *To find the number* $N(n)$ *of partitions, without repetition, of a number* n: $(7 = 6 + 1 = 5 + 2 = 4 + 3 = 4 + 2 + 1, N(7) = 5)$.

The problem might also be formulated: *To find the number of solutions of* $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n$, $x_i = 0, 1$. But, as already indicated, we possess no means of treating this Diophantine equation directly.

Euler attacks the question as follows without, it may be stated, paying any attention to questions of convergence, etc.:

Let

$$\prod_{i=1}^{\infty}(1 + x^i) = (1 + x^1)(1 + x^2)(1 + x^3) \cdots = 1 + c_1x^1 + c_2x^2 + \cdots + c_nx^n + \cdots.$$

The number of partitions required is clearly given by c_n (for example, $c_7x^7 = x^7 + x^{6+1} + x^{5+2} + \cdots + x^{4+2+1} = 5x^7$, $c_7 = 5$).

It goes without saying that Euler did not possess modern function-theoretic methods for determining the value of c_n, nor did he possess any other powerful methods of dealing with this and other partition problems. To us is of great interest the fact that he was apparently the inventor of this method of "generating functions."

II. *To find the number of partitions of* n, *with repetition.* For example, $p_5 = 7$. The corresponding Diophantine equation would be $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n$, $x_i \geqq 0$. This is again useless. Euler starts out with

$$\begin{aligned}\prod_{i=1}^{\infty}\frac{1}{1 - x^i} &= (1 + x^1 + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 \\ &\qquad\qquad + x^6 + x^9 + \cdots) \cdots \textit{in inf.} \\ &= 1 + x^{1\cdot1} + (x^{2\cdot1} + x^{1\cdot2}) + (x^{3\cdot1} + x^{1\cdot2+1\cdot1} + x^{1\cdot3}) + \cdots \\ &= c_0 + c_1x^1 + c_2x^2 + c_3x^3 + \cdots,\end{aligned}$$

and, again, the coefficient c_n of x^n gives the required number $p(n)$. Of course, this number grows tremendously large with increasing n. This problem we

shall discuss in some detail (sections **12–17**). At present we only state that one of Hardy's (and Ramanujan's) first results of the new theory was to obtain an accurate expression for $p(n)$, and one which can really be handled. In this problem, $\Pi(1 - x^i)^{-1}$ is the generating function, and Hardy and Ramanujan teach how to deal with this very complicated infinite product, as an analytic function of (complex) x, in the neighborhood of the natural boundary of the function.

In order to show the difference between a problem of partition and the corresponding problem of composition, we treat the question:

III. *To find the number of compositions of* n,[1] *with repetition.* We determine first the number of compositions of n into exactly s positive summands. As generating function we may choose

$$\left(\frac{x}{1-x}\right)^s = (x + x^2 + x^3 + \cdots)^s = a_s x^s + a_{s+1}x^{s+1} + \cdots.$$

It is found that

$$\left(\frac{x}{1-x}\right)^s = x^s\left(1 + \frac{s}{1}x + \frac{s(s+1)}{1\cdot 2}x^2 + \cdots + \frac{s(s+1)\cdots(s+i-1)}{1\cdot 2\cdots i}x^i + \cdots\right) = \sum_{n=s}^{\infty}\binom{n-1}{s-1}x^n$$

and therefore $\binom{n-1}{s-1}$ is the number of compositions of n into exactly s positive summands. Hence the total number of compositions of n into positive summands is

$$\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} = 2^{n-1}.$$

For example, for $n = 3 : 3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$. In order to pass from this problem to II, it would be necessary first to know how many partitions of n have a group of exactly n_1 summands alike, another group of n_2 summands alike, etc., when $n_1, n_2, \cdots$ are any positive integers such that $n_1 + n_2 + \cdots = n$. This would be a very difficult problem.

IV. *To find the integers which are the sum of exactly two odd primes* p_i. We may choose as generating function

$$(x^{p_1} + x^{p_2} + \cdots)^2 = c_1 x^{\alpha_1} + c_2 x^{\alpha_2} + \cdots,$$

where $p_1 < p_2 < \cdots$ are the successive odd primes. In particular, Goldbach's theorem (see section **5**) would be proved if we could show that for every even exponent the corresponding coefficient does not vanish. Of this theorem, E. Landau, author of the standard work on prime numbers, the "Handbuch" already referred to, said, in 1912, that he considered this problem "unangreifbar beim gegenwärtigen Stande der Wissenschaft."[2] Nothing perhaps can better

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[1] (**1**), (**2**).

[2] International Congress, Cambridge, 1912: Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung.

emphasize the importance of the new work than the fact that this problem is, in a perfectly definite sense, brought under the range of Hardy and Littlewood's heavy artillery so that we may, without undue optimism, hope to see it solved in our generation.

Hardy and Littlewood actually prove[1] that every sufficiently large odd positive integer is the sum of at most three primes, provided that we admit a certain generalization of a famous assumption, which has never been either proved or disproved, concerning the zeros of the Riemann ζ-function, of which an expansion is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

V. For a special purpose, we also indicate Jacobi's proof (not historically the first one) of the theorem that every positive integer is the sum of four non-negative squares. As generating function one might choose $(1 + x^{1^2} + x^{2^2} + \cdots)^4$. In his *Fundamenta nova*,[2] Jacobi—who always kept his eye open for mathematical by-products—made use of the fact that he was able to express a certain quantity entering in his theta-series in two different manners:

$$\begin{aligned}\left(\frac{2K}{\pi}\right)^2 &= (1 + 2q^1 + 2q^4 + 2q^9 + \cdots)^4 \\ &= 1 + 8\left\{\frac{q}{1-q} + \frac{q^2}{1+q^2} + \frac{q^3}{1-q^3} + \frac{q^4}{1+q^4} + \cdots\right\}.\end{aligned}$$

Obviously, $(1 + 2q^1 + 2q^4 + \cdots)^4$ may be used as generating function instead of $(1 + q^1 + q^4 + \cdots)^4$. The argument is now as follows:

$$\begin{aligned}&(1 + 2q^1 + 2q^4 + 2q^9 + \cdots)^4 \\ &\quad = 1 + 8(q + q^2 + q^3 + q^4 + q^5 + q^6 + + + \cdots) \\ &\quad + 8(+ q^2 - q^4 + q^6 - + - \cdots) \\ &\quad + 8(+ q^3 + q^6 + + + \cdots) \\ &\quad + 8(+ q^4 - + - \cdots) \\ &\quad + 8(+ q^5 + + + \cdots) \\ &\quad + 8(+ q^6 - + - \cdots) \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad = 1 + 8(1\cdot q + 2\cdot q^2 + 2\cdot q^3 + 1\cdot q^4 + 2\cdot q^5 + 4\cdot q^6 + \cdots),\end{aligned}$$

and it is immediately clear that, for α prime, q^α has the coefficient 2, that is, a value different from zero. Therefore, certainly every *prime* number is the sum of four or fewer positive squares, and using a classical elementary identity,[3] every positive integer is seen to possess this property. In this case we succeed because we know enough concerning the properties of the function represented

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[1] **(21)**, **(22b)**. For summary of **(22b)** see footnote on first page of Part II.

[2] *Fundamenta nova theoriae functionum ellipticarum*, 1829, p. 188.

[3] $$(a^2 + b^2 + c^2 + d^2)\cdot(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = A^2 + B^2 + C^2 + D^2,$$

where

$$\begin{aligned}A &= a\alpha - b\beta - c\gamma + d\delta, & B &= a\beta + b\alpha - c\delta - d\gamma, \\ C &= c\alpha + a\gamma + d\beta + b\delta, & D &= d\alpha - a\delta - c\beta + b\gamma.\end{aligned}$$

by $(1 + 2q^1 + 2q^4 + \cdots)^4$. Except for certain simple types of generating functions (rational functions) we must admit that this is something very exceptional. Some of Hardy's important papers on the new theory[1] are closely connected with certain related problems. He gives also references to the many important papers which have been written on the problem of breaking up a number into a given number of positive squares, based on the theory of theta-series.

VI. As a last example of generating functions we select Waring's problem. In 1770, Waring stated,[2] without any attempt at proof, an extension to higher powers of the theorem that every number is equal to the sum of four non-negative squares. Waring's statement is: *For every positive integral exponent k there exists a positive integer g_k, depending only on k, such that every positive integer is the sum of g_k or fewer positive kth powers.* He stated that for example every natural number is the sum of nine or fewer cubes; of nineteen or fewer fourth powers, and other cases. That at least nine cubes are required follows from the fact that $23 = 2 \cdot 2^3 + 7 \cdot 1^3$ cannot be represented by fewer cubes; similarly, $79 = 4 \cdot 2^4 + 15 \cdot 1^4$ requires nineteen fourth powers.

Starting about the middle of last century, the finiteness of g_k was gradually proved by means of elementary, but sometimes very complicated, modifications of the theorem $g_2 = 4$ and of similar theorems, first for $k = 4$, then for $k = 3, 5, 6, 7, 8, 10, 12, 14$,[3] and finite upper bounds for the g_k in these cases were obtained or indicated. The general existence proof of g_k for all positive integers k was established by Hilbert, in 1909, in a famous paper.[4] The main part of Hilbert's proof is of transcendental character, since it is based on the transformation of a certain quintuple integral (25-fold in the first presentation, in the *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen). The transcendental character of Hilbert's proof has been gradually eliminated by various authors, but it still has the character of a pure existence proof. It is in connection with Waring's problem that Hardy and Littlewood have made the most fundamental and revolutionizing progress.[5] If we are to use generating functions in Waring's problem, the problem would, for the case $k = 3$, present itself as follows:

In $f(x) = (1 + x^{1^3} + x^{2^3} + \cdots)^s = 1 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots,$ *how large must the exponent s be chosen in order that all c_n shall be* > 0?

As a matter of fact, for reasons already mentioned, one is much more interested in the number s which is large enough to ensure that all c_n for sufficiently large n are positive, thus determining a number G_3 such that every *sufficiently large*

[1] (14), (17).

[2] Waring, *Meditationes Algebraicae*, 1770.

[3] (1), (2), (18) or Kempner, *Das Waringsche Problem*, Diss. Göttingen, 1912.

[4] Mathematische Annalen, vol. 67 (1909), p. 281. Hilbert's proof does not give a method of actually determining an upper bound for g_k, however large, for any single k. (It is true that Hilbert gives some indications as to how an upper bound might be determined from his paper, but these suggestions have never been carried out. The results which might be thus obtained would certainly be incomparably less precise than the results obtained by Hardy and Littlewood in their new work.)

[5] (15), (18), (19), (20), (22), also (23), (24), (25), (26), (27), (28), (30), (31).

number may be represented as the sum of not more than G_3 cubes (and similarly for kth powers). From its definition, $G_k \leqq g_k$. This inequality is very interesting. There can be, from divers considerations, little doubt that for all $k \geqq 3$ really $G_k < g_k$, but the only cases for which this question is settled, even using all the new results of Hardy and Littlewood, is $G_2 = g_2 = 4$; $G_3 \leqq 8$, $g_3 = 9$.[1] If we were in possession of a theory of the series $(1 + x^{1^3} + x^{2^3} + \cdots)^s$ in the sense in which we happen to possess a theory of the series $(1 + 2x^{1^2} + 2x^{2^2} + \cdots)^4$ in the theory of elliptic functions, we might hope to discover some other expansion of the function from which we could learn something about our coefficients in $\sum c_n x^n$. However, we have no such theory; the elliptic function theory employs only functions of the type mentioned last. Hardy's and Littlewood's main contributions[2] to Waring's problem may be briefly said to consist in the following:

(1) They invent means of proving that, for $s > (k - 2)2^{k-1} + 5$, at most a finite number of coefficients of $(1 + x^{1^k} + x^{2^k} + \cdots)^s = \sum c_n x^n$ vanish, that is, they prove that all sufficiently large numbers are a sum of $(k - 2)2^{k-1} + 5$ or fewer positive kth powers.

(2) Beyond this they determine the "order of magnitude" of c_n as a function of n, thus determining for any given exponent k the *order of magnitude* of the number of different manners in which a given large number n may be written as the sum of s or fewer kth powers. It is reasonably safe to say that nobody had dreamed that our mathematical power was great enough to even attempt a problem like this.

7. We naturally ask whether no work was done, with the exception of the large but very special body of results connected with the theta-series, on the basis of Euler's generating functions during the century and a half between Euler's *Introductio* and Hardy's and Littlewood's and Ramanujan's accomplishments. The answer is that an enormous amount of work was done in this direction, first of a formal "Combinatory Analysis" character, incorporated by MacMahon into (**5**), then about the middle of last century powerful algebraic methods were applied by, among others, Cayley and Sylvester.[3] Also, in simple cases, the Cauchy calculus of residues was employed to determine the coefficient c_n of the expanded power series. At present, we illustrate these algebraic methods by an example which has been often used for this purpose, so by Hardy.[4] We do this to bring out two points.

[1] Since a cube is of one of the three forms $9k + 1$, $9k - 1$, $9k$, it is clear that every number of the form $9k \pm 4$ requires at least four positive cubes, *i.e.*, $G_3 \geqq 4$. A situation similar to the one revealed by the inequality $G_3 < g_3$ has been known to exist, since a century, in another problem. We consider the polygonal numbers of order m: $c_\mu = \frac{1}{2}\cdot\mu(\mu - 1)(m - 2) + \mu$; $c_0 = 0$, $c_1 = 1$, $c_2 = m, \cdots$. It is clear that the number $m - 1$ requires $m - 1$ summands from the set c_μ. Yet, Legendre (*Théorie des nombres*, vol. 2, 3d ed., 1830, p. 350) proved that every even number $> 28m^3$ is a sum of *four or fewer* c_μ's, every odd number $> 28m^3$ a sum of *five or fewer* c_μ's of which at most four need be chosen > 1. The extra summand 1 which appears in the statement for odd numbers is, not unlikely, only due to the method of proof.

[2] Hardy and Littlewood really do more; we refrain from quoting the formulæ embodying their results.

[3] (**1**), (**2**), (**3**), (**4**), (**5**), (**6**), (**7**).

[4] (**13**), (**18**).

Consider the number $N(n)$ of partitions, with repetition, of n into summands 1, 2, 3. For example, $N(5) = 5$.

A generating function is clearly

$$f(x) = (1 + x^1 + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)$$
$$= [(1 - x)(1 - x^2)(1 - x^3)]^{-1}.$$

Breaking up into partial fractions:

$$f(x) = \frac{1}{6(1-x)^3} + \frac{1}{4(1-x)^2} + \frac{17}{72(1-x)} + \frac{1}{8(1+x)} + \frac{1}{9(1-\omega x)}$$
$$+ \frac{1}{9(1-\omega^2 x)}, \qquad \omega^2 + \omega + 1 = 0,$$

$$f(x) = \tfrac{1}{12}(1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots) + \tfrac{1}{4}(1 + 2x + 3x^2 + \cdots)$$
$$+ \tfrac{17}{72}(1 + x + x^2 + \cdots) + \tfrac{1}{8}(1 - x + x^2 - \cdots)$$
$$+ \tfrac{1}{9}(1 + \omega x + \omega^2 x^2 + \cdots) + \tfrac{1}{9}(1 + \omega^2 x + \omega^4 x^2 + \cdots),$$

from which

$$c_n = \frac{17}{72} + \frac{1}{8}(-1)^n + \frac{\alpha}{9} + \frac{1}{4}(n+1) + \frac{1}{12}(n+1)(n+2),$$

where $\alpha = \omega + \omega^2$, $\omega^2 + \omega$, $1 + 1$, according as $n \equiv 1, 2, 0 \bmod 3$, resp.;

$$c_n = -\frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3} + \frac{1}{12}(n^2 + 6n + 5 + 4)$$
$$= -\frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3} + \frac{1}{12}(n+3)^2.$$

This is the formula for $N(n)$, and it therefore represents a positive integer. Hence, from

$$\left| -\frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3} \right| < \frac{1}{2},$$

$N(n) =$ *integer closest to* $\frac{1}{12}(n+3)^2$.

The number which we have found is Sylvester's "denumerant,"

$$N(n) = \frac{n}{\overline{\overline{1,\ 2,\ 3}}}\,.$$

Similarly, the denumerant

$$N(n) = \frac{n}{\overline{\overline{a_1,\ a_2,\ \cdots,\ a_m}}}$$

represents the number of partitions of n into summands chosen from $a_1, a_2, \cdots, a_m$. The rudiments of a calculus of denumerants exist, but the formulæ are for the most part recurrence formulæ which do not give a satisfactory answer to the general problem of determining the value of $N(n)$. Jacobi, who, as we have already seen, also worked in the field of partitions, used for $\dfrac{n}{\overline{\overline{1,\ 2,\ 3}}}$ the notation $N(n = 1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3)$, $x_i \geqq 0$.

The following quotation from Sylvester will show the importance he attached to the theory of partitions: "Partitions constitute the sphere in which analysis lives, moves and has its being; and no power of language can exaggerate or point out too forcibly the importance of this till recently almost neglected but vast, subtle and universally permeating element of algebraic thought and expression." (Quoted by Bachmann, (1) p. 104.)

8. The two points to which we wish to call attention are these:

I. $\dfrac{n}{\overline{1, 2, 3}}$ involves in a natural and, indeed, probably unavoidable manner complex roots of unity.

II. It consists of parts of two different types, one part monotonically increasing with n, the other part of oscillatory character.

These latter form essentially the famous "waves" of Sylvester.[1] In a rather formal manner these denumerants have been carefully examined and we must, in justice, say that the problem of finding the total number $p(n)$ of partitions of a number into positive summands was, in a certain sense, solved by such methods in 1914 by Csorba.[2] The formula is of elementary character, containing, as it does, only finite summations and products. Little more than this can be said in its favor. It would probably be impossible to obtain from it any information concerning the order of magnitude of $p(n)$; hopeless to attempt to read from it when or whether $p(n)$ has a value $\neq 0$; impossible to carry through the computations for any but the very smallest values of n.

9. Hardy, in his beautiful inaugural lecture (18), illustrates the new method of procedure in problems of partition by a brief discussion of the problem treated in section 7. It is first mentioned that the following method must have suggested itself not to one or two, but to many mathematicians. It amounts to the determination of the residue of an analytic function. We consider

$$f(x) = \frac{1}{(1 - x^1)(1 - x^2)(1 - x^3)} = 1 + c_1x^1 + c_2x^2 + \cdots + c_nx^n + \cdots$$

as an analytic function of the complex variable x. Its sole singularities are simple poles at certain points of the circumference of the unit circle, namely, at $1, -1, \omega, \omega^2$ $(\omega^2 + \omega + 1 = 0)$. Therefore, by Cauchy's integral, c_n, the number of partitions required, is given by

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx,$$

where the integration is carried around a circle Γ of radius $r < 1$ about the origin. This integral can be evaluated in the following manner: we deform the path as in Figures 1, 2, 3 (see Part II). Then

$$\int_\Gamma = \int_{\Gamma_1} - \int_{C_1} - \int_{C_2} - \int_{C_3} - \int_{C_4},$$

[1] (1), (2).

[2] *Mathematische Annalen*, 1914, vol. 75, pp. 545–568.

where we may choose the circle Γ_1 as large as we like, and where we assume that all integrals are taken in a positive sense (thus accounting for the four minus signs in the last formula). But on account of the denominator x^{n+1} in the integrand we are sure that, for sufficiently large values of n, $|\int_{\Gamma_1}| < \epsilon$, $\epsilon > 0$ arbitrarily small, so that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(x)}{x^{n+1}}\,dx = -\frac{1}{2\pi i}\left\{\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right\}\frac{f(x)}{x^{n+1}}\,dx.$$

The problem thus reduces to an exercise in Cauchy's Calculus of Residues.

10. *Cannot this method be extended to all problems of partition?* For example, to the problem II of section 6 (total number of partitions of n, with the generating function $\Pi_{i=1}^{\infty}(1-x^i)^{-1}$), or to the problem VI (Waring's problem, with a generating function $(1+x^{1^k}+x^{2^k}+\cdots)^s$)? We do not have to seek very far for an answer to this question. In cases where our generating function has a finite number of singularities, of so mild a character that we may hope to integrate $f(x)/x^{n+1}$ around each one separately, and if $f(x)$ exists in the entire complex plane and behaves for large values of $|x|$ in such manner that we may expect to integrate $f(x)/x^{n+1}$ around a large circle about the origin (or around any large closed contour extending far enough in all directions), we may hope to succeed by this method.

Unfortunately, these prerequisites are satisfied only in the simplest cases, notably when $f(x)$ is a simple rational function. This represents but a very small subclass of the totality of partition problems, and these not the most interesting or important.

Consider the two problems last mentioned (II and VI of section 6). In problem II, $f(x) = [(1-x^1)(1-x^2)(1-x^3)\cdots]^{-1}$ has obviously a singular point at every point on the unit circle representing a root of unity, that is, the singular points are everywhere dense on the circumference so that the unit circle is a natural boundary for the function, across which the function can in no plausible way be extended, quite apart from the fact that it would not be possible to integrate around the individual singularities. Similarly, in problem VI, it is known from general theorems in the theory of analytic functions that $(1+x^{1^k}+x^{2^k}+\cdots)$ and therefore also $f(x) = (1+x^{1^k}+x^{2^k}+\cdots)^s$ represents, for $k > 1$, a function which has the unit circle for a natural boundary. It is therefore to be expected that an attack based on the evaluation of the contour integral—up to this point the work can usually be carried, except for possible minor modifications—will require essentially novel methods and a most searching examination of the integrand and its behavior on or near the natural boundary.[1] This is exactly what Hardy, with Ramanujan and Littlewood, and making use of an independent investigation by Weyl, has succeeded in doing.[2]

[1] Compare with this the simplicity of the problem of finding the number of *compositions* of n. The generating function $x^s\cdot(x-1)^{-s}$ has $x = 1$ as sole singularity in the finite part of the plane.

[2] It is true that Hardy and Littlewood arrange their proofs in the manner that they start out from an arbitrary n (the number to be partitioned) which is then held fixed, so that the number of summands is limited ($\leqq n$) for any mode of partition. Therefore, for the given n, the gener-

11. As the authors insist, there is nothing novel in their plan of attack up to the point of evaluating the integral, or, to be accurate, of determining its order of magnitude as a function of n for large values of n. To quote again Landau, an enthusiastic admirer of the English mathematicians and possibly the world's foremost authority on the subject of the analytic theory of numbers: *The attempt to use Cauchy's integral seemed almost "schülerhaft,"* in view of the obstacles which were to be expected along this road. The work is evidence enough that genius of the highest order and most dogged perseverance were required to overcome these difficulties. Hardy tells us that in the treatment of Waring's problem Littlewood and he were at one point held up for two years, and that they finally succeeded only by linking up their work with the independent investigation by Weyl mentioned before.

Hardy's and Littlewood's later work which is still in full progress tends to smooth off somewhat the roughest and most painful bumps of their first trail. That the work is in its final shape seems hardly probable. Already parts have been simplified by Landau,[1] Weyl,[1] Ostrowski.[1] Besides, Landau[2] and Siegel[2] have extended Hardy's and Littlewood's results to more general problems. The papers in which the theory is developed are scattered in many journals, as is seen from the literature list. The rudiments of the theory appear to go back to at least the Cambridge Mathematical Congress of 1912. For this, compare two long papers by Hardy and Littlewood.[3] The influence of Hardy on Ramanujan and the influence of Ramanujan on Hardy can be traced from Hardy's very appreciative article in honor of the memory of the Indian Mathematician.[4] The first papers dealing with the theory from the standpoint of analytic functions of a complex variable were (**9**), (**10**), (**11**), (**12**), (**13**). Before the theory was farther developed, Ramanujan died (1920). The most delicate parts of the new theory were worked out by Hardy and Littlewood in joint papers.

Up to the present time co-workers in the new field seem to comprise, as far as can be judged by the articles printed, German mathematicians exclusively (Weyl, Landau, Ostrowski, Siegel). Before sketching in greater detail the contents of one of the papers (**13**), we quote two more passages from Landau's

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ating function might have been chosen as $f_1(x) = [(1 - x^1)(1 - x^2) \cdots (1 - x^n)]^{-1}$, instead of allowing for an infinite product, and the singularities (on the circumference of the unit circle) would now be poles (and nowhere dense). It may therefore seem unnecessary to consider generating functions with the unit circle as natural boundary; but, on the other hand, the set of singular points of $f_1(x)$ depends on n, while the set of singular points of $f(x)$ is independent of n. It is likely that the apparently simpler case is really the harder one to treat, in somewhat the same sense in which it is ordinarily harder to deal with a large number of terms of a series than to deal with the infinite series and harder to deal with a finite summation than with a definite integral. In the same way, in problem VI for a given n we might think of choosing the rational function $(1 + x^{1^k} + x^{2^k} + \cdots + x^{n^k})^s$, or a still simpler function, instead of the transcendental function $(1 + x^1 + x^{2^k} + \cdots \text{ in inf.})^s$.

[1] (**25**), (**26**), (**30**), (**31**), (**27**).

[2] (**26**), (**28**).

[3] Hardy and Littlewood, "Some problems of Diophantine approximation," *Acta Mathematica*, vol. 37, 1914, pp. 155–191 and pp. 192–239.

[4] *Proceedings of the London Mathematical Society*, 2d ser., vol. 19 (1921), xl–lviii. See also this MONTHLY (*1920*, 316, 338; *1921*, 136, 224, 458).

excellent report,[1] already referred to: *Etwas ganz Gewaltiges ist es, was meine Englischen Freunde geleistet haben,* and: *Ich freue mich, erlebt zu haben, dass die additive Zahlentheorie Methoden erhalten hat.*

(*To be concluded in the next issue.*)

A CONTRIBUTION OF LEIBNIZ TO THE HISTORY OF COMPLEX NUMBERS.[2]

By R. B. McCLENON, Grinnell College.

One of the most important and fascinating chapters in the history of mathematics is the development of the concept of complex numbers. Certain parts of this development have not yet been adequately treated by writers on the history of mathematics; and among these is to be mentioned the work of Leibniz.

It may be worth while to recall that neither the Hindu nor the Arabian algebraists, nor the medieval Europeans, had recognized any possibility of attaching a meaning to a square root of a negative number; indeed it was only the exceptional writer who recognized even *negative* roots of equations.[3] In the sixteenth century, Tartaglia and Cardan, in the formula for the roots of the cubic $x^3 + ax = b$, viz.,

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}},$$

noticed that in case $(b^2/4) + (a^3/27)$ were negative, the value of x would involve an "impossible" expression; and accordingly this case came to be known as the "irreducible case," a term which persists down to the present time. Vieta (1540–1603), the greatest algebraist of his time, contented himself with working out a trigonometric solution for the cubic in this case.[4] Descartes, in connection with his "rule of signs," mentioned the existence of imaginary roots in an algebraic equation, but did not enter upon any discussion of them.[5]

It is now almost exactly 250 years since Leibniz, then a young man of 25, first entered upon the serious study of the possibility of getting some clear meaning out of these so-called "impossible" quantities. The inspiration for this work came to him through the study of Bombelli's *Algebra,* a standard work which had been published at Bologna in 1572 and reprinted in 1579. Leibniz was not at all satisfied with Bombelli's discussion of the "Cardan" formula for the

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[1] (24).

[2] Read before the Iowa Section of the Association, April 27, 1923.

[3] For example, Leonardo of Pisa. *Scritti* (ed. Boncompagni), Rome, 1857, II, pp. 239, 243.

[4] Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert,* Leipzig, 1903, pp. 91, 117; Cantor, *Vorlesungen über Geschichte der Mathematik,* vol. II, 2d ed., Leipzig, 1900, p. 636. Vieta's construction is given in detail in Montucla, *Histoire des Mathématiques,* vol. I, 2d ed., Paris, 1799, p. 605 ff.

[5] Cantor, *loc. cit.,* p. 795.

solution of the cubic equation, especially in the irreducible case. In a letter to Huygens,[1] he expresses his dissatisfaction with Bombelli[2] for not accepting Cardan's formula as adequate in this case; and proceeds to make these three assertions: (1) that Cardan's formula is universally valid, (2) that by means of this formula every cubic equation can be solved, and (3) that roots of all even degrees can be formed which contain imaginaries and yet which are themselves real. As an example of this last, Leibniz mentions that

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}. \tag{1}$$

He also says in this same letter that he has found "a method for extracting, either exactly or approximately, the roots of binomials where imaginaries enter."[3] In reply to this communication, Huygens expresses his astonishment at the relation (1) in these words: "The remark which you make concerning roots that can not be extracted, and containing imaginary quantities which when added together give none the less a real quantity, is surprising and entirely new. One would never have believed that $\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}$ would make $\sqrt{6}$, and there is something hidden in this which is incomprehensible to us."[4]

Leibniz evidently spent considerable time and effort on the question of the meaning of imaginary expressions, and the possibility of securing reliable results by applying to them the ordinary laws of algebra; for Gerhardt found among Leibniz's papers a discussion of the solution of algebraic equations which, although undated, bears every evidence of having been written at about this time (1675).[5] It is published in the *Briefwechsel*, pp. 550–564, and although it is one of the first significant documents in the history of complex numbers, it has not hitherto, so far as I know, been described by historians of mathematics.[6] A rather full description of this paper will accordingly be worth while; and not alone because it has historical importance, but also because the clear-cut way in which Leibniz presents many of his points offers valuable suggestions to the teacher of the present day.

After stating the condition under which a quadratic equation will have real roots, Leibniz continues, "But if now a simple, that is, a linear equation, is multiplied by a quadratic, a cubic equation will result, which will have three

[1] *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern*, ed. C. J. Gerhardt, vol. 1, Berlin, 1899, pp. 547, 548.

[2] This dissatisfaction may perhaps not have been altogether justified, as Bombelli in reality shows considerable skill in handling imaginary expressions. Cf. his *L'Algebra, parte maggiore dell'Arithmetica divisa in tre libri*, Bologna, 1572, pp. 292–293.

[3] *Loc. cit.*, p. 549.

[4] *Loc. cit.*, p. 566.

[5] Gerhardt gives the date 1673; but the editors of Huygens's Works think 1675 more probable. The difference is of no moment in this connexion.

[6] Cantor apparently refers to it (*Geschichte*, vol. III, p. 105 (1894 ed.)) but gives no detailed information. He undervalues the significance of equation (1) above when he says: "Leibniz's formula was to be sure a striking example, but it did not bring the question of its real meaning a single step farther." The obvious advance shown in Leibniz's paper, summarized in this article, shows that Cantor's dictum is in this case too severe.

real roots if the quadratic is possible, or two imaginary roots and only one real one if the quadratic is impossible." He then points out that it is exactly in the case where all three roots of the cubic are real that the difficulty in the use of Cardan's formula lies: the roots of $y^3 + qy - r = 0$ being

$$y = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}\,.$$

"How can it be," he says, "that a real quantity, a root of the proposed equation, is expressed by the intervention of an imaginary? For this is the remarkable thing, that, as calculation shows, such an imaginary quantity is only observed to enter those cubic equations that have no imaginary root, all their roots being real or possible, as has been shown by trisection of an angle, by Albert Girard and others.[1] * * * This difficulty has been too much for all writers on algebra up to the present, and they have all said that in this case Cardan's rules fail."

Realizing clearly, then, the nature and difficulty of the problem, involving as its solution did a decisive step in advance of all his predecessors, Leibniz set to work to get to the bottom of the matter. He was led to the solution of the problem by an analogy in a similar situation.[2] "It will be useful to mention how my mind was led to the solution of this problem. I once came upon two equations of this kind: $x^2 + y^2 = b$, $xy = c$, whence $x^2 = \frac{c^2}{y^2}$, and so $\frac{c^2}{y^2} + y^2 = b$ and $y^4 - by^2 + c^2 = 0$, or $y^2 = \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}$, $y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}}$. Substituting therefore this value of y^2 in $x^2 + y^2 = b$, I wrote $x^2 - \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} = 0$, or $x = \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}$. But c was greater than b, and therefore $\sqrt{\frac{b^2}{4} - c^2}$ was an imaginary quantity. However, I knew otherwise that the sum of the unknowns $x + y$ was a real quantity and equal to a certain line d, which puzzled me greatly, for inasmuch as I had deduced from the preceding calculation that $d = x + y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}$, I did not understand how such a quantity could be real, when imaginary or impossible numbers were used to express it. I therefore began to retrace the steps of my calculation, suspecting an error; but in vain, for the result was always the same. At length it occurred to me to try this operation: put

$$d = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}} = A + B;$$

[1] Girard, *Invention nouvelle en l'Algebre*, Amsterdam, 1629. Vieta had given the solution by trigonometry earlier than Girard. (See note 4, p. 369.)

[2] Leibniz, *loc. cit.*, pp. 553 ff. Leibniz uses the sign ⊓ for equality, but otherwise his symbols are as here given. The sentences in quotation marks are literal translations from the Latin original, and all omissions in the midst of a quotation are indicated by asterisks.

hence, squaring both sides,

$$d^2 = A^2 + B^2 + 2AB = \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2} + 2c. \; * * *$$

Therefore $d^2 = b + 2c$, and $d = \sqrt{b + 2c}$. Therefore, equating the two values of d,

$$\sqrt{b + 2c} = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}.$$

If we put $b = 2$ and c also $= 2$, there results

$$\sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}.$$

I do not remember to have noted a more singular and paradoxical fact in all analysis; for I think I am the first one to have reduced irrational roots, imaginary in form, to real values without extracting them."

Thus Leibniz was led to what he called a *sixth arithmetical operation*, viz., the reduction of imaginary expressions to real form. He then proceeds to apply this operation to the Cardan form of the roots of a cubic equation. And first, he extends the principle of the preceding work with square roots to cube roots, as follows:

"Let $2b$ be a certain quantity: it can be written also in this way: $b + \sqrt{-ac} + b - \sqrt{-ac}$. For although $\sqrt{-ac}$ is an imaginary quantity, yet the sum is none the less real, since the imaginaries are destroyed. Let this formula be divided into two parts, the binomial $b + \sqrt{-ac}$ and the 'apotome'[1] $b - \sqrt{-ac}$, and let us investigate the cube of each separately: the cube of $b + \sqrt{-ac}$ will be

$$b^3 - ac\sqrt{-ac} - 3bac + 3b^2\sqrt{-ac},$$

and the cube of $b - \sqrt{-ac}$ will be

$$b^3 + ac\sqrt{-ac} - 3bac - 3b^2\sqrt{-ac},$$

and therefore

$$\sqrt[3]{\begin{matrix} b^3 - ac\sqrt{-ac} \\ -3bac + 3b^2\sqrt{-ac} \end{matrix}} + \sqrt[3]{\begin{matrix} b^3 + ac\sqrt{-ac} \\ -3bac - 3b^2\sqrt{-ac} \end{matrix}},$$

or

$$\sqrt[3]{\begin{matrix} b^3 \\ -3bac \end{matrix} + \sqrt{\begin{matrix} -a^3c^3 \\ +6a^2c^2b^2 \\ -9b^4ac \end{matrix}}} + \sqrt[3]{\begin{matrix} b^3 \\ -3bac \end{matrix} - \sqrt{\begin{matrix} -a^3c^3 \\ +6a^2c^2b^2 \\ -9b^4ac \end{matrix}}} = 2b,$$

or $b + \sqrt{-ac} + b - \sqrt{-ac}$.

"But if now from a binomial of this kind the cube root can always be extracted, as it can from this one, then certainly the imaginaries can always be

[1] This is the designation used by Euclid in Book X of the *Elements*. (Heath's edition, vol. III, pp. 5–7.) Cf. Heath, *History of Greek Mathematics*, Oxford, 1921, vol. I, p. 407.

removed from a binomial and an 'apotome' when they are joined together. But since it can not always be extracted from a given expression in the form

$$\sqrt[3]{\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}},$$

such as cubic equations give, that is, since the given quantity $r/2$ can not always be separated into two, $b^3 - 3bac$, nor the given quantity $(r^2/4) - (q^3/27)$ into three, $-a^3c^3 + 6a^2c^2b^2 - 9b^4ac$, without another equation, equally as difficult as the given one, therefore it happens that we can not always eliminate the imaginaries from real quantities.

"But it will be useful to give examples in rational numbers. Take the equation, which also Albert Girard used:[1] $x^3 - 13x - 12 = 0$, whose true root is 4. From the formulas of Scipio Ferro or Cardan,

$$x = \sqrt[3]{6+\sqrt{\frac{-1225}{27}}}+\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}.$$

I will prove that this expression is correct and real, and must be admitted. Put $x = 2 + \sqrt{-\tfrac{1}{3}} + 2 - \sqrt{-\tfrac{1}{3}}$, and certainly x will be equal to 4, as the equation postulated. Now let us see if the Cardan formula can be derived from this. Certainly by cubing and applying the above formula $b + \sqrt{-ac} + b - \sqrt{-ac}$ to this, making $b = 2$, and $ac = \tfrac{1}{3}$, we shall have for the cube of $2 + \sqrt{-\tfrac{1}{3}}$ this formula:[2]

$$\begin{matrix} +8 \\ -3,2,\tfrac{1}{3} \end{matrix} = -2 + \sqrt{-}\left\{\begin{array}{l} -\dfrac{1}{27} \\ +6,\dfrac{1}{9},\ 4 = \dfrac{72}{27} \\ -9,\ 16,\dfrac{1}{3} = \dfrac{-1296}{27} \end{array}\right.$$

or, adding up, $6 + \sqrt{\frac{-1225}{27}}$. In the same way the cube of $2 - \sqrt{-\tfrac{1}{3}}$ will be $6 - \sqrt{\frac{-1225}{27}}$, and hence $\sqrt[3]{6+\sqrt{\frac{-1225}{27}}}$ will be $2 + \sqrt{-\tfrac{1}{3}}$ and $\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}$ will be $2 - \sqrt{-\tfrac{1}{3}}$ and, by joining the binomial to the 'apotome,' x or $\sqrt[3]{6+\sqrt{\frac{-1225}{27}}}+\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}$ will be the same as $2 + \sqrt{-\tfrac{1}{3}} + 2 - \sqrt{-\tfrac{1}{3}}$, that is, will be 4, as was proposed to show."

Leibniz adds an example where a negative number (-6) is a root of the

[1] *Loc. cit.*

[2] Leibniz, it will be observed, here uses commas to indicate multiplication; he later introduced the dot which has been universally adopted. Cf. Tropfke, *Geschichte der Elementarmathematik*, vol. 2 (2d ed.), Berlin and Leipzig, 1921, p. 24.

cubic, $x^3 - 48x - 72 = 0$, and establishes the fact that

$$\sqrt[3]{36 + \sqrt{-2800}} + \sqrt[3]{36 - \sqrt{-2800}} = -6.$$

He finally takes the bull by the horns, substitutes in the cubic equation $x^3 - qx - r = 0$ the expression for x given by the Cardan formula, and shows by actually carrying out the algebraic reductions that the equation is thus satisfied.

The rest of the memoir is devoted to a discussion of the great difficulty of extending the methods of solution to the 5th, 6th, and higher degree equations, with emphasis upon the necessity of doing this. The concluding sentences are as follows:[1] "For this evil I have found a remedy and obtained a method, by which without experimentation the roots of such binomials can be extracted, imaginaries being no hindrance, and not only in the case of cubics but also in higher equations. This invention rests upon a certain peculiarity which I will explain later. Now I will add certain rules derived from the consideration of irrationals (although no mention is made of irrationals), by which a rational root can easily be extracted from them."

Here the manuscript breaks off; no doubt Leibniz became convinced that he could not carry his "method" as far as he had at first supposed, and thus the essay was left unfinished. But the influence of this work of Leibniz is seen in the writings of Tschirnhausen on the one hand and of John Bernoulli on the other, each of whom received stimulation and valuable assistance from Leibniz in the field of algebra. Thus this particular memoir on complex numbers, although remaining unpublished for two centuries, is an interesting and important document in the history of mathematics.

VECTOR ANALYSIS OF A SURFACE.

By J. B. REYNOLDS, Lehigh University.

A variable vector $\mathbf{r}$ depending upon two independent scalars u and v may be taken as defining a surface. If v is constant, the vector $\mathbf{r}$ defines one of a family of curves lying on the surface called u-lines, while if u is constant, we have one of the family of v-lines lying upon the surface. If there is a relation $\phi(u, v) = 0$, we have a general curve upon the surface.

Now $\mathbf{t} = (d\mathbf{r}/du)$ is a vector parallel to a tangent to the general curve, and $\boldsymbol{t}_u = (\partial \mathbf{r}/\partial u)$ and $\boldsymbol{t}_v = (\partial \mathbf{r}/\partial v)$ are vectors parallel, respectively, to tangents to the u- and v-lines.

The vector $\boldsymbol{n}'' = \boldsymbol{t}_u \times \boldsymbol{t}_v$ is parallel to the surface normal, while the vector $d\boldsymbol{t}_1/du$, $\boldsymbol{t}_1$ being a unit vector parallel to the tangent, is parallel to the principal normal to the curve, and lies in the osculatory plane

Again, $\boldsymbol{n}' = \boldsymbol{t}' \times \boldsymbol{t}$ is a vector parallel to the binormal, $\boldsymbol{t}'$ being $d\boldsymbol{t}/du$.

[1] *Briefwechsel*, p. 564.

Considering v as a function of u, and calling $p = (dv/du)$, we have

$$\boldsymbol{t} = \frac{d\boldsymbol{r}}{du} = \boldsymbol{t}_u + p\boldsymbol{t}_v$$

in which $\boldsymbol{t}$ is parallel to the tangent to the curve and will be defined for a chosen value of p.

Now if ω is the angle between the principal normal to the curve and the normal to the surface at any point, we have, using the notation above,

$$\boldsymbol{n}\cdot\boldsymbol{n}'' = \boldsymbol{n}_0\boldsymbol{n}_0'' \cos \omega,$$

the subscript zero indicating the absolute value of the vector, or

$$\frac{d\boldsymbol{t}_1}{du}\cdot(\boldsymbol{t}_u \times \boldsymbol{t}_v) = \left(\frac{d\boldsymbol{t}_1}{du}\right)_0 (\boldsymbol{t}_u \times \boldsymbol{t}_v)_0 \cos \omega$$

But, R being the radius of curvature of the curve, we have

$$R = \frac{\boldsymbol{t}_0}{\left(\frac{d\boldsymbol{t}_1}{du}\right)_0},$$

so we may write

$$\frac{\cos \omega}{R} = \frac{\frac{d\boldsymbol{t}_1}{du}\cdot(\boldsymbol{t}_u \times \boldsymbol{t}_v)}{\boldsymbol{t}_0(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}. \tag{1}$$

If $\omega = 0$ or π, R becomes the radius of curvature ρ of a normal section of the surface, giving $R = \pm \rho \cos \omega$, which is Meusnier's theorem.

We have, therefore, for a normal section

$$\frac{1}{\rho} = \frac{\left(\frac{d\boldsymbol{t}_1}{du}\right)\cdot(\boldsymbol{t}_u \times \boldsymbol{t}_v)}{\boldsymbol{t}_0(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}. \tag{2}$$

From $\boldsymbol{t} = \boldsymbol{t}_u + p\boldsymbol{t}_v$ it follows that $\boldsymbol{t} \times \boldsymbol{t}_v = \boldsymbol{t}_u \times \boldsymbol{t}_v$ since $\boldsymbol{t}_v \times \boldsymbol{t}_v = 0$ so that

$$\left(\frac{d\boldsymbol{t}_1}{du}\right)\cdot(\boldsymbol{t}_u \times \boldsymbol{t}_v) = \frac{d\boldsymbol{t}_1}{du}\cdot(\boldsymbol{t} \times \boldsymbol{t}_v) = \left(\frac{d\boldsymbol{t}_1}{du} \times \boldsymbol{t}\right)\cdot\boldsymbol{t}_v = S\boldsymbol{n}'\cdot\boldsymbol{t}_v$$

because $(d\boldsymbol{t}_1/du) \times \boldsymbol{t}$ is parallel to the binormal.

To determine the value of the scalar multiplier S, we note that our assumption is

$$\frac{d\boldsymbol{t}_1}{du} \times \boldsymbol{t} = S\boldsymbol{n}' = S(\boldsymbol{t}' \times \boldsymbol{t}).$$

Now

$$\boldsymbol{t}_1 = \frac{\boldsymbol{t}}{\boldsymbol{t}_0} = \frac{\boldsymbol{t}}{\sqrt{\boldsymbol{t}\cdot\boldsymbol{t}}},$$

therefore

$$\frac{dt_1}{du} = \frac{(t\cdot t)t' - t(t\cdot t')}{(t\cdot t)^{3/2}},$$

whence

$$\frac{dt_1}{du} \times t = \frac{(t\cdot t)(t' \times t) - 0}{(t\cdot t)^{3/2}} = \frac{t' \times t}{t_0},$$

whence $S = 1/t_0$, giving

$$\frac{1}{\rho} = \frac{1}{t_0^2} \frac{n'\cdot t_v}{(t_u \times t_v)_0} = \frac{(t' \times t)\cdot t_v}{t_0^2(t_u \times t_v)_0}. \tag{3}$$

Since the direction of the curve at a point is determined by a constant value of p, we have, from $t = t_u + pt_v$,

$$t' = t_{uu} + 2pt_{uv} + p^2 t_{vv}$$

so that

$$(t' \times t)\cdot t_v = \{t_{uu} + 2pt_{uv} + p^2 t_{vv}\}\cdot(t_u \times t_v),$$

whence

$$\frac{1}{\rho} = \frac{\{t_{uu} + 2pt_{uv} + p^2 t_{vv}\}\cdot n''}{\{t_u\cdot t_u + 2pt_u\cdot t_v + p^2 t_v\cdot t_v\}n_0''}. \tag{4}$$

To find the maximum and minimum values of ρ we put $(\partial/\partial p)(1/\rho) = 0$, resulting in the equation

$$(t_u\cdot t_v t_{vv} - t_v\cdot t_v t_{uv})\cdot n'' p^2 + (t_u\cdot t_u t_{vv} - t_v\cdot t_v t_{uu})\cdot n'' p + (t_u\cdot t_u t_{uv} - t_u\cdot t_v t_{uu})\cdot n'' = 0.$$

If p and p' are the roots of this equation,

$$p + p' = \frac{(t_v\cdot t_v t_{uu} - t_u\cdot t_u t_{vv})\cdot n''}{(t_u\cdot t_v t_{vv} - t_v\cdot t_v t_{uv})\cdot n''}$$

and

$$pp' = \frac{(t_u\cdot t_u t_{uv} - t_u\cdot t_v t_{uu})\cdot n''}{(t_u\cdot t_v t_{vv} - t_v\cdot t_v t_{uv})\cdot n''}, \tag{5}$$

whence

$$t_v\cdot t_v pp' + t_v\cdot t_u(p + p') + t_u\cdot t_u = 0 \tag{6}$$

or

$$(t_u + pt_v)\cdot(t_u + p't_v) = 0.$$

That is, sections of maximum and minimum curvature are perpendicular to each other.

We have then, if ρ_1 and ρ_2 are the principal radii of curvature, simplifying (4) by (6) and the expression for $p + p'$,

$$\frac{1}{\rho_1} = \frac{(t_{uv} + pt_{vv})\cdot n''}{(t_u\cdot t_v + pt_v\cdot t_v)n_0''} \quad \text{and} \quad \frac{1}{\rho_2} = \frac{(t_{uv} + p't_{vv})\cdot n''}{(t_u\cdot t_v + p't_v\cdot t_v)n_0''}, \tag{7}$$

and adding, we obtain for the mean curvature at any point by means of equations

(5) and (6):

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{(2t_u\cdot t_v t_{uv}-t_v\cdot t_v t_{uu}-t_u\cdot t_u t_{vv})\cdot n''}{n_0''[(t_u\cdot t_v)^2-(t_u\cdot t_u)(t_v\cdot t_v)]}.$$

But since

$$\begin{aligned} n_0''^2 &= (t_u\times t_v)\cdot(t_u\times t_v)=t_u\cdot t_v\times(t_u\times t_v)\\ &= t_u\cdot[t_u(t_v\cdot t_v)-t_v(t_v\cdot t_u)]\\ &= (t_u\cdot t_u)(t_v\cdot t_v)-(t_u\cdot t_v)^2, \end{aligned}$$

we have

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{(t_v\cdot t_v t_{uu}+t_u\cdot t_u t_{vv}-2t_u\cdot t_v t_{uv})\cdot n''}{n_0''^3}. \tag{8}$$

In like manner we find for the total curvature

$$\frac{1}{\rho_1\rho_2}=\frac{(t_{uu}\cdot n'')(t_{vv}\cdot n'')-(t_{uv}\cdot n'')^2}{n_0''^4}. \tag{9}$$

At umbilics the curvature of all normal sections is the same, so by (4)

$$\frac{t_{uu}\cdot n''}{t_u\cdot t_u}=\frac{t_{uv}\cdot n''}{t_u\cdot t_v}=\frac{t_{vv}\cdot n''}{t_v\cdot t_v} \tag{10}$$

for the determination of such points.

If $p = 0$ and $p' = \infty$, we get from (4) for the curvature of principal sections

$$\frac{1}{\rho_1}=\frac{t_{uu}\cdot n''}{(t_u\cdot t_u)n_0''},\qquad \frac{1}{\rho_2}=\frac{t_{vv}\cdot n''}{(t_v\cdot t_v)n_0''}. \tag{11}$$

If the curves of reference are lines of curvature, we must have the conditions that they intersect perpendicularly, and that $p' = \infty$ when $p = 0$. Considering the quadratic for p we may write for these conditions

$$t_u\cdot t_v = 0 \qquad\text{and}\qquad n''\cdot t_{uv}=0. \tag{12}$$

So, basing our analysis on lines of curvature, we have by (4)

$$\frac{1}{\rho_1}=\frac{(t_{uu}+p^2 t_{vv})\cdot n''}{(t_u\cdot t_u+p^2 t_v\cdot t_v)n_0''}$$

and

$$\frac{1}{\rho_2}=\frac{(t_{uu}+p'^2 t_{vv})\cdot n''}{(t_u\cdot t_u+p'^2 t_v\cdot t_v)n_0''},$$

whence, since the sections are perpendicular and

$$pp'=\frac{-t_u\cdot t_u}{t_v\cdot t_v},$$

we find

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{t_{uu}\cdot n''}{t_u\cdot t_u n_0''}+\frac{t_{vv}\cdot n''}{t_v\cdot t_v n_0''}$$
$$=\frac{1}{n_0''}\left(\frac{t_{uu}}{t_u\cdot t_u}+\frac{t_{vv}}{t_v\cdot t_v}\right)\cdot n''.$$

That is, the sum of the curvatures of mutually perpendicular sections is constant.

Again, using lines of curvature as reference lines, in general, by (4)

$$\frac{1}{\rho}=\frac{(t_{uu}+p^2t_{vv})\cdot n''}{(t_u\cdot t_u+p^2t_v\cdot t_v)n_0''}.$$

If θ is the angle between the plane of the normal section in question and the plane of the normal section containing the tangent to the u-line at the point, we have

$$\tan\theta=\frac{(t\times t_u)_0}{t\cdot t_u}$$
$$=\frac{[(t_u+pt_v)\times t_u]_0}{(t_u+pt_v)\cdot t_u}$$
$$=\frac{p(t_v\times t_u)_0}{t_u\cdot t_u}$$
$$=\frac{pn_0''}{t_u\cdot t_u},$$

whence

$$p^2=\frac{(t_u\cdot t_u)^2}{n_0''^2}\tan^2\theta=\frac{t_u\cdot t_u}{t_v\cdot t_v}\tan^2\theta,$$

because $n_0''^2=(t_u\cdot t_u)(t_v\cdot t_v)-(t_u\cdot t_v)^2$ and, in this case, $t_u\cdot t_v=0$, so that we have

$$\frac{1}{\rho}=\frac{t_{uu}\cdot n''}{n_0''t_u\cdot t_u}\cos^2\theta+\frac{t_{vv}\cdot n''}{n_0''t_v\cdot t_v}\sin^2\theta$$
$$=\frac{\cos^2\theta}{\rho_1}+\frac{\sin^2\theta}{\rho_2},$$

which is Euler's Theorem.

Since for elliptic points the curvatures of both principal sections have the same sign, we have by (9) for such points

$$(t_{uu}\cdot n'')(t_{vv}\cdot n'')>(t_{uv}\cdot n'')^2$$

and (13)

$$(t_{uu}\cdot n'')(t_{vv}\cdot n'')\overline{\gtrless}(t_{uv}\cdot n'')^2$$

for parabolic and hyperbolic points respectively.

As in the first part of this article, considering the angle ω between the surface normal n and the principal normal n'' of a curve on the surface, we may write

$(\boldsymbol{n} \times \boldsymbol{n}'')_0 = \boldsymbol{n}_0\boldsymbol{n}_0'' \sin \omega$ whence

$$\sin \omega = \frac{\left[\frac{d\boldsymbol{t}_1}{du} \times (\boldsymbol{t}_v \times \boldsymbol{t}_u)\right]_0}{\left(\frac{d\boldsymbol{t}_1}{du}\right)_0 (\boldsymbol{t}_v \times \boldsymbol{t}_u)_0}$$

$$= \frac{R\left\{\frac{d\boldsymbol{t}_1}{du} \times (\boldsymbol{t}_v \times \boldsymbol{t}_u)\right\}_0}{\boldsymbol{t}_0(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}.$$

Now $\frac{d\boldsymbol{t}_1}{du} =$ is parallel to $\boldsymbol{t} \times (\boldsymbol{t} \times \boldsymbol{t}')$ whence

$$\frac{d\boldsymbol{t}_1}{du} = s\{\boldsymbol{t} \times (\boldsymbol{t} \times \boldsymbol{t}')\},$$

s being a scalar. To determine s, we have, from $\boldsymbol{t}_1 = \frac{\boldsymbol{t}}{\sqrt{\boldsymbol{t}\cdot\boldsymbol{t}}}$,

$$\frac{d\boldsymbol{t}_1}{du} = \frac{(\boldsymbol{t}\cdot\boldsymbol{t})\boldsymbol{t}' - \boldsymbol{t}(\boldsymbol{t}\cdot\boldsymbol{t}')}{(\boldsymbol{t}\cdot\boldsymbol{t})^{3/2}}$$

and

$$\boldsymbol{t} \times (\boldsymbol{t} \times \boldsymbol{t}') = \boldsymbol{t}(\boldsymbol{t}\cdot\boldsymbol{t}') - \boldsymbol{t}'(\boldsymbol{t}\cdot\boldsymbol{t}),$$

that

$$s = -\frac{1}{(\boldsymbol{t}\cdot\boldsymbol{t})^{3/2}} = -\frac{1}{\boldsymbol{t}_0^3}$$

and finally

$$\frac{\sin \omega}{R} = \frac{[(\boldsymbol{t}_v \times \boldsymbol{t}_u) \times \{\boldsymbol{t} \times (\boldsymbol{t} \times \boldsymbol{t}')\}]_0}{\boldsymbol{t}_0^4(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}. \tag{14}$$

If $\omega = 90°$ we have for R the radius of the projection of the curve in the tangent plane at the point; hence, if R_t is the tangential or geodesic radius of curvature, we may write

$$\frac{1}{R_t} = \frac{[(\boldsymbol{t}_v \times \boldsymbol{t}_u) \times \{\boldsymbol{t} \times (\boldsymbol{t} \times \boldsymbol{t}')\}]_0}{\boldsymbol{t}_0^4(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}. \tag{15}$$

By two applications of the vector formula $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ to this we get

$$\frac{1}{R_t} = \frac{\{\boldsymbol{t}_u[(\boldsymbol{t}_v\cdot\boldsymbol{t})(\boldsymbol{t}\cdot\boldsymbol{t}') - (\boldsymbol{t}_v\cdot\boldsymbol{t}')(\boldsymbol{t}\cdot\boldsymbol{t})] - \boldsymbol{t}_v[(\boldsymbol{t}_u\cdot\boldsymbol{t})(\boldsymbol{t}\cdot\boldsymbol{t}') - (\boldsymbol{t}_u\cdot\boldsymbol{t}')(\boldsymbol{t}\cdot\boldsymbol{t})]\}_0}{\boldsymbol{t}_0^4(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}.$$

Now, referring to lines of curvature as reference lines, in which case $\boldsymbol{t}_u\cdot\boldsymbol{t}_v = 0$, we find, since $\boldsymbol{t} = \boldsymbol{t}_u + p\boldsymbol{t}_v$,

$$\frac{1}{R_t} = \frac{[p(\boldsymbol{t}_v\cdot\boldsymbol{t}_v)(\boldsymbol{t}_u\cdot\boldsymbol{t}') - (\boldsymbol{t}_u\cdot\boldsymbol{t}_u)(\boldsymbol{t}_v\cdot\boldsymbol{t}')]\sqrt{\boldsymbol{t}_u\cdot\boldsymbol{t}_u + p^2\boldsymbol{t}_v\cdot\boldsymbol{t}_v}}{\boldsymbol{t}_0^4(\boldsymbol{t}_u \times \boldsymbol{t}_v)_0}. \tag{16}$$

To find the equation determining geodesics, we have that the normal to a

geodesic coincides with the normal to the surface giving $\omega = 0$ whence

$$\frac{1}{R_t} = \frac{\sin \omega}{R} = 0$$

so that for geodesics we have

$$(\boldsymbol{t}_u \cdot \boldsymbol{t}_u)(\boldsymbol{t}_v \cdot \boldsymbol{t}') = p(\boldsymbol{t}_v \cdot \boldsymbol{t}_v)(\boldsymbol{t}_u \cdot \boldsymbol{t}'). \tag{17}$$

This results in a second order third degree differential equation to determine such curves, because in general

$$\boldsymbol{t}' = \boldsymbol{t}_{uu} + 2p\boldsymbol{t}_{uv} + p^2\boldsymbol{t}_{vv} + \frac{dp}{du}\boldsymbol{t}_v.$$

Application. Consider the torus whose rectangular equation is

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2)$$

generated by revolving the circle $(x - c)^2 + z^2 = a^2$ about the z-axis.

Letting u be the angle between the radius to any point on a circular section and the xy-plane, and v the angle between the plane of the section and the xz-plane, we may write for a vector equation of this surface

$$\boldsymbol{r} = (c + a \cos u) \cos v\boldsymbol{i} + (c + a \cos u) \sin v\boldsymbol{j} + a \sin u\boldsymbol{k},$$

$\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ being mutually perpendicular unit vectors parallel to the x-, y-, and z-axes respectively.

From this we obtain

$$\begin{aligned}
\boldsymbol{t}_u &= - a \sin u \cos v\boldsymbol{i} - a \sin u \sin v\boldsymbol{j} + a \cos u\boldsymbol{k},\\
\boldsymbol{t}_v &= - (c + a \cos u) \sin v\boldsymbol{i} + (c + a \cos u) \cos v\boldsymbol{j},\\
\boldsymbol{t}_{uu} &= - a \cos u \cos v\boldsymbol{i} - a \cos u \sin v\boldsymbol{j} - a \sin u\boldsymbol{k},\\
\boldsymbol{t}_{vv} &= - (c + a \cos u) \cos v\boldsymbol{i} - (c + a \cos u) \sin v\boldsymbol{j},\\
\boldsymbol{t}_{uv} &= a \sin u \sin v\boldsymbol{i} - a \sin u \cos v\boldsymbol{j},\\
\boldsymbol{n}'' &= \boldsymbol{t}_u \times \boldsymbol{t}_v = - a(c + a \cos u) \cos u \cos v\boldsymbol{i} - a(c + a \cos u) \cos u \sin v\boldsymbol{j}\\
&\qquad - a(c + a \cos u) \sin u\boldsymbol{k}.
\end{aligned}$$

To ascertain by (12) whether our lines of reference are lines of curvature, we have

$$\boldsymbol{t}_u \cdot \boldsymbol{t}_v = a(c + a \cos u) \sin u \sin v \cos v - (c + a \cos u) \sin u \sin v \cos v = 0,$$

$$\begin{aligned}
\boldsymbol{n}'' \cdot \boldsymbol{t}_{uv} &= - a^2(c + a \cos u) \sin u \sin v \cos u \cos v\\
&\qquad + a^2(c + a \cos u) \sin u \sin v \cos u \cos v = 0,
\end{aligned}$$

so that these conditions are met.

To determine the total curvature by (9) we have

$$\begin{aligned}
\boldsymbol{t}_{uu} \cdot \boldsymbol{n}'' &= a^2(c + a \cos u); & \boldsymbol{t}_{vv} \cdot \boldsymbol{n}'' &= a(c + a \cos u)^2 \cos u\\
\boldsymbol{t}_{uv} \cdot \boldsymbol{n}'' &= 0; & \boldsymbol{n}_0''^2 &= a^2(c + a \cos u)^2
\end{aligned}$$

giving the expression

$$\frac{1}{\rho_1\rho_2} = \frac{\cos u}{a(c + a \cos u)}$$

so that, according to (13), if $c > a$ we have elliptic points, $-(\pi/2) < u < (\pi/2)$; parabolic, $u = \pm(\pi/2)$; and hyperbolic, $(\pi/2) < u < (3/2)\pi$.

If $c < a$ we have elliptic points for $-(\pi/2) < u < (\pi/2)$ and for $\pi - \cos^{-1}(c/a) < u < \pi + \cos^{-1}(c/a)$; parabolic, $u = \pm(\pi/2)$; infinite curvature, $u = \pi \pm \cos^{-1}(c/a)$; and hyperbolic points for $(\pi/2) < u < \pi - \cos^{-1}(c/a)$ and for $\pi + \cos^{-1}(c/a) < u < (3\pi/2)$.

By (10) for umbilics we have

$$(c + a \cos u)^3 = a \cos u(c + a \cos u)^2$$

so that if $c > a$ there are no umbilics, while if $c < a$ there are umbilics at $u = \pi + \cos^{-1}(c/a)$ which are two points on the z-axis. If $c = 0$ every point is an umbilic, and the surface is a sphere.

By (11) we get for the curvature of the principal sections

$$\frac{1}{\rho_1} = \frac{\boldsymbol{t}_{uu}\cdot\boldsymbol{n}''}{(\boldsymbol{t}_u\cdot\boldsymbol{t}_u)\boldsymbol{n}_0''} = \frac{a^2(c + a \cos u)}{a^3(c + a \cos u)} = \frac{1}{a},$$
$$\frac{1}{\rho_2} = \frac{\boldsymbol{t}_{vv}\cdot\boldsymbol{n}''}{(\boldsymbol{t}_v\cdot\boldsymbol{t}_v)\boldsymbol{n}_0''} = \frac{a(c + a \cos u)^2 \cos u}{a(c + a \cos u)^3} = \frac{\cos u}{c + a \cos u},$$

so the one principal section is a circle of radius a; the radius of the other varying from $a \pm c$ to ∞, being ∞ for $u = (\pi/2)$ and zero for $a = \pi - \cos^{-1}(c/a)$ when $c < a$.

To determine geodesics on this surface we have from

$$\boldsymbol{t}' = \boldsymbol{t}_{uv} + 2p\boldsymbol{t}_{uv} + p\boldsymbol{t}_{uv} + \frac{dp}{du}\boldsymbol{t}_v$$

that

$$\boldsymbol{t}'\cdot\boldsymbol{t}_u = ap^2(c + a \cos u) \sin u,$$
$$\boldsymbol{t}'\cdot\boldsymbol{t}_v = (c + a \cos u)[(c + a \cos u)\frac{dp}{du} - 2pa \sin u] + (c + a \cos u)\frac{dp}{du}$$

and the resulting differential equation is

$$a(c + a \cos u)\frac{dp}{du} - 2pa^2 \sin u = (c + a \cos u)^2 \sin up^3.$$

Two particular solutions are evident.

If $p = 0$, $(dp/du) = 0$, that is $v = c_1$, showing that the u-lines are geodesics.

If $p = \infty$ and $u = 0$ or π, we have two particular v-lines, circles of radii $c \pm a$ as geodesics.

If we let $(c + a \cos u)p = a \tan \theta$, the differential equation becomes

$$\cot\theta\, d\theta - \frac{a \sin u}{c + a \cos u}\, du = 0,$$

whence $(c + a \cos u) \sin\theta = c_1$, or

$$p = \frac{dv}{du} = \frac{a \tan\theta}{c + a \cos u} = \frac{ac_1}{(c + a \cos u)\sqrt{(c + a \cos u)^2 - c_1^2}}$$

or, in general, for geodesics on the torus

$$v = ac_1 \int \frac{du}{\sqrt{(c + a \cos u)^4 - c_1^2(c + a \cos u)^2}} + c_2.$$

If $c_1 = 0$ we have the first solution noted above.

If $c = 0$

$$v = c_1 \int \frac{\sec^2 u\, du}{\sqrt{(a^2 - c_1^2) - c_1^2 \tan^2 u}} = \sin^{-1} \frac{c_1 \tan u}{\sqrt{a^2 - c_1^2}} - \beta$$

or

$$\sin (v + \beta) = \tan \alpha \tan u,$$

where

$$\tan \alpha = \frac{c_1}{\sqrt{a^2 - c_1^2}},$$

giving for geodesics on the sphere

$$\mathbf{r} = \frac{a \tan \alpha \cos v}{\sqrt{\tan^2 \alpha + \sin^2 (v + \beta)}}\, \mathbf{i} + \frac{a \tan \alpha \sin v}{\sqrt{\tan^2 \alpha + \sin^2 (v + \beta)}}\, \mathbf{j} + \frac{a \sin (v + \beta)}{\sqrt{\tan^2 \alpha + \sin^2 (v + \beta)}}\, \mathbf{k},$$

α and β being arbitrary constants. This is a curve lying in the plane

$$z \tan \alpha = x \sin \beta + y \cos \beta$$

and is therefore a great circle of the sphere.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

REPLY.

48 (*1923*, 136). In what quadratic realms of rationality and for what values of p (which is a prime integer) is the function $\frac{x^p - 1}{x - 1}$ factorable?

Reply by LOUIS WEISNER, University of Rochester.

The group G of the cyclotomic equation $(x^p - 1)/(x - 1) = 0$, where p is an odd prime, is a regular cyclic group of order $p - 1$, generated by the substitution $s = (\omega\, \omega^2 \cdots \omega^{p-1})$, where ω is a primitive root modulo p. If $\varphi(\epsilon, \epsilon^2, \cdots \epsilon^{p-1})$, where $\epsilon, \epsilon^2, \cdots \epsilon^{p-1}$ denote the roots of the equation, is a function belonging to the subgroup H generated by s^2, the numerical value of φ is of the form $a + b\sqrt{c}$, where a, b and c are rational numbers and c is not a perfect

square. Thus the equation is reducible in $R(\sqrt{c})$ and in no other quadratic realm, since H is the only subgroup of index 2 in G.

To determine c, consider the function

$$\varphi_1 = \epsilon^{\omega} + \epsilon^{\omega^3} + \cdots + \epsilon^{\omega^{p-2}}$$

which belongs to H and under G takes the second value

$$\varphi_2 = \epsilon + \epsilon^{\omega^2} + \cdots + \epsilon^{\omega^{p-3}}.$$

The numbers φ_1, φ_2 satisfy the equation

$$\varphi^2 - (\varphi_1 + \varphi_2)\varphi + \varphi_1\varphi_2 = 0$$

whose coefficients are rational numbers which we proceed to evaluate. We have at once $\varphi_1 + \varphi_2 = -1$, and [1]

$$\varphi_1\varphi_2 = \sum \epsilon^{\omega^\alpha + \omega^\beta} \qquad \begin{pmatrix} \alpha = 1, 3, \cdots p-2 \\ \beta = 2, 4, \cdots p-3 \end{pmatrix}.$$

If possible, let $\omega^\alpha + \omega^\beta \equiv 0 \bmod p$. Then $\omega^{2\alpha} \equiv \omega^{2\beta} \bmod p$, whence $2\alpha \equiv 2\beta \bmod p - 1$ and $\alpha \equiv \beta \bmod (p-1)/2$. Since α is odd while β is even, and α, $\beta < p$, this congruence is satisfied only if $\alpha = \beta \pm (p-1)/2$. This requires that $(p-1)/2$ be odd.

(1) Hence if $(p-1)/2$ is even, $\varphi_1\varphi_2$ is a sum of $\frac{1}{4}(p-1)^2$ imaginary pth roots of unity. Since this sum equals a rational number, it must be a multiple of $\epsilon + \epsilon^2 + \cdots + \epsilon^{p-1}$. Thus the $\frac{1}{4}(p-1)^2$ terms, when combined, give $p-1$ distinct terms with the same coefficient, which is therefore $(p-1)/4$. Hence $\varphi_1\varphi_2 = -(p-1)/4$. It follows that

$$\varphi^2 + \varphi - \frac{p-1}{4} = 0,$$

whence

$$\varphi_1, \varphi_2 = \frac{-1 \pm \sqrt{p}}{2}.$$

(2) If $(p-1)/2$ is odd, let us take for β any positive even integer less than p, and $\alpha = \beta \pm (p-1)/2$, the sign being chosen so that α is positive and less than p. Then α is an odd number and

$$\omega^\alpha \equiv \omega^\beta \omega^{\pm\frac{p-1}{2}} \equiv -\omega^\beta \bmod p,$$

so that there are $(p-1)/2$ values of α and β for which $\omega^\alpha + \omega^\beta \equiv 0 \bmod p$. Therefore $\varphi_1\varphi_2$ is equal to $(p-1)/2$ plus a sum of $(p-1)^2/4 - (p-1)/2$ imaginary pth roots of unity. This sum equals the rational number

$$\frac{p-1}{4} - \frac{1}{2} = \frac{p-3}{4}.$$

[1] For a different evaluation of $\varphi_1\varphi_2$, see Netto, *Substitutionentheorie und ihre Anwendung auf die Algebra* (1882), page 186.

Hence

$$\varphi_1\varphi_2 = \frac{p-1}{2} - \frac{p-3}{4} = \frac{p+1}{4}.$$

Therefore

$$\varphi^2 + \varphi + \frac{p+1}{4} = 0,$$

whence

$$\varphi_1, \varphi_2 = \frac{-1 \pm \sqrt{-p}}{2}.$$

We therefore have the following theorem: *The function* $(x^p - 1)/(x - 1)$, *where p is an odd prime, is factorable in* $R(\sqrt{(-1)^{(p-1)/2}p})$, *and is not factorable in any other quadratic realm.*

DISCUSSIONS

I. The Infinite and Imaginary in Algebra and Geometry: A Reply.[1]

By W. L. G. Williams, Cornell University.

"The traditional treatment of imaginary and infinite elements in algebra and geometry," says Professor R. M. Winger in a recent number of this Monthly, "has curiously resisted the reform movement in American text-book writing." What this reform movement is, where it exists, and what text-books its adherents have written, we do not know and until we can examine its products we should hesitate to call it a reform movement.

However, the two questions that Professor Winger raises, the question of the introduction into our elementary college courses in algebra of a number ∞ and the question of considering, in our elementary courses in analytic geometry, imaginary as well as real points, are so different that they require separate examination.

1. Professor Winger points out the service of a line at infinity in rendering the principle of duality universally valid and in preserving a one-to-one correspondence between a figure and its projection. This idea of a line at infinity, which Poncelet introduced into geometry just a century ago, has nevertheless given rise to many misconceptions and errors. One form of the equation of the line at infinity is the one which appears in the article under discussion, viz.,

$$0x + 0y + 1 = 0,$$

and another is

$$1 = 0,$$

equations satisfied by no real or imaginary points. For mathematics as it now exists these two equations are identical and equally meaningless as equations of lines or of any other loci. The second of these equations is evidently quite independent of x and y, but it has been proposed by Professor Winger and

[1] *V.* Infinite and Imaginary Elements in Algebra and Geometry, this Monthly (*1922*, 290).

is an imaginary circle, when they might be more original and entertaining and say that it represents the two quaternion straight lines

$$x + yi + j = 0,$$
$$x - yi - j = 0?$$

To most people, of whom the writer is one, as long as we lack a space of four dimensions to enable us to plot all the points, real and imaginary, which satisfy an equation in two variables, it will be more interesting to confine themselves to points in the real plane, and they will not consider that they are telling only part of the truth when they say that

$$x^2 + y^2 = 0$$

represents only a single point.

II. On Teaching the Slide-Rule.

By A. A. Bennett, University of Texas.

For historical and other reasons the majority of teachers of mathematics who teach the use of logarithms are themselves unfamiliar with the use of the slide-rule. The slide-rule is rightly regarded as a necessary adjunct to an engineer's outfit of tables and instruments, but facility in handling this practical computing device is not universally acknowledged as essential to an education in pure mathematics. Most teachers who do instruct classes in the slide-rule find that the students take to it readily, and that slide-rule and logarithm table complement each other in developing a concrete notion of the value of significant figures. One of the most serious handicaps in teaching the use of the slide-rule to a large class is the difficulty of coördinating the work of teacher and student. The manufacturers of engineers' slide-rules usually have in stock large size "demonstration slide-rules" that are well fitted for the purposes of instruction. These are made of cheap material and serve also as advertisements and for these reasons may be had at a low price. These are not a new undertaking but since they are not listed in the catalogs, many instructors may have been deterred from taking up the subject in class by ignorance of the facilities available. A so-called "student's slide-rule" identical in marking with the more expensive instruments, and made of durable but cheaper material, serves all the needs of the average student in his individual study.

RECENT PUBLICATIONS.

REVIEWS.

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen. By Heinrich W. E. Jung. Berlin and Leipzig, Walter de Gruyter & Co., 1923. 246 pages. Price $1.40.

Let there be given an irreducible polynomial in x and y of degree in y greater than zero. The equation obtained by setting such a polynomial equal to zero

will be satisfied by a function $y(x)$, termed an algebraic function of x. The domain of the independent variable x is taken as the plane of complex numbers. Let $R(x, y)$ be any rational function of x and y. If y be restricted to the values $y(x)$, $R(x, y)$ becomes a function of x, namely $R[x, y(x)]$ for which the notation $r(x)$ will be exclusively used. Fundamental in the theory of algebraic functions is the study of the nature of the zeros and poles of functions $r(x)$, and the determination of the extent to which these zeros and poles can be arbitrarily prescribed. A very important class of functions also studied in this theory is the class of the so-called abelian integrals, the integrals with respect to x of the above functions $r(x)$.

The theory of algebraic functions has been developed in several distinct ways. Riemann defined the fundamental types of abelian integrals by means of their characteristic properties, and made their existence depend upon the truth of the Dirichlet principle of the potential function theory. The fundamental existence theorems of the potential function theory necessary for a rigorous foundation of this part of Riemann's theory have only been established in recent years. With the fundamental types of abelian integrals as a basis, Riemann was then ready to develop the general theory of algebraic functions.

Clebsch and Jordan, followed by Brill and Noether, developed the theory from the point of view of algebraic geometry, first studying the intersections of algebraic curves and families of such curves, and, with this as an aid, attacking the problem of the determination of the extent to which the zeros and poles of a function $r(x)$ can be prescribed.

Weierstrass, using the powerful instruments of function theory, went directly to the problem of characterizing the functions $r(x)$. He obtained what was analogous to a partial fraction representation of such functions in terms of a simple set of such functions. He then derived the theory of abelian integrals.

While Weierstrass and others characterized the functions $r(x)$, through their principal parts at their poles, Dedekind and Weber, followed by Hensel and Landsberg, sought to determine the functions $r(x)$ by giving their zeros as well as poles. The present treatise has followed the lead of Hensel and Landsberg. Termed the "arithmetische Methode" it is essentially algebraic, with a strong tendency toward the formal. The main body of the theory is developed entirely without the aid of algebraic geometry. Use is made of Riemann surfaces throughout. The classical function theory is practically unused. For example the reviewer has found no use whatsoever made of the Cauchy formula or integral law. The basic instrument of the treatise is the power series. Such series are manipulated formally. The reader is evidently supposed competent to answer questions of convergence for himself. Although the reviewer himself prefers to develop the theory of algebraic functions from the point of view of the complex function theory, yet he found Professor Jung's treatment strikingly suggestive and unified.

The book has thirteen chapters. It starts with a chapter on the behavior of an algebraic function in the neighborhood of a point, treated with a leaning

toward strictly algebraic methods, followed by a chapter on Riemann surfaces and on the diagram of Puiseux, carefully done and illustrated by a profusion of well-selected examples. The major and distinctive portion of the book is devoted to the marshalling of theorems preparatory to the final results concerning the prescription of the zeros and poles of a function $r(x)$. First the prescribed zeros and poles are taken as finite points with nothing required as to the nature of the zeros or poles at the point at infinity. Of the functions so obtained those are sought which have prescribed properties at the point at infinity also. The existence of a desired function, say $r_1(x)$, is made to depend on the existence of a basis, that is, a finite set of functions of the type $r(x)$, upon which a fundamental requirement is that every function $r_1(x)$ be expressible as a sum of a finite number of terms, each term consisting of a member of the basis multiplied by a proper rational function of x. Matrices whose elements are functions are an essential medium in the presentation. The theory of the combination and transformation of such matrices is presented in a separate chapter. The Riemann-Roch theorem is one of the important theorems in which this part of the work culminates.

The tenth and eleventh chapters are respectively on algebraic curves in non-homogeneous coördinates, and in homogeneous coördinates. These chapters seem to the reviewer vague. There does not seem to be an exact enough connection between definitions and theorems, in particular as regards the infinite domain. The twelfth chapter is on the analysis situs of Riemann surfaces carefully written along the lines usual in connection with algebraic functions. The final chapter is on abelian integrals. The existence of the canonical integrals of the three kinds is easily established with the aid of the earlier theorems. The book ends with a proof about a page long of the fact that any abelian integral can be expressed as a sum of constant multiples of the canonical integrals plus a function $r(x)$.

The style is simple, direct, and easy to follow, although at times it seems unnecessarily formal, especially in the use of "divisors." If we except the chapters on algebraic geometry it is obvious that the writer has endeavored to give all of the proof of each of the theorems stated, and with the exception of a few places where the reader can fill in, has succeeded in doing so. A distinctive and very helpful side of the book is the numerous sets of examples worked out in a manner which splendidly supplements the main body of the theory. The book will be a valuable aid to all those who wish to take up the study of algebraic functions from the algebraic point of view.

H. M. MORSE.

Inleiding tot de Analytische Meetkunde van het platte vlak. By J. WOLFF. Groningen, P. Noordhoff, 1922. viii + 296 pages. Price f. 5.25.

Inleiding tot de Analytische Meetkunde; deel I, Het platte vlak. By J. G. RUTGERS. Groningen, P. Noordhoff, 1923. xvi + 299 pages. Price f. 7.25.

In Holland as in most countries of continental Europe, university studies are pursued almost exclusively by those who wish to prepare themselves for the

learned professions. A student who wants to take a doctor's degree in any of the natural sciences or who wants to take an engineering degree has a course in analytical geometry during his first year at the University. In the secondary school he has had thorough courses in the elementary subjects, including trigonometry, solid geometry, and sometimes mechanics, while a well-rounded general education has usually made him a person of considerable intellectual maturity.

It is for this reason that the two books under review, which present the material for this course, are a good deal more extensive, both as regards the topics included and the completeness of their treatment, and are written from a more advanced point of view than can be the case with our usual college texts in analytical geometry. While not conceived on as ample a basis as Salmon's works, they belong to the type of which the well-known text of Briot and Bouquet and the excellent book of G. Kowalewski are examples.

While these books would not therefore be suitable for our freshmen and sophomores, they are excellent companions for the more advanced student or for the exceptional undergraduate who is looking for a more complete treatment than his college course offers him. The introduction of homogeneous coördinates of points and lines, the treatment of projective properties, the discussion of general theorems on algebraic curves and the use of the theory of determinants are features in both texts. Professor Wolff devotes a final chapter to a study of projective transformations and makes general use, throughout his book, of the differential calculus; Professor Rutgers takes up some topics of the modern geometry of the triangle. More than 350 problems are scattered throughout the first of these books; there are over 200 in the second. And many of these are suitable material for students in an American college; particularly those problems which are not of the milk chocolate variety that frequently forms too large a share of the student's present diet.

A. DRESDEN.

Introduction à l'étude des Fonctions Elliptiques à l'usage des étudiants des Facultés des Sciences. By PIERRE HUMBERT. Paris, J. Hermann. 1922. 8vo. 38 pages. Price 3 francs.

Contents: Preface, p. 5; Introduction: A brief survey of some fundamental notions and results of the theory of a complex variable, pp. 7–9; Chapter I: Formation of an elliptic function, pp. 10–19; Chapter II: General remarks on elliptic functions, pp. 20–24; Chapter III: Properties of Weierstrass's functions $p(u)$, $\zeta(u)$ and $\sigma(u)$, pp. 25–37; Additional Note, pp. 37–38.

This short pamphlet may be called to the favorable attention of those who are familiar with an elementary course in complex variable theory and are not yet acquainted with the elliptic functions. Being concise and well written, it will help the reader in his study of the theory of elliptic functions as treated in the special textbooks. The existence of doubly periodic functions is shown in a very simple manner: the author considers the inverse of an elliptic integral in an analogous way to that of the study of sin u as the inverse of $u = \int_0^z (dz/\sqrt{1 - z^2})$. Jacobi's functions are not to be found.

D. KAZARINOFF.

ARTICLES IN CURRENT PERIODICALS.

Acta Mathematica, volume 44, no. 1, 1923: "Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes" by G. H. Hardy and J. E. Littlewood, 1–70. [Quotation: "It was asserted by Goldbach, in a letter to Euler dated 7 June, 1742, that every even number $2m$ is the sum of two odd primes, and this proposition has generally been described as 'Goldbach's Theorem.' There is no reasonable doubt that the theorem is correct, and that the number of representations is large when m is large; but all attempts to obtain a proof have been completely unsuccessful.

"Our main result may be stated as follows: if a certain hypothesis (a natural generalization of Riemann's hypothesis concerning the zeros of his Zeta-function) is true, then every large odd number n is the sum of three odd primes."]—nos. 2 and 3, 1923: "Mémoire sur le calcul aux différences finies" by N. E. Nörlund, 71–212; "Sur l'intégrale $\int_{x_0}^{x} f(y)df(x)$ où x et y sont liés par une relation symétrique" by P. Appell, 213–215.

American Journal of Mathematics, volume 44, October, 1922 [published May, 1923]: "On the kernel of the Stieltjes integral corresponding to a completely continuous transformation" by C. A. Fischer, 237–246; "Equivalence and reduction of pairs of Hermitian forms" by Mayme I. Logsdon, 247–260; "Plane cubics with a given quadrangle of inflexions" by B. M. Turner, 261–278; "Normal ternary continued fraction expansions for the cube roots of integers" by P. H. Dans, 279–296; "Concerning compact Kürschák fields" by V. D. Gokhale, 297–316.

American Journal of Psychology, volume 34, April, 1923: "The equation of the learning function" by M. F. Meyer and F. G. Eppright, 203–222.

Annales de l'École Normale Supérieure, volume 58, February and March, 1923: "Sur les modules des zéros des polynomes" by P. Montel, 33–34 (continuation and conclusion); "Sur les formules d'interpolation de Stirling et de Newton" by N. E. Nörlund, 35–54 (continued from volume 57); "Sur les systèmes cycliques de triples de Steiner" by S. Bays, 55–95.—volume 58, April, 1923: "Sur une classe d'équations fonctionnelles" by G. Julia, 97–128.

Annals of Mathematics, 2d series, volume 23, March, 1922: "Dirichlet's problem" by G. E. Raynor, 183–197; "Annihilators of modular invariants and covariants" by Olive C. Hazlett, 198–211; "Systems of linear inequalities" by W. B. Carver, 212–220; "Euler squares" by H. F. MacNeish, 221–227; "Geometric aspects of Einstein's theory" by J. Pierpont, 228–254; "Cauchy's paper of 1814 on definite integrals" by H. J. Ettlinger, 255–270; "Arithmetical deduction of Kronecker's class-number relations" by G. H. Cresse, 271–279; "Cyclotomic heptasection for the prime 43" by O. Upadhyaya, 280–281; "Summation of a double series" by T. H. Gronwall, 282–285.

Annals of Mathematics, 2d series, volume 23, June, 1922: "On the positions of the imaginary points of inflexion and critic centers of a real cubic" by B. M. Turner, 287–291; "Frequency distributions obtained by certain transformations of normally distributed variates" by H. L. Rietz, 292–300; "The associated point of seven points in space" by H. S. White, 301–306; "Common solutions of two simultaneous Pell equations" by A. Arwin, 307–312; "On the complete independence of Hurwitz's postulates for Abelian groups and fields" by B. A. Bernstein, 313–316; "On power series with positive real part in the unit circle" by T. H. Gronwall, 317–332; "Algebraic surfaces, their cycles and integrals. A correction" by S. Lefschetz, 333.

Bulletin of the American Mathematical Society, volume 29, March, 1923: "The twenty-ninth annual meeting of the Society" by R. G. D. Richardson, 97–116; "The Evanston meeting of the Society" by A. Dresden, 117–124; "The fifteenth regular meeting of the Southwestern Section" by E. B. Stouffer, 125–127; "A set of axioms for line geometry" by M. G. Gaba, 128–138; Reviews: by J. W. Glover of P. J. Richard and E. Petit, *Théorie mathématique des Assurances* (2 vols., 2d ed., Paris, 1922), 139–140; and by E. P. Lane and H. J. Davis of E. Cartan, *Leçons sur les Invariants Integraux* (Paris, 1922); Notes, 141–142; New publications, 143–144.

Bulletin of the American Mathematical Society, volume 29, April, 1923: "The February meeting of the Society" by R. G. D. Richardson, 145–153; "Relations between kindred Riemannian P and Q functions" by R. D. Curtis, 154–160; "The reduction of singularities of plane curves by birational transformation" [presidential address delivered before the American Mathematical Society, December 28, 1922] by G. A. Bliss, 161–183; Reviews: by R. B. McClenon of C. Tweedie, *James Stirling* (Oxford, 1922), 184–185; by J. W. Young of W. A. Granville, *The Fourth Dimension and the Bible* (Boston, 1922), 185; and by J. E. Rowe of T. Vahlen, *Ballistik* (Berlin, 1922), 186–187; Notes, 188–189; New Publications, 190–192.—May, 1923:

"The April meeting of the San Francisco Section" by B. A. Bernstein, 193–196; "The April meeting of the Society in Chicago" by A. Dresden, 197–203; "The April meeting of the Society in New York" by R. G. D. Richardson, 204–218; "An elementary proof of a fundamental lemma concerning the limit of a sum" by H. J. Ettlinger, 219–223; "Klein's collected papers, volume II" [review of Felix Klein: *Gesammelte mathematische Abhandlungen, Vol. II* (Berlin, 1922)] by V. Snyder, 224–229; "Lévy on functionals" [review of P. Lévy, *Leçons d'Analyse Fonctionnelle* (Paris,1922)] by C. A. Fischer, 229–231; Reviews: by F. Cajori of H. Andoyer, *L'Oeuvre scientifique de Laplace* (Paris, 1922), 232; by J. W. Young of F. Michel and M. Potron, *La Composition des Mathématiques dans l'Examen D'Admission à l'École Polytechnique de 1901 à 1921* (Paris, 1922), 232; by A. Emch of G. H. Ling, G. Wentworth and D. E. Smith, *Elements of Projective Geometry* (Boston, 1922), 233; by J. B. Shaw of C. E. Weatherburn, *Elementary Vector Analysis* (London, 1921), 233; by L. W. Dowling of R. Fricke, *Die elliptischen Funktionen und ihre Anwendungen, Zweiter Teil* (Leipzig, 1922), 234; and by E. P. Adams of P. Pringsheim, *Fluoreszenz und Phosphoreszenz im Lichte der neueren Atomtheorie* (Berlin, 1921), 234; Notes, 235–237; New publications, 238–240.

Bulletin des Sciences Mathématiques, second series, volume 47, April, 1923: Review by G. Valiron of E. Borel, *Méthodes et Problèmes de la Théorie des Fonctions* (Paris, 1922), 129–134; Review by H. Andoyer of Bureau des Longitudes, *Annuaire pour l'an 1923*, 135–137; Review by S. Lefschetz of Manning, *Primitive Groups. Part 1* (Stanford University, 1921), 138–139; Review by S. Lefschetz of J. B. Shaw, *Vector Calculus with Applications* (New York, 1922), 139–140; "Sur deux formules de Lagrange" by B. Niewenglowski, 141–146; "Sur l'abaissement du degré de l'équation modulaire" by J. Plemelj, 146–153; "Le passage à la limite des équations aux différences aux équations différentielles dans les problèmes aux limites" by M. Plancherel, 153–160.—May, 1923: Review by M. Brillouin of J. Villey, *Les divers aspects de la théorie de la relativité* (Paris, 1923), 161–164; Review by A. Corvisy of J. J. Thomson, *Les rayons d'électricité positive et leur application aux analyses chimiques*, 164–167; Review by R. d'Adhémar of A. E. Kennelly, *Les applications élémentaires des fonctions hyperboliques à la science de l'ingénieur électricien* (Paris, 1922), 167–170; "Le passage à la limite des équations aux différences aux équations différentielles dans les problèmes aux limites" by M. Plancherel, 170–177 (concluded); "Sur un théorème de M. Hadamard" by G. Valiron, 177–192.

Bulletin of the West Virginia Scientific Association (West Virginia University Bulletin, series 22, number 5, part 2), volume 2, number 1, April, 1923: "A slide rule representation of Einstein's restricted theory of relativity" by C. N. Reynolds, Jr., 34–41.

L'Enseignement Mathématique, volume 22, no. 6, published May, 1923: "Démonstration du théorème de Staeckel par l'élimination du temps entre les équations de Lagrange" by E. Turrière, 337–343; "Démonstration d'un théorème de Morley" by B. Niewenglowski, 344–346; "Récréation mathématique. Le Jeu de cloche et marteau" by M. Jéquier, 347–357; Mélanges et correspondance: "Fonctions triplement périodiques d'une seule variable indépendante" by M. Winants, 358–359 [Concluding sentences: "La triple périodicité d'une structure cristalline est un fait qu'aucun cristallographe ne conteste plus. Elle pourrait, et même devrait suggérer au mathématicien l'étude systématique des fonctions à trois périodes. Cette étude nous paraît fort difficile, d'autant plus que nous ne connaissons encore aucune analyse à trois dimensions, pouvant être considérée comme une généralisation de l'analyse complexe. On ne soutiendra certainement pas que la théorie des quantités complexes rentre dans celle des quaternions comme un cas particulier dans un cas général. Néanmoins, nous croyons qu'aujourd'hui l'on peut ne plus mettre en doute l'existence des fonctions triplement périodiques d'une seule variable indépendante.]; "Sur le théorème de la progression arithmétique de Dirichlet" by L. Aubry, with reply by A. M. Bedarida, 360–361; Chronique, 362–382 [contains reports of meetings of the French association for the advancement of the sciences, of the Swiss mathematical society and of the Swiss society of professors of mathematics]; Bibliographie, 382–397; Bulletin bibliographique, 398–408; Index to volume 22, 209–416.

Mathematics Teacher, volume 16, January, 1923: "The extension of concepts in mathematics" by A. J. Kempner, 1–23; "An experiment in classification of pupils in algebra" by (Miss) L. Price, 24–28; "Permutations in the 16th century Cabala" by M. Turetsky, 29–34; "The cultural value of secondary mathematics" by J. H. Minnick, 35–40; "The teaching of algebra" by H. F. Richards, 41–47; "Concerning the intercommunion of mathematics and astronomy" by Agnes G. Rowlands, 48–51; "The art of questioning" by C. G. Gould, 52–56; New books, 57–58; News notes, 59–62.—February: "Objectives in teaching of mathematics in secondary schools" by Gertrude E. Allen, 65–77; "Our geometry in Egypt and China" by W.

A. Austin, 78–86; "Mechanics" by G. R. Mirick, 87–93; "The place of the history and recreations of mathematics in teaching algebra and geometry" by L. G. Simons, 94–101; "A study of the cultivation of space imagery in solid geometry through the use of models" by E. W. Schreiber, 102–111; "Some aspects of correlation theory" by L. E. Mensenkamp, 112–122; "Concerning the disciplinary value of mathematics" by C. J. Keyser, 123; News and notes, 124–128.—March: "The Thorndike philosophy of teaching the processes and principles of arithmetic" by M. A. Bailey, 129–140; "Probability applied to grades" by E. J. Moulton, 141–149; "The cultural value of mathematics" by Helen E. Howarth, 150–156; "Defects remaining in the notation and nomenclature of elementary mathematics" by J. V. Collins, 157–161; "The logic of mathematical processes" by H. F. Sloctemyer, 162–169; "The way mathematicians work" by J. B. Shaw, 170–174; "Informal tests for diagnosis and remedial teaching in mathematics" by P. L. Spencer, 175–182; "Recent symbolisms for decimal fractions" by F. Cajori, 183–187; News and notes, 188–190; Discussion, 191–192.

Mathematics Teacher, volume 16, April, 1923: "Psychological tests of mathematical ability and educational guidance" by Agnes L. Rogers, 193–205; "Textbooks in unified mathematics for college freshmen" by Vera Sanford, 206–214 [Quotation from conclusion: "If one may make a guess, the future development of this work will be less and less along the conventional line. As one studies these textbooks, he almost invariably finds that the work in analytics is arid in comparison to the rest. One questions whether the normal probability curve, though younger, may not in time usurp some of the attention paid to conics. . . . The more radical of the books succeed in convincing us that freshman college mathematics may be made to present many contacts with the real world, and that, even though these contacts may be through problems that have only the appearance of reality, they at least have the virtue of showing the things that 'mathematics has done and is doing for mankind.' "]; "Teaching the algebraic language to Junior High School pupils" by J. R. Overman, 215–217; "The unitary organization of the mathematics of the seventh, eighth and ninth grades" by E. R. Breslich, 228–235; "Mechanics" by G. R. Mirick, 236–241; "A brief study in non-mathematical logic" by N. J. Lennes, 242–246; "Classification of positive integers as regards the ultimate sum of their digits" by G. A. Miller, 247–248; "Team work in elementary algebra" by J. B. Hawley, 248–250; News and notes, 251–256.—May, 1923: "Empirical theorems in Diophantine analysis" by R. D. Carmichael, 257–265; "The Pennsylvania state course of study in mathematics" by J. A. Foberg, 266–273; "Fate and Freedom" by A. Korzybski, 274–290; "Mechanics" by G. A. Mirick, 291–294; "Varieties of minus signs" by F. Cajori, 295–301; "Correlation of the mathematical subjects develops mathematical power" by C. A. Stone, 302–310; News and notes, 311–314; New books, 315–319.

Mathematische Annalen, volume 88, nos. 3–4 (published February 17, 1923): "Bemerkungen über unendliche Folgen und ganze Funktionen" by G. Pólya, 169–183; "Additive Theorie der Zahlkörper II" by C. L. Siegel, 184–210; "Ueber ein Problem von A. Hurwitz, quaternäre quadratische Formen betreffend" by H. Brandt, 211–214; "Ueber S. Lies Geometrie der Kreise und Kugeln" (continuation) by E. Study, 215–241; "Ueber die invariante Darstellung algebraischer Funktionen einer Veränderlichen" by K. Petri, 242–289; "Beiträge zur allgemeinen Topologie I" by H. Tietze, 290–312.

Mathematische Annalen, volume 89, nos. 1–2, published April 24, 1923: "Die Flächen vierter Ordnung mit gewundener Doppelkurve" by H. Mohrmann, 1–31; "Ueber Kurven von Maximal-Klassenindex. Ueber Kurven von Maximalindex" by J. Sz. Nagy, 32–75; "Ueber Axiomensysteme für beliebige Satzsysteme, Teil II. Sätze Höheren Grades" by P. Herz, 76–102; "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I" by K. Löwner, 103–121; "Ueber die Gewinnung summierbarer Polynomreihen aus summierbaren Fourierreihen" by K. Popoff, 122–125; "Bemerkung zu der Arbeit des Herrn Popoff 'Ueber die Gewinnung summierbarer Potenzreihen aus summierbaren Fourierreihen' " by O. Blumenthal, 126–129; "Oszillationstheoreme oberhalb der Stieltjesschen Grenze" by O. Haupt and E. Hilb, 130–146; "Ueber die Zetafunktionen gewisser algebraischer Zahlkörper" by E. Artin, 147–156; "On a formula of transformation" by P. O. Upadhyaya, 157–159.

Mathematische Zeitschrift, volume 13, nos. 3–4: "Axiomatische Begründung der transfiniten Kardinalzahlen, I. (Herrn K. Hensel zum sechzigsten Geburtstag.)" by A. Fraenkel, 153–188; "Kurven auf algebraischen Flächen" by H. W. E. Jung, 189–201; "Ebene Schnitte und Berührungskegel einer algebraischen Fläche" by H. W. E. Jung, 202–216; "Die Tangentenlosigkeit der von Kochschen Kurve" by F. Apt, 217–222; "Die windschief involutorischen Paarungen in einer linearen Strahlenkongruenz und die beiden Arten windschiefer involutorischer linearer Strahlenkongruenzen" by S. Jolles, 221–262; "Zur Theorie der nichtlinearen Differential-

gleichungen" by J. Horn, 263–282; "Ueber die Riemannsche Funktionalgleichung der ζ-Funktion. (Dritte Mitteilung.) Die Funktionalgleichung der L-Reihen" by H. Hamburger, 281–311; Mathematische Preisaufgabe der Fürstlich Jablonowskischen Gesellschaft: (Quoted) "Der Gausssche Algorithmus des 'arithmetisch-geometrischen Mittels' kann dahin verallgemeinert werden, dass an Stelle der beiden Mittelbildungen irgend zwei Funktionen φ und ψ von zwei Veränderlichen treten. Man kommt auf diese Weise, ausgehend von zwei beliebigen Zahlwerten x und y, auf zwei unendliche Folgen

$$\begin{array}{lllll} x, & x_1, & x_2, & x_3, & \cdots, \\ y, & y_1, & y_2, & y_3, & \cdots, \end{array}$$

welche durch die rekurrenten Gleichungen

$$\begin{aligned} x_n &= \varphi(x_{n-1}, y_{n-1}) \\ y_n &= \psi(x_{n-1}, y_{n-1}) \end{aligned} \qquad (n = 1, 2, 3, \cdots),$$

beherrscht sind. Im allgemeinen werden dann keine Grenzwerte jener Folgen existieren, während doch die Folgen selbst besondere Eigenschaften besitzen können.

" Die Gesellschaft wünscht:

" Die Untersuchung der genannten Folgen für irgendwelche einfache Funktionen $\varphi(x, y)$, und $\psi(x, y)$, wobei namentlich auch eine independente Darstellung der rekurrent definierten Glieder der Reihen ins Auge zu fassen wäre.

"Einlieferung bis zum 31. Oktober 1924; Preis 3000 Mark," 312.

Mathematische Zeitschrift, volume 16, April, 1923: "Untersuchungen über die Konvergenz der limitärperiodischen Jacobi-Ketten beliebiger Ordnung" by E. Schlechter, 173–206; "Sur le théorème d'unicité des solutions des équations différentielles ordinaires" by J. Tamarkine, 207–213; "Sur la méthode de C. Störmer pour l'intégration approchée des équations différentielles ordinaires" by J. Tamarkine, 214–219; "Momentprobleme für ein endliches Intervall" by F. Hausdorff, 220–248; "Beiträge zur Theorie der Kurven konstanter geodätischer Krümmung auf krummen Flächen" by R. Weyrich, 249–272; "Ueber die Gestalt der Integralkurven einer Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes" by O. Perron, 273–295; "Ueber eine Integralgleichung in der Theorie der heterogenen Gleichgewichtsfiguren" by J. Lense, 296–300; "Ueber die Lösungen der Riemannschen Funktionalgleichung" by E. Hecke, 301–307; "Ueber die Parallelverschiebung in Riemannschen Räumen" by H. Tietze, 308–317; "Ganzzahlige Gleichungen ohne Affekt" by M. Bauer, 318–319.

Mathematische Zeitschrift, volume 17, May, 1923: "Verallgemeinerung eines Minkowskischen Satzes" by R. Remak, 1–34; "Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen" by H. Prüfer, 35–61; "Untersuchungen über die Figur der Himmelskörper. Vierte Abhandlung. Zur Maxwellschen Theorie der Saturnringe" by L. Lichtenstein, 62–110; "Ueber die Bianchische Identität für symmetrische Uebertragungen" by J. A. Schouten, 111–115; "Ueber die Maximalzahl der Doppelpunkte bei ebenen Polygonen von gerader Seitenzahl" by E. Steinitz, 116–129; "Ueber das Wachstum analytischer Funktionen in Halbstreifen und ähnlichen Gebieten" by L. Neder, 130–143; "Ueber affine Geometrie: Zur Theorie der Affingesimsflächen" by E. Salkowski, 144–148; "Ueber einen Grenzwertsatz" by O. Perron, 149–152; "Bilineare Transformation quadratischer Formen" by H. Brandt, 153–160.

Messenger of Mathematics, volume 52, October, 1922: "The transformation of the elliptic function of the seventh order" by G. Greenhill, 81–89; "A proof of Burnside's formula for log $\Gamma(x + 1)$ and certain allied properties of Riemann's ζ-function" by J. R. Wilton, 90–93; "A direct proof of the binomial theorem for a rational index" by T. W. Chaundy, 94–96.—November, 1922: "A direct proof of the binomial theorem for a rational index" (continued) by T. W. Chaundy, 97–98; "Adaptation of curvilinear isothermal coördinates to integrate the equations of equilibrium of elastic plates" by B. G. Galerkin, 99–109; "Sur quelques séries et produits infinis" by S. P. Sorensen, 109–112.—No. 8, December, 1922: "The connexion between the sum of the squares of the divisors and the number of the partitions of a given number" by P. A. MacMahon, 113–116; "Electromagnetism and dynamics" by H. Bateman, 116–128.

Nature, volume 110, November 25, 1922: "The time-triangle and time-triad in special relativity" by R. A. P. Rogers, 698–699; "Space-time geodesies" by H. T. H. Piaggio, 699.—December 16: "Space-time geodesics" by A. A. Robb, 809–810.—December 23: "A quantum theory of optical dispersion" by C. G. Darwin, 841–842.—Volume 111, January 6, 1923: Notices by W. E. H. B. of A. MacLeod, *Introduction à la géométrie non-Euclidienne* (Paris, 1922) and of L. E. Dickson, *History of the Theory of Numbers* (Vol. II, Washington, 1920).— January 20:

"Early mathematical instruments in Oxford" [review of R. T. Gunther's *Early Science in Oxford* (London, 1922)], 75–76.—February 3: "Greek geometry with special reference to infinitesimals" by T. L. Heath, 152–153; "The finitistic theory of space" [notice of N. M. Poppovich's *Die Lehre vom diskreten Raum in der neueren Philosophie* (Vienna and Leipzig, 1922). "The theory leads Dr. Petronievics to affirm the absoluteness of Euclidean space"].—February 10: "The definition of limiting equality" by G. H. Bryan, 183–184 [Quotation: "I consider that the proper test of limiting equality is that the difference between two quantities should become (numerically) less than any assignable fraction of one of the quantities, however small, in other words that $x - a < ae$ where e is any fraction of unity, however small (instead of $x - a < e$ where e is any quantity, however small), the present definition being assumed to hold good even if the two quantities vanish or become infinite at the limit"]—February 17: "Stirling's theorem" by J. Satterly, 220; "Definitions and laws of motion in the 'Principia'" by G. Greenhill, 224–226.—February 24: "Absolute measure and the C. G. S. units" by G. Greenhill, 259–261; "The teaching of elementary geometry," 271.—March 3: "Sequence in school geometry," 277–278.—March 17: Notice of J. L. Heiberg, *Mathematics and Physical Science in Classical Antiquity* (trans. D. C. Macgregor, London, 1922), 355–356; Obituary notice of A. N. Favaro, by F. Cajori, 368.—March 24: Notice by E. G. C. Poole of J. Edwards, *A Treatise on the Integral Calculus* (Vol. 2, London, 1922), 391; "Definitions and laws of motion in the 'Principia'" by W. Peddie, 395, and by F. E. Hackett, 395–396; "Stirling's theorem" by J. Strachan, 397; Obituary notice of James Gow, by T. L. Heath, 403.—March 31: "The Fourier-Bessel function" [review of A. Gray and G. B. Mathews, *A Treatise on Bessel Functions and their Applications to Physics* (2d ed., prefaced by A. Gray and T. M. Macrobert, London, 1922), and of G. M. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge, 1922)] by G. Greenhill, 422–425.—April 28: Notice of E. Study, *Mathematik und Physik* (Braunschweig, 1923), 565; of H. P. Few, *Elementary Determinants for Electrical Engineers* (London and New York, 1922), 566.

Nature, volume 111, May 5, 1923: "Stirling's theorem" by H. E. Soper, 601.—May 12: "Martin's equations for the epidemiology of immunizing diseases" by A. J. Lotka, 633–634.—May 26: "New works on relativity" [review of ten recent books, including L. Silberstein, *The Theory of General Relativity and Gravitation* (Toronto, 1922), and F. D. Murnaghan, *Vector Analysis and the Theory of Relativity* (Baltimore, 1922)], 697–699.—June 2: "An Einstein paradox" by R. W. Genese, 742.—June 9, 1923: "Actuarial mathematics" by W. E. H. B. [review of A. Henry, *Calculus and Probability for Actuarial Students* (London, 1922), and of E. F. Spurgeon, *Life Contingencies* (London, 1922)], 769–770; "A puzzle paper band" by C. V. Boys, 774.—June 16: "Martini's equations for the epidemiology of immunizing diseases" by G. N. Watson, 808; "The tercentenary of Blaise Pascal" by H. W. Carr, 814–816; "Curve fitting," 824.—June 23: "The teaching of the calculus" by G. H. B. [review of G. W. Brewster, *Common Sense of the Calculus* (London, 1923)], 837–838.

Nouvelles Annales de Mathématiques, fifth series, volume 1, May, 1923: "Sur les systèmes de quadriques ayant mèmes projections de leurs lignes de courbure sur un plan principal commun" by M. d'Ocagne, 277–284; "Sur la formule d'Holditch et les applications qu'on peut en déduire" by R. Esteve, 284–300; "Sur un invariant intégral se rattachant à la Mécanique statistique" by J. Haag, 301–302.—June, 1923: "Sur les fonctions analytiques d'une variable réelle" by G. Valiron, 321–329; "Sur une forme remarquable de l'intégrale de l'équation des cordes vibrantes" by S. Zaremba, 330–338; "La construction du centre de courbure des coniques d'après Mannheim démontrée par le théorème de Pascal" by J. Larras, 338–339; "Sur les cercles focaux" by H. Lebesgue, 340–350; "Sur un probleme de choc avec frottement" by H. Villat, 351–357.

Pedagogical Seminary, volume 30, March, 1923: "Mathematics in current literature" by Martha MacLear, 48–50; "Self-taught arithmetic from the age of five to the age of eight" by Sophie R. Altshiller-Court, 51–68.

Popular Astronomy, volume 31, May, 1923: "Leap year rules and calendar accuracy" by C. F. Marvin, 298–308. [Quotation: "The present rule (Gregorian calendar) had best be continued to the year 3200 A.D., when the error of reckoning will be about 0.9 day. Then introduce a rule to omit 5 leap years in 600 years, resulting in a calendar which will operate until the year 17600 with errors well under one day throughout the whole period."]

Proceedings of the Edinburgh Mathematical Society, volume 40, 1922: "Determination of the arbitrary constants which appear in the asymptotic expansions for the functions of the elliptic cylinder" by W. Marshall, 2–8; "Newton's conception of a limit as interpreted by Jurin and Robins respectively" by G. A. Gibson, 9–20; "Notes on Everett's interpolation

formula" by G. J. Lidstone, 21–26; "On Mathieu functions of higher order" by P. Humbert, 27–33; "The vibrations of a particle about a position of equilibrium—Part 2. The relation between the elliptic function and series solutions" by B. B. Baker, 34–49; "A dissection of the extended form of Pythagoras' Theorem" by G. D. C. Stokes, 50–54; "A new form for the sum of a trigonometric series" by G. J. Lidstone, 54–56; "To construct an ellipse whose focus F is given, which shall pass through three given points A, A', A'' (Halley's Problem)" by R. F. Davis, 56–57; "Common Logarithms calculated by simple multiplication" by R. F. Muirhead, 57–60; "Trigonometrical Ratios of $(A \pm B)$" by A. D. Russell, 60–61; "On Sylvester's Dialytic Method of Elimination" by E. T. Whittaker, 62–63.

Proceedings of the National Academy of Sciences of the U. S. A., vol. 8, December, 1922: "Projective and affine geometry of paths" by O. Veblen, 347–350; "A further note on the mathematical theory of population growth" by R. Pearl and L. J. Reed, 365–368.—Volume 9, January, 1923: "On Riemann spaces conformal to Euclidean space" by H. W. Brinkmann, 1–3; "Equiaffine geometry of paths" by O. Veblen, 3–4; "Affine geometries of paths possessing an invariant integral" by L. P. Eisenhart, 4–7; "Closed connected sets which are disconnected by the removal of a finite number of points" by J. R. Kline, 7–12; "Some extensions in the mathematics of hydromechanics" by R. S. Woodward, 13–18.—February: "Sets of conjugate cycles of a substitution group" by G. A. Miller, 52–54.—March: "Note on the relative abundance of the elements in the earth's crust" by A. J. Lotka, 87–90; "Continuous transformations of manifolds" by S. Lefschetz, 90–93; "A lemma on systems of knotted curves" by J. W. Alexander, 93–95.—April: "Concerning the cut-points of continuous curves and of other closed and connected point-sets" by R. L. Moore, 101–106; "On Riemann spaces conformal to Einstein spaces" by H. W. Brinkmann, 172–174.—Vol. 9, June, 1923: "Another interpretation of the fundamental gauge-vector of Weyl's theory of relativity" by L. P. Eisenhart, 175–178; "Tensor analysis without coördinates" by G. Y. Rainich, 179–183; "Geometric aspects of the Abelian modular functions of genus four (III)" by A. B. Coble, 183–187.

School and Society, volume 17, January 29, 1923: "A mathematician on the present status of the formal discipline controversy" by N. J. Lennes, 63–71 [presented at the Rochester meeting of the Association, September, 1922].—March 31: "Mathematics the project instrument" by T. Lindquist, 343–348.—May 19, 1923: "The Ph.D. degree and honesty" by G. A. Miller, 553.

School Review, volume 31, May, 1923: "Diagnostic algebra tests and remedial measures" by T. M. Deam, 376–379.

Science, new series, volume 57, April 20, 1923: "The Pi Mu Epsilon mathematical fraternity" by E. D. Roe, Jr., 461–462.—May 4: "An Egyptian mathematical papyrus in Moscow" by L. C. Karpinski, 528–529; "The international work of scientific synthesis" [a review of the international journal *Scientia*] by R. D. Carmichael, 529–532.—May 11: "Note on the equations for tortuosity in Thomson and Tait's *Natural Philosophy*" by C. Barus, 562.—May 25: "The American Mathematical Society" [report of the meeting of the Chicago Section, April 13–14, 1923] by A. Dresden, 620.

Science, new series, volume 57, June 1, 1923: Review by R. D. Carmichael of A. Einstein, *The Meaning of Relativity* (Princeton lectures translated into English by E. P. Adams, Princeton, 1922), 642–643. [Quotation: "Full details cannot be given in so short an exposition. But the author can and does succeed in making clear the general ideas and in setting before the reader the spirit and trend of the argument and in leading him to see much of the detail of the whole theory. The exposition is thus a very useful one."]—June 8: "Mathematical propaganda" by G. A. Miller, 663.—June 22: "The new volumes of the Encyclopedia Britannica" by R. D. Carmichael, 721–724.

Tôhoku Mathematical Journal, volume 21, July, 1922: "A new proof of a theorem of Taylor's" by Y. Tanaka, 1–2; "On the theory of double sequences" by T. Kojima, 3–14; "On the curvature of the closed convex curve" by T. Kojima, 15–20; "Notes on closed convex curves" by T. Kubota, 21–25; "On the Fourier constants and a special system of non-homogeneous linear equations in infinitely many unknowns" by S. Takenaka, 26–42; "Note on the rapidly convergent series for $N!$ due to Prof. Burnside" by S. Nakajima, 43–45; "Cyclotomic sexe-section for the primes 19 and 31" by P. O. Upadhyaya, 46–50; "Über gewisse Infinitesimaloperationen der höheren Operationsstufen" by R. E. Moritz, 51–64; "Pentaspherical geometries in noneuclidean space, II" by T. Takasu, 65–137; "On the multiplication table in China" by K. Ueno, 138–147; "The logarithm in Japanese mathematics" by T. Hayashi, 148–190.—Volume 22, December, 1922: "On a Diophantine equation" by K. Oishi, 1–13; "Sur le champ de gravitation dans l'espace vide" by K. Ogura, 14–37; "Untersuchungen über das Poissonsche

Integral auf der Kugel und seine Ableitungen" by P. E. Strässle, 38–76; "An extension of a theorem of Salmon" by Y. Sawayama, 77–78; "Über eine arithmetische Eigenschaft gewisser Reihenentwicklungen" by G. Pólya, 79–81; "Sur certains systèmes d'équations différentielles" by P. Sibirani, 82–86; "Über die Mittelwerte analytischer Funktionen" by G. Szegö, 87–98; "Axiomatic investigation on number-systems, I" by K. Yoneyama, 99–137; "On multiplicative and enumerative properties of numerical functions" by E. D. Pepper, 138–152; "On geometrical construction by a ruler of finite length and compasses of finite aperture" by M. Shibayama, 153–154; "On a sextic" by P. O. Upadhyaya, 155–157; "Cyclotomic sexe-section for the prime 61" by P. O. Upadhyaya, 158–162; "Second paper on tautochronous motion" by P. O. Upadhyaya, 163–164; "On the integral $\int_0^\infty \frac{\sin^n x}{x^m} dx$ with an appendix on its application to theory of approximation of a function" by T. Hayashi, 165–170.

Transactions of the American Mathematical Society, volume 22, October, 1921 [published April, 1923]: "On certain numerical invariants of algebraic varieties with application to Abelian varieties" by S. Lefschetz [concluded], 407–482; "On restricted systems of higher indeterminate equations" by E. T. Bell, 483–488; "Maximum modulus of some expressions of limited analytic functions" by S. Kakeya, 489–504; "Differential variations in ballistics, with applications to the qualitative properties of the trajectory" by T. H. Gronwall, 505–525; "An expansion theorem for a system of linear differential equations of the first order" by W. A. Hurwitz, 526–543.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 54, no. 2, published May 5, 1923: "Vom Ptolemäischen zum Kopernikanischen System" by W. Lietzmann, 65–70; "Zur Bestimmung der Asymptoten einer ebenen Kurve" by V. Kommerell, 70–75; "Zu den arithmetischen Operationen" by N. Gennematás, 75–78; "Ueber die Behandlung der Parallelenlehre im Unterricht" by K. Fladt, 78–84; "Beiträge zur Berechnung des Kugeldreiecks im Falle des casus ambiguus" by Ruoss, 84–89; "Ueber den Zusammenhang der Keplerschen Gesetze untereinander" by H. Teege, 89–94; "Ein elementares Verfahren, die relativistischen Aberrationsgesetze unmittelbar aus dem Diagramm der Aberration abzulesen" by H. Meurer, 95–96; "Die Winkelmessung der Artilleristen" by P. Luckey, 97–99; "Sphärisch-astronomische Relation zwischen den Neigungswinkeln der regulären Körper" by H. Michnik, 99; Aufgabenrepertorium, 100–105; Berichte, 106–110; "Aus der Forschung. Eine amerikanische Geschichte der Zahlentheorie" [review of L. E. Dickson, *History of the Theory of Numbers* (Vols. I and II, Washington, 1919–1920)] by H. Wieleitner, 110–111; Bücherbesprechungen und Zeitschriftenschau, 113–127; "Die Lehre von den Wurzeln im elementaren Unterricht" by Brettar, 127–128.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Tex.

CLUB ACTIVITIES.

The Junior Mathematics Club of the University of Chicago, Chicago, Ill. [1921, 324.]

Officers elected for the year 1922–1923 were: President (fall and winter), C. E. Van Horn, Gr.; president (spring), L. M. Graves, Gr.; secretary and treasurer, Margaret Mauch, Gr.; chairman of program committee, Mrs. J. P. Ballantine, Gr.; chairman of social committee, Esther Weaver, Gr. Two social meetings were held during the year. At the regular meetings the following papers were presented:

October 25, 1922: "Difference quotients" by J. P. Ballantine, Gr.
November 8: "Mathematics, mathematicians and humor" by M. H. Ingraham, Gr.
December 6: "Non-euclidean geometry" by L. M. Graves, Gr.
January 10, 1923: "Some simple methods of deriving series" by Mrs. J. P. Ballantine, Gr.
January 24: "Classes, sets and some paradoxes" by C. E. Van Horn, Gr.
February 7: "The fourth dimension" by D. S. Patton, Gr.
February 21: "Coefficients of correlation" by H. R. Phalen, Gr.
April 11: "The game of Nim" by Esther Weaver, Gr.
April 25: "Caustic curves" by D. L. Holl, Gr.

(Reported by Miss Mauch.)

THE MATHEMATICS CLUB OF HAMLINE UNIVERSITY, St. Paul, Minn.

This club was organized in the spring of 1923, Professor Roger A. Johnson acting as president, and Victor Westman '24 as secretary. The following papers were presented:
March 28, 1923: "Nim and other mathematical recreations" by Mr. C. A. Rupp, instructor.
April 25: "The theory of relativity" by Stanley Poukey '23.
May 23: "Some aspects of non-euclidean geometry" by Professor Johnson.

(Reported by Professor Johnson.)

THE MATHEMATICS CLUB OF THE STATE UNIVERSITY OF IOWA, Iowa City, Ia. [1922, 355.]

As officers for the year 1922–1923 the following were elected: President, Orley Brown, graduate assistant; secretary, Glenn Aldrich, graduate assistant. The meetings, all open to the public, were well attended. Papers were read as follows:
October 12, 1922: "Purposes of the Mathematics Club" by Professor W. H. Wilson.
November 2: "Magic squares" by Helen Moon, Gr.
November 16: "Some phases of projective geometry" by Orley Brown, Gr.
December 7: "Mathematical recreations" by Iona Reger '23.
January 18, 1923: "Mathematical tricks and puzzles" by Glenn Aldrich, Gr.
February 1: "The development of algebraic notation" by Frances Baker '23.
February 15: "The cubic equation" by Eric Erickson, Gr.
March 1: "The theory of probability" by Henry Pollard, Gr.
March 15: "The ellipse" by Ruth Balluf '23.
April 12: "Various kinds of averages" by Clarence Balof, Gr.
May 3: "Logarithms of complex numbers" by Howard Hughes '23. Election of officers: President, Eric Erickson, Gr.; secretary, Howard Hughes '23.

(Reported by Mr. Aldrich.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3040. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Give the radius, R, of a sphere rolling down two intersecting straight lines including the angle 2α and equally inclined to the horizon; show that the locus of the center of the sphere is an ellipse of semi-axes $R \csc\alpha$, R.

3041. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line passing through a fixed point.

3042. Proposed by C. N. SCHMALL, New York City.

Given the bases and the sum of the areas of several triangles that have a common vertex; show that the locus of the vertex is a straight line.

3043. Proposed by O. D. KELLOGG, Harvard University.

Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)dxdy = 0 \tag{1}$$

for all functions V with continuous second derivatives and vanishing on the boundary of T, U is harmonic, *i.e.*, satisfies Laplace's equation $(\partial^2 U/\partial x^2) + (\partial^2 U/\partial y^2) = 0$. (It is understood that if U becomes infinite in the neighborhood of a boundary point of T, V is so further restricted that the integral shall have a sense.)

SOLUTIONS.

2981 [1922, 313]. Proposed by H. P. MANNING, Providence, R. I.

Find the envelope of the circle on which two diametrically opposite points divide in a given ratio the focal radii of a variable point on an ellipse or hyperbola.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let C be a point on a central conic with foci A and B, and center O. If Q and Q' are points on BC and AC such that $BQ/BC = AQ'/AC = k$, then the diameter, QQ', of the circle of the problem is equal to $(1-k)AB = 2(1-k)ae$, CO passes through the center C' of the circle, and $OC' = kOC$. Thus C' describes a conic similar to the original, and the envelope of the equal circles must consist of two branches parallel to the conic C'. If the coördinates of C, in case of the ellipse, are $a\cos\varphi$, $b\sin\varphi$, then the coördinates of C' are $ka\cos\varphi$, $kb\sin\varphi$. Let ψ be the inclination of the normal at C', then $\tan\psi = a\sin\varphi/b\cos\varphi$, $\sin\psi = \sin\varphi/\sqrt{1-e^2\cos^2\varphi}$. Hence

$$x = ka\cos\varphi \pm \frac{(1-k)be\cos\varphi}{\sqrt{1-e^2\cos^2\varphi}},$$
$$y = kb\sin\varphi \pm \frac{(1-k)ae\sin\varphi}{\sqrt{1-e^2\cos^2\varphi}}.$$

The hyperbola may be treated in a similar manner.

Also solved by A. PELLETIER, J. B. REYNOLDS and J. K. WHITTEMORE.

2988 [1922, 356]. Proposed by PHILIP FRANKLIN, Harvard University.

Prove, geometrically, that if in an ellipse the tangent at P cuts the directrices in Z, Z' and the remaining tangents from Z and Z' to the ellipse meet at T, PT is the normal to the ellipse at P. (An analytic proof is given in the *Journal of the Indian Mathematical Society*, vol. 13, 1921, p. 234.)

SOLUTION BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

The line QQ' joining the points of contact Q, Q' of the tangents TZ, TZ' with the given ellipse (E) is the polar of the point T, hence PT is the polar of the point $R \equiv (QQ', PZZ')$; therefore, the four points Q, Q', R, $S \equiv (QQ', PT)$ are harmonic, and so is the pencil $P(QQ', RS)$. Now the lines PQ, PQ' are the polars of Z, Z' with respect to (E), and since Z, Z' lie on the directrices of (E), the lines PQ, PQ' pass through the foci of (E). Thus the line PT is separated harmonically from the tangent PZZ' by the focal radii PQ, PQ' of P, *i.e.*, PT is the normal to (E) at P. For $\angle QPZ = \angle Q'PZ$, a well-known property, and hence $\angle QPS = \angle Q'PS$.

Also solved by J. W. CLAWSON, R. M. MATHEWS, A. PELLETIER, MABEL M. YOUNG and the PROPOSER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Because the Mathematical Association meets at the University of Cincinnati in December in affiliation with the American Association for the Advancement of Science, railroad rates are available for a fare and a half. The reduced rates should prove attractive to the Association members and guarantee a large attendance from the Middle West, as well as from points farther west and east. President Carmichael's presidential retiring address and other important papers will be given at the first session on Thursday afternoon, December 27. A second session will be held on Friday morning and a joint session on Friday afternoon, in conjunction with Section A of the American Association and the Chicago Section of the American Mathematical Society, at which time addresses in commemoration of the seventy-fifth anniversary of the American Association will be given by Professors G. A. Miller and A. B. Coble, and, at the invitation of the Chicago Section and this Association, Professor L. E. Dickson will speak on "Arithmetics and their Algebras."

The following 19 doctorates with mathematics as major subject were conferred by American Universities in the academic year 1921–1922; the university and the title of the dissertation are given with each name. C. R. ADAMS, Harvard, "The general theory of linear partial q-difference equations and of the linear partial difference equation of the intermediate type"; B. H. BROWN, Harvard, "The equilong transformations of Euclidean space"; MARGARET BUCHANAN, Bryn Mawr, "Systems of two linear integral equations with two parameters and symmetrisable kernels"; E. H. CARUS, Chicago, "Invariants as products: a vector interpretation of the symbolic method"; W. E. CEDERBERG, Wisconsin, "On the solution of the differential equations of motion of a double pendulum"; H. S. EVERETT, Chicago, "Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials"; V. D. GOKHALE, Chicago, "Concerning compact Kürschák Fields"; CLARIBEL KENDALL, Chicago, "Certain congruences determined by a given surface"; R. E. LANGER, Harvard, "I. Developments associated with a boundary problem not linear in the parameter. II. The boundary problems and developments associated with a system of linear differential equations of the first order"; C. C. MACDUFFEE, Chicago, "Invariantive characteristics of linear algebras with the associative law not assumed"; E. D. MEACHAM, Chicago, "Properties of surfaces whose osculating ruled surfaces belong to linear complexes"; EUGENIE M. MORENUS, Columbia, "Geometric properties completely characterizing the set of all the curves of constant pressure in a field of force"; ANNA M. MULLIKIN, Pennsylvania, "Certain theorems relating to plane connected point sets"; ELEANOR PAIRMAN, Radcliffe, "Expansion theorems for solution of a Fredholm's linear homogeneous integral equation of the second kind with kernel of special

non-symmetric type"; H. P. PETTIT, Illinois, "A general cyclide with special reference to the quintic cyclide"; J. F. REILLY, Iowa, "On certain generalizations of osculatory interpolation"; W. P. UDINSKI, Illinois, "On a series of rational functions formally analogous to Fourier's series"; B. C. WONG, California, "A study and classification of ruled quartic surfaces by means of a point-to-line transformation"; E. B. ZEISLER, Chicago, "Definite integral representation of invariants."

The following five doctorates, conferred in the latter half of the calendar year 1922, are listed in the same way as the preceding. E. H. CLARKE, Chicago, "On the minimum of the sum of a definite integral and a function of a point"; J. D. ESHLEMAN, Chicago, "The Lagrange problem in parametric form in the calculus of variations"; W. S. KIMBALL, Chicago, "Scattering of particles by an Einstein curve"; HARRY LANGMAN, Columbia, "Conformal transformations of period n and groups generated by them"; J. S. TURNER, Chicago, "Fundamental system of formal invariants of a modular group of transformations."

The following four doctorates in mathematical physics or mathematical astronomy were conferred during the calendar year 1922; the university, the major subject, and the title of the dissertation are given with each name. F. E. CARR, Chicago, mathematical astronomy, "A solution of the problem of two bodies one of which is a rotating oblate spheroid"; H. C. LEVINSON, Chicago, mathematical astronomy, "The gravitational field of masses relatively at rest according to Einstein's theory of gravitation"; LOUIS SLICHTER, Wisconsin, mathematical physics, "An experimental study of an acoustic system"; WARREN WEAVER, Wisconsin, mathematical physics, "A summary of the analytic formulation of the theory of electrodynamics."

The many friends of SAMUEL DOUGLAS KILLAM, professor of mathematics at the University of Alberta, will be shocked to learn of his death by drowning at Lake Annis, Nova Scotia, on July 22, 1923. He was born in Yarmouth, Nova Scotia, May 9, 1888, and received the degrees of B.A. and M.A. from Mount Allison University in 1908 and 1910 respectively. He spent the winter semester of 1908–1909 at the University of Berlin but then went to the University of Göttingen where he received the Ph.D. degree in 1912. His thesis (28 pp. and 15 plates) was entitled *Ueber graphische Integration von Funktionen einer komplexen Variabeln mit speziellen Anwendungen.* In 1912 he was appointed instructor in mathematics at the University of Rochester; in the following year he went as instructor in mathematics to the University of Alberta where he remained ever since except for a period of war service in the Canadian Artillery. In 1915 he was promoted to be assistant professor of applied mathematics and he was promoted to a full professorship in 1921. In May of last year he presented to the Royal Society of Canada a paper on "The solution of plane triangles by nomographic charts." He published an article on "Bond schedules for the amortization of a premium or accumulation of a discount" in the *Canadian Chartered Accountant,* April, 1920. He has been a member of the Association since 1920.

JOHN GASTON LEATHEM died March 19, 1923. He was born in Belfast, Ireland, on the 5th of May, 1871. After studying at Queen's College, Belfast, he went to Cambridge in 1891 and was fourth wrangler in the tripos of 1894. He held the Isaac Newton studentship in astronomy and physical optics from 1896–99 and was mathematical lecturer at St. John's College, Cambridge, afterwards becoming university lecturer. In 1905 he took up, in conjunction with E. T. Whittaker, the editorship of the well-known series of mathematical tracts to which he contributed two volumes. His main interests were electrodynamical theory and the application of the theory of conformal mapping to problems in physics.

In his recent presidential address to the Institute of Physics, Sir J. J. THOMSON gave an account of some of his impressions during his recent visit to America. Of interest to teachers of mathematics is his statement that American Universities do not seem designed to produce a large number of men qualified to take up advanced research work. Few of the science students, he says, have the necessary equipment in mathematics, while the stern training which a good student in a great English University has to go through appears to be unknown. In this regard, Great Britain has a distinct advantage which is sorely needed if America is to hold its own in competition. In these remarks the distinguished physicist and mathematician has undoubtedly pointed out one of the weakest points in our current mathematical instruction: the failure to impress sufficiently upon the first rate student of science the absolute necessity of advanced mathematical training.

SUBSCRIPTIONS TO FORTSCHRITTE AND JAHRESBERICHT

Members of the ASSOCIATION are already aware of the extortionate prices demanded for *Jahrbuch über die Fortschritte der Mathematik,* vol. 45 ($19.20), and *Jahresbericht der deutschen Mathematiker-Vereinigung,* vol. 31, 1922 ($14.00). As the result of extensive correspondence, Professors Bieberbach and Lichtenstein have communicated the following offer to members of the ASSOCIATION: Volume 46 of the *Fortschritte* will be supplied at 25 per cent. discount on the published price, postage and packing being extra. The first Heft of this volume (550 pages, 1923), published at $5.45, therefore costs $4.20. The *Jahresbericht,* vol. 32, 1923, will be sent for $2.25 which includes the dues for the Vereinigung.

In case any *Library,* not in a College or University already an institutional member of the ASSOCIATION, wishes to avail itself of the above rate, it may do so by joining the Vereinigung for which the annual fee is, for the time being, not more than fifty cents per year.

Orders under the above plan should be sent to Professor Dr. L. Bieberbach, Marienbaderstr. 9, Berlin-Schmargendorf, Germany. Payment should be made in dollars, when ordering, by draft either on New York or Berlin.

We are informed that if a considerable number of new subscribers results from these very reasonable rates, a still further reduction in the price per page of the *Fortschritte* may be expected. The number of subscribers now is 130 less than before the war.

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or September, 1918—fifty cents; September, 1915— seventy-five cents; May, 1915—one dollar (See MONTHLY, March, 1921, p. 152)

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum with out going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the..

....Fund, and to be used

for { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witnesses: Signature..

....

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Annual Meeting, University of Cincinnati, December 27–28, 1923.

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Knox College, Galesburg, May 4–5

IOWA, Des Moines, November; Ames, April, 1924

KANSAS, Topeka, January 20

KENTUCKY, Center College, April, 1924

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Annapolis, December 8, 1924

MINNESOTA, Northfield, May 19

MISSOURI, University of Missouri, Columbia, November 30–December 1

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April

SOUTHEASTERN, Agnes Scott College, Decatur, March 10

TEXAS, Fort Worth, November 30–December 1, 1924

INDEX TO VOLUME XXX, 1923.

By MILDRED E. CARLEN, Brown University.

Misprinted names in the text are corrected, and missing initials supplied, in this index.

PAPERS, REPORTS OF MEETINGS.

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QUESTIONS AND DISCUSSIONS—QUESTIONS.

RECENT PUBLICATIONS—REVIEWS.

RECENT PUBLICATIONS—NOTES—BOOKS.

RECENT PUBLICATIONS—PERIODICALS.

MATHEMATICAL CLUBS—ACTIVITIES.

PROBLEMS—AUTHORS.

Numbers refer to pages, black-face type indicating a problem solved and solution published, italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS.

Numbers in black-face type refer to problems, those in light-face to pages.

NOTES AND NEWS.

PERSONAL MENTION.

[In order to find a complete list of references to any individual, the alphabetical lists under "Papers, Reports of Meetings," "Questions and Discussions—Discussions," "Recent Publications—Reviews," "Recent Publications—Notes—Books," and "Problems—Authors" should also be consulted.]

PERSONAL MENTION—NECROLOGY.

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THE EIGHTH SUMMER MEETING OF THE ASSOCIATION.

The eighth summer meeting of the Mathematical Association of America was held by invitation at Vassar College on Wednesday and Thursday, September 5–6, 1923, in conjunction with, and immediately preceding, the summer meeting of the American Mathematical Society. One hundred fifteen were present at the meeting, including the following seventy members of the Association:

H. L. Agard, Williams College.
N. Altshiller-Court, University of Oklahoma.
R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.
Clara L. Bacon, Goucher College.
Ida Barney, Yale Observatory.
Mrs. Ethelwynn R. Beckwith, Harvard University.
Suzan R. Benedict, Smith College.
B. A. Bernstein, University of California.
Harry Birchenough, N. Y. State College for Teachers.
E. W. Brown, Yale University.
H. S. Brown, Hamilton College.
J. A. Bullard, U. S. Naval Academy.
W. G. Bullard, Syracuse University.
R. W. Burgess, Brown University.
W. D. Cairns, Oberlin College.
G. M. Conwell, N. Y. State College for Teachers.
Lennie P. Copeland, Wellesley College.
G. W. Creelman, Hotchkiss School.
Louise D. Cummings, Vassar College.
C. E. Dimick, U. S. Coast Guard Academy.
John Eiesland, University of West Virginia.
W. B. Ford, University of Michigan.
O. E. Glenn, University of Pennsylvania.
W. C. Graustein, Harvard University.
C. C. Grove, Metropolitan Life Insurance Company.
Ruth H. Hall, Rosemary Hall, Greenwich, Conn.
E. R. Hedrick, University of Missouri.
Anna M. Howe, Newcomb Memorial College.
E. V. Huntington, Harvard University.
Nelle L. Ingels, Interstate Commerce Commission.
Myra I. Johnson, Syracuse University.
D. K. Kazarinoff, University of Michigan.
H. A. Kiess, Albright College.
J. R. Kline, University of Pennsylvania.
W. D. Lambert, U. S. Coast and Geodetic Survey.
Florence P. Lewis, Goucher College.
J. J. Luck, University of Virginia.
C. C. MacDuffee, Princeton University.
R. M. Mathews, Wesleyan University.
Gertrude I. McCain, Westminster College.
J. F. Messick, Emory University.
C. N. Moore, University of Cincinnati.
Eugenie M. Morenus, Sweetbriar College.
C. C. Morris, Ohio State University.
A. D. Pitcher, Western Reserve University.
E. L. Post, New York City.
Susan M. Rambo, Smith College.
Arthur Ranum, Cornell University.
H. W. Reddick, Cooper Union.
L. J. Reed, Johns Hopkins University.
R. G. D. Richardson, Brown University.
E. D. Roe, Jr., Syracuse University.
J. B. Rosenbach, Carnegie Institute of Technology.
Mary E. Sinclair, Oberlin College.
H. E. Slaught, University of Chicago.
H. L. Slobin, New Hampshire College.
Clara E. Smith, Wellesley College.
D. E. Smith, Columbia University.
W. M. Smith, Lafayette College.
May J. Sperry, Syracuse University.
Bird M. Turner, University of West Virginia.
H. W. Tyler, Massachusetts Institute of Technology.
H. S. Vandiver, Cornell University.
Oswald Veblen, Princeton University.
Mary E. Wells, Vassar College.
V. H. Wells, Williams College.
J. K. Whittemore, Yale University.
Euphemia R. Worthington, Southern Branch, University of California.
J. W. Young, Dartmouth College.

The local committee under the chairmanship of Professor H. S. White made ample provision for the convenience and comfort of the guests. Those in attendance were housed in Main Hall, the central building of Vassar College, and took their meals in the commodious dining hall. The officers of the two

mathematical organizations were assigned to commodious guest rooms, the retiring president of the Association enjoyed the distinction of using a bed formerly the possession of Matthew Vassar, the founder of the College. The joint dinner of the two organizations was held on Thursday evening, about one hundred twenty being present. Professor E. W. Brown acted as toastmaster and short talks were made by President MacCracken, Professors White, Slaught, and Huntington. A number of features added much to the pleasure of the guests during the meetings. Tea was served each afternoon at the home of Professor and Mrs. White. On Wednesday afternoon at the close of the program the majority of those in attendance made a tour of the campus and buildings, including a walk to Sunset Hill just outside the campus. On motion of the Secretary, at the session on Thursday morning a vote of thanks was given by the members to Professor and Mrs. White for their hospitality, to President MacCracken and the authorities of Vassar College for their generosity in making possible the facilities of Vassar College during the vacation season, to the local committee for its effective and untiring efforts, and to the program committee under the chairmanship of Professor Lennie P. Copeland for the interesting and profitable papers which were prepared under their direction.

At the joint session of the Association and Society on Thursday afternoon, a vote of sympathy was extended to the Physico-Mathematical Society of Japan, expressing our deep sorrow and sincere sympathy in the great calamity that has befallen the Japanese people, and expressing the hope that the later reports may be more favorable than those at first received, and that the indomitable will that has always characterized the Japanese nation may serve to establish conditions as nearly normal as such a catastrophe can permit.

In the absence of the president and vice-presidents, Professor Slaught presided at the session on Wednesday afternoon, and Professor Huntington at the session on Thursday morning. Professor Veblen, the president of the Society, presided at the joint session on Thursday afternoon. The program of the meeting follows, accompanied by abstracts of the papers and discussions, the numbers corresponding to the numbers in the list of titles. A telegram to the secretary announced that, by reason of illness, Dr. J. G. Coffin could not present his paper on "Mathematical training for laboratory men."

Session of the Association.

(1) "Elliptic geometry" by Professor James Pierpont, Yale University, by invitation.

(2) "The influence of engineering on mathematical teaching" by Professor E. R. Hedrick, University of Missouri.

(3) "The honor student in mathematics" by Professor Suzan R. Benedict, Smith College.

1. Professor Pierpont's address will appear in full in an early number of the Monthly.

2. It has often been assumed that the effect of engineering on mathematical

teaching in colleges has been detrimental to the best interest of the subject, and has tended to suppress its cultural and scientific values. Professor Hedrick reviewed in some detail the major subjects in mathematics taught to engineers, and pointed out some of the more important changes that have taken place in the period since engineering has exerted its greatest influence. Among these changes are the earlier introduction of the ideas of the calculus, the emphasis upon functional representation as opposed to geometric properties of conic sections in analytic geometry, the widening of the range of applications of the calculus, the encouragement of the use of tables of integrals in place of formal solutions of formal problems in integration, and the greater emphasis that has been placed on differential equations. It is pointed out that the cultural and scientific values of the resulting course are at any rate as great as in the older courses, and that the training of the student in functional thinking is now far better than under the older scheme. Attention is directed also to the introduction of vector methods in engineering work, and to the increasing demand for reasonable training in the determination of empirical formulas from scientific experimental data.

3. For the last two years Smith College has offered a special honors course to be substituted for the ordinary curriculum in the junior and senior year. In this course a very small group of students, selected partly because of high standing and partly because of evident capacity for independent thinking, are released from all class room responsibilities. Under the direction of members of the faculty they pursue one line of work chosen by them, their progress being judged by conference with the teachers in charge, a paper submitted at the end of the course, and comprehensive final examinations.

There is a question as to whether mathematics is a subject which a student so young, and necessarily so untrained, could follow profitably without more supervision and assistance than this system contemplates. If so a second question arises as to the reading to be suggested.

If the work is divided into eight units, two of which are equivalent to the entire time of the student, the program contemplated is as follows:

2 units Advanced Calculus, Differential Equations, Algebra;

2 units Modern Geometry, Functions of a Real and of a Complex Variable;

2 units Applications;

2 units Review and correlation of the work covered and preparation of a paper.

This paper is submitted with the hope that the department of mathematics at Smith may have the opinion of this Association upon the two questions before us, that is, the suitability of mathematics for this kind of specialization and the choice of subjects which might be pursued profitably by the undergraduate.

Professor Clara Smith stated that honor students at Wellesley are required to concentrate in a field, within which, however, there may be several lines closely connected. This concentrated program can begin with the junior or with the senior year. For example, two students recently took two 3-hour courses together with courses in physics, such students being allowed three hours of their

own choosing in the senior year, if they so desire, and if the department thinks best. These students are asked to make such reports as would lead them to read up on certain courses not covered in the regular courses, topics suited to this purpose having been assigned. An oral examination is held for each candidate, covering both the regular class courses and the independent work.

Professor Mordell stated that he has been used to practically the same scheme in the English universities. The honor student is quite at liberty to read all the books on a certain subject, although it is the exceptional student who can wisely direct his own effort, for a lecturer can put the material in a proper perspective for the students, whereas an unguided student may get a wrong or incomplete view of the subject. Most students prefer to attend lectures regularly and to take very full notes, even to the entire disregard of text books. They may elect special subjects even from the beginning of their university course.

Professor Burgess spoke of the type of English honor system which prevails at Oxford. He concluded from his own observations that this system is not applicable to the United States unless accompanied by the tutorial system, the scholarship system, and the examination system. He mentioned too that most of these students have won their scholarships through close competition coming up from the secondary schools.

Professor Slaught raised a question as to the wisdom of turning undergraduates loose in independent effort and of allowing them to spend all of the two years in a given subject. In reply Miss Benedict reiterated that the students were quite at liberty in the independent work of their last two years, that they were first of all very carefully selected, and that they report fortnightly to their Faculty advisors.

Session of the Association

(4) "A mathematical formulation of the law of growth" by Professor L. J. Reed, Johns Hopkins University.

(5) "Mathematical methods in economic research" by Professor C. C. Morris, Ohio State University.

(6) "Geophysics and mathematics" by W. D. Lambert, U. S. Coast and Geodetic Survey.

(7) "Mathematics applied and misapplied" by Dr. C. C. Grove, Metropolitan Life Insurance Company.

4 and 5. The papers by Professors Reed and Morris will appear in early issues of the Monthly.

6. After dealing briefly with the scope of geophysics, Mr. Lambert called attention to the fact that geophysics is more dependent on mathematical reasoning than physics in the narrower sense. No one person could discuss all the mathematical problems arising in so wide and varied a field as geophysics. The speaker confined his remarks to two topics of a general nature:

(1) the use of elementary mathematical methods; and

(2) the importance of numerical computation.

Elementary methods are used rather in clearing the ground of vague and

unprofitable speculations that hinder the advance line than in making the advance itself. Such speculations are uncommonly plentiful in geophysics, and many of them can be disposed of by a little careful thought combined with an elementary knowledge of mathematics and physics. Instances of such were given. In conclusion the speaker emphasized the matter of numerical computation both as a subject for instruction and as a necessary complement to the analytical treatment of problems having a practical application.

7. Aside from a plea for teachers who are alive and enthusiastic for their subject, and are in the profession as a life work, Dr. Grove presented the need for more emphasis on the calculus of finite differences in our colleges, especially for students intending to specialize in science, in the economic and social sciences as well as in the physical sciences. He pointed out that especially in statistical work we have much to do with quantities that can vary only by units, and that therefore much of the work would not naturally be cast into the forms of the infinitesimal calculus. In *Bulletin No. 27, 1921, Bureau of Education*, the speaker emphasized combinations and permutations as leading up to the treatment of mathematical probability and thus to the study of empirical probability and frequency distributions, also finite differences which we use in many practical ways in connection with statistical series to detect mistakes in observation or in recording; to interpolate and extrapolate terms, to graduate series, to show where series are really algebraic, and of what order, to form series, etc. He referred also to the direct and clear presentation made in "*Calculus and Probability for Actuarial Students*," by Alfred Henry, C. & E. Layton, London, 1922.

He discussed as a concrete example a table giving the probability of dying eventually of cancer at various ages, with a view to fitting a curve to the changing probabilities. He also instanced numerous ways in which mathematics was misused or misinterpreted. He mentioned, for example, that there is apt to be too little consideration of the distinction between a qualitative and quantitative study of any subject, those making a qualitative study and embellishing it with a few arithmetical or other mathematical stunts thinking that they are thus making a quantitative study. It is his opinion that in any quantitative study of the stock market, for instance, the approach should be through the theory of probabilities rather than by other methods recently used, and that in the treatment of business cycles there should be a greater use of trigonometric series, of the theory of probability and of differential equations, rather than such a mass of what seem to be cycles and epicycles. He closed his presentation by an extended treatment of the significance of the decline or rise in deaths from certain causes from year to year, pointing out with Yule that "the theory of simple sampling cannot apply to the variations of the death rate in localities or populations of different age and sex compositions, nor to the death rates in a mixture of healthy and unhealthy districts, nor to death rates in successive years during a period of continuously improving sanitation."

Joint Session of the Association with the American Mathematical Society.

(8) "An introductory account of the arithmetical theory of algebraic numbers and its recent development" by L. J. Mordell, Fielden Professor of Pure Mathematics, University of Manchester, England, by invitation of the American Mathematical Society.

(9) Presidential retiring address: "Mathematicians and music" by Professor R. C. Archibald, Brown University.

8. Professor Mordell's paper will appear in an early issue of the Bulletin of the American Mathematical Society. It was a great pleasure and inspiration to our members to have Professor Mordell present at the meetings, to listen to his able summary of the development of the arithmetical theory of algebraic numbers, and to have his keen comments at various points in the discussions.

9. The presidential retiring address by Professor Archibald will be printed in full in an early issue of the Monthly.

Meeting of the Board of Trustees of the Association.

Seven members of the Board were present at this meeting. The following fifty-nine persons and one institution were elected to membership on applications duly certified:

To individual membership.

Eli Allison, M.S. (Wesleyan). Prof., Brenau Coll., Gainesville, Ga.
Kathleen Begley, A.B. (Oklahoma). Teacher, High School, Henryetta, Okla.
R. T. Belcher, B.A. (Queens Univ., Ireland). Asst. prof., Pomona Coll., Claremont, Calif.
Undine Butler, A.B. (Oklahoma). Teacher, Central High School, Oklahoma City, Okla.
Helen Calkins, A.M. (Columbia). Asst. prof., Knox Coll., Galesburg, Ill.
P. A. Caris, A.M. (Bucknell). Teacher, W. Phila. High School for Boys, Philadelphia, Pa.
Bro. E. Charles, A.M. (Catholic Univ.). 1240 N. Broad St., Philadelphia, Pa.
F. C. Clement, A.B. (St. Olaf). Instr., St. Olaf Coll., Northfield, Minn.
Emaline Collins, A.B. (E. Central St. Teachers Coll.). Teacher, High School, Davis, Okla.
G. W. Creelman, A.B. (Harvard). Hotchkiss School, Lakeville, Conn.
H. B. Curtis, Ph.D. (Cornell). Instr., Northwestern Univ., Evanston, Ill.
H. M. Dadourian, Ph.D. (Yale). Prof., Trinity Coll., Hartford, Conn.
Mrs. Armaud P. Daspit, A.B. (La. State Univ.). Instr., La. State Univ., Baton Rouge, La.
R. D. Daugherty, Ph.B. (Iowa Wesleyan). Prof., Ia. State Teachers Coll., Cedar Falls, Ia.
D. M. Garrison, Grad. U. S. Naval Acad. Head of dept. of math., St. John's Coll., Annapolis, Md.
E. F. Gee, Ph.B. (Michigan). Instr., Detroit Jr. Coll., Detroit, Mich.
Harry Gwinner, M.E. (Maryland). Vice Dean, Eng. Coll., Univ. of Maryland, College Park, Md.
R. L. Hicks, B.S. (Lebanon); A.B. (Oklahoma). Dale, Okla.
C. M. Howard, E. Mines (Ala. Polytech. Inst.). Prof. of math. and registrar, Texas Woman's Coll., Fort Worth, Tex.
A. R. Jerbert, M.S. (Washington). Assoc., Univ. of Washington, Seattle, Wash.
Bro. F. John, A.M. (Catholic Univ.). 1240 N. Broad St., Philadelphia, Pa.
Myra I. Johnson, Ph.B. (Syracuse). Grad. student, Syracuse Univ., Syracuse, N. Y.
Odessa R. Lastrapes, A.B. (Tulane). New Orleans, La.
Katharine Leonard, A.M. (Vermont). Head of dept. of math., State Teachers Coll., Moorehead, Minn.
Peysah Leyzerah, Ph.D. (Clark). Instr., Lehigh Univ., Bethlehem, Pa.
A. V. Martin, A.B. (Hampden-Sidney). Prof., Presbyterian Coll. of S. C., Clinton, S. C.

LIDA B. MAY, A.M. (Texas). Prof., Kidd-Key Coll., Sherman, Tex.
E. H. McALISTER, A.M. (Oregon). Prof., Mech. and astr., Univ. of Oregon, Eugene, Ore.
T. F. McBEATH, A.B. (W. Ky. Normal). Head of dept. of math., Miss. State Coll. for Women, Columbus, Miss.
INA L. McBEE, A.B. (Okla.). Teacher, Jr. High School, Duncan, Okla.
W. H. McEWEN, M.S. ( ). Asst., Univ. of Minnesota, Minneapolis, Minn.
G. E. MEADOR, A.M. (Okla.). Prof., Oklahoma City Coll., Oklahoma City, Okla.
Rev. J. H. MEYER, S.J. Church of Immaculate Conception, New Orleans, La.
E. D. MOUZON, Jr., A.B. (Southern Meth. Univ.). Instr., Southern Meth. Univ., Dallas, Tex.
Rev. PAUL MUEHLMAN, A.M. (St. Louis Univ.). Instr., Loyola Univ., Chicago, Ill.
HERMANCE MULLEMEISTER, Ph.D. (Utrecht, Holland). Instr., Univ. of Washington, Seattle, Wash.
W. K. NELSON, E.E. (Colorado). Asst. prof. of eng. math., Univ. of Colorado, Boulder, Colo.
Sister MARY PAULA, M.S. (Notre Dame). Prof., St. Mary's Coll., Monroe, Mich.
L. E. PETTY, A.M. (Peabody). Pres., Silliman Coll., Clinton, La.
T. I. PORTER, A.B., B.S. in Educ. (Missouri). Instr. in math. and physics, Univ. of Omaha, Omaha, Nebr.
E. L. POST, Ph.D. (Columbia). 280 Manhattan Ave., New York, N. Y.
JULIA P. PROSSER, A.M. (South Carolina). Saint Mary's School, Raleigh, N. C.
Mrs. EFFIE M. RALLS, A.B. (Tri State Ind.). Head of dept. of math., High School, Coalgate, Okla.
W. E. ROBERTSON, A.B. (Okla.). Asst. instr., Univ. of Okla., Norman, Okla.
FLORENCE C. ROHRBAUGH, A.B. (Okla.). Teacher, High School, Norman, Okla.
LILLIAN ROSANOFF, Ph.D. (Clark). Teacher, Commercial High School, Brooklyn, N. Y.
C. A. RUPP, A.M. (Harvard). Instr. in math. and astr., Hamline Univ., St. Paul, Minn.
R. Q. SEALE, A.M. (Columbia). Instr., Southern Meth. Univ., Dallas, Tex.
HARRY SLAWTER, A.B. (W. Va. Wesleyan). Bridgeport, W. Va.
T. F. SMITH, A.M. (Manhattan Coll.). Prof., Little Rock Coll., Little Rock, Ark.
J. B. STEED, B.S. in Educ. (Missouri). Prof. of educ., Miami State Sch. of Mines, Miami, Okla.
W. P. SUESMAN, LL.B. (Lake Forest). Treas., Household Furniture Co., Providence, R. I.
J. M. TAYLOR, M.S. (Adrian). Instr., Univ. of Washington, Seattle, Wash.
ROBERT TORREY, Ph.B. (Mississippi). Asso. prof., Univ. of Mississippi, University, Miss.
R. O. WEBB, A.B. (Okla.). Prin., High School, Wilson, Okla.
J. D. WHITNEY, A.B. (Okla.). Fellow, Univ. of Oklahoma, Norman, Okla.
W. M. WHYBURN, A.M. (Texas). Head of dept. of math., South Park Jr. Coll., Beaumont, Tex.
L. E. WILLIAMS, A.M. (Chicago). Dean, Woman's Coll. of Alabama, Montgomery, Ala.
R. E. WITT, C.E. (Washington & Lee). Lexington, Va.

To institutional membership.

THE UNIVERSITY OF DELAWARE, Newark, Del., Prof. G. A. Harter, Official representative.

It was voted to hold no summer meeting of the Association in 1924 because of the mathematical meetings to be held at Toronto, and to urge our members to join with all other mathematicians of the country in attending these important meetings. For this reason the Trustees requested President Carmichael to give his Presidential retiring address at the Cincinnati meeting in December. (The plans for the Cincinnati meeting are announced in another place in the MONTHLY.)

Professor Archibald and the Secretary were asked to send a letter of appreciation to Professor H. P. Manning recognizing his valuable and self-sacrificing labors in the editorial work of the MONTHLY.

The Trustees transacted other items of business in a preliminary way concerning the finances of the Association, and the question of places for the summer meetings.

W. D. CAIRNS, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE TEXAS SECTION.

The second annual meeting of the Texas Section of the Association was held in the Sunday School room of the First Presbyterian Church in Houston, Texas, on December 1–2, 1922, in conjunction with a meeting of the Mathematics Section of the Texas State Teachers' Association. A joint meeting of the two associations was held on the afternoon of the first and the Texas Section met alone the following morning. The secretary of the Section was unable to be present. The following officers were elected for the following year: Chairman, L. R. Ford; Vice-Chairman, A. A. Bennett; Secretary-Treasurer, H. J. Ettlinger.

There were forty-two in attendance including the following sixteen members of the Association:

A. A. Bennett, J. E. Burnam, P. J. Daniell, Elizabeth Dice, H. J. Ettlinger, G. C. Evans, L. R. Ford, H. Halperin, A. J. Hargett, E. H. Jones, J. O. Mahoney, H. Porter, C. P. Rockwell, E. R. Tucker, P. H. Underwood, C. N. Wunder.

The following papers were read:

(1) "Address on the principles of relativity" by Professor H. Halperin;

(2) "Operations with negative numbers, formal and intuitional justification" by Miss Lel Red (by invitation);

(3) "A series of rational functions analogous to Fourier series" by Professor W. P. Udinski (by invitation);

(4) "Some applications of Duhamel's theorem" by Professor H. J. Ettlinger;

(5) "Descriptive geometry with applications to axomometry and photogrammetry" by Miss Elizabeth Dice;

(6) "Significant figures" by Professor A. A. Bennett;

(7) "Interest and annuities" by Professor E. H. Jones;

(8) "History and theory of workmen's compensation insurance" by Mr. C. P. Rockwell;

(9) "Simple examples of variable annuities" by Professor L. R. Ford;

(10) "Training in mathematics as preparation for studies in our schools of commerce" by Professor L. H. Fleck (by invitation);

(11) "Mathematical principles of economics" by Professor G. C. Evans.

Abstracts of these papers follow below, the number corresponding to the number in the list of titles.

1. The extended address delivered by Professor Halperin at the invitation of the Section covered the following points:

a. Unsatisfactory state of pre-Einsteinian physics, where various natural phenomena could not be explained except by means of very strained artificial assumptions.

b. Classical relativity, according to which no absolute uniform motion of an observer can be detected by methods involving mechanical laws of nature only.

c. Einstein's special theory of relativity, according to which the absolute uniform motion of an observer cannot be detected by any methods, whether of a mechanical or of an electromagnetic and optical nature.

d. The two principles of the special theory of relativity and the results that follow from them:

(*a*) Relativity of simultaneity of two events.
(*b*) Relativity of the magnitude of an interval of time.
(*c*) Relativity of the distance between two points, and of the shape of material bodies.
(*d*) Absolute nature of the quantity $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$.
(*e*) Relativity of the mass of a moving material body.
(*f*) Incorporation of the law of conservation of mass into that of conservation of energy.
(*g*) Law of composition of velocities.
(*h*) The absolute nature of the velocity of light, and the fact that no material body can move with such a velocity.

e. Physical explanation of above facts. On the basis of the fundamental principle of relativity each one of a group of observers in relative uniform motion with regard to one another considers himself, perfectly naturally, to be at rest and makes his space and time measurements accordingly.

f. Unsatisfactory features of the special theory of relativity. Desirability of having a generalized theory. Difficulties met with in the attempts of advancing such a theory. Einstein's way of overcoming these difficulties. Similarity between gravitational and centrifugal forces.

g. The generalized principle of relativity. The principle of equivalence. The validity of the special theory of relativity in an infinitely small region of a gravitational field.

h. Results of the generalized theory of relativity. Irregularity in the motion of Mercury. Deflection of the rays of light in passing near the sun. Shift of the lines of the spectrum of light coming from the sun or from the stars, as compared with those of light of terrestrial origin.

2. Miss Red explained some of the psychological difficulties involved in the teaching of negative numbers in a first course in algebra. She gave numerous familiar examples that appeal to the scholar which suggest the feasibility of introducing negative numbers and which at the same time indicate the rules of operation that must be adopted if negative numbers are to prove a useful concept.

3. The system of difference equations and boundary conditions analogous to that which in the differential case leads to Fourier series is found to lead to an expansion problem in the difference calculus involving rational functions. The region of convergence is determined in the ordinary sense and under Cesàro summability of given order.

4. Professor Ettlinger pointed out many applications of Duhamel's theorem selected from various fields of analysis, in particular, from differential geometry, mechanics, and the theory of maxima and minima.

5. Miss Dice emphasized with numerous examples the aid afforded by the methods of descriptive geometry in the study of geometric configurations in space. In view of the simplicity and power of its methods, she made a plea for its wider study.

6. Professor Bennett pointed out many common elementary errors due to a mistaken notion of the character of mathematical accuracy, and urged that the concept of "significant figures" with all its connotation be kept more constantly in mind in mathematical publication and in the instruction of the young.

7. Professor Jones defined the concepts and developed in systematic manner the formulas of the elementary mathematics of finance.

8. Mr. Rockwell, the state actuary of Texas, gave a comprehensive survey of the present and historical aspects of the Workmen's Compensation Insurance as effective in Texas. There are many items of mathematical interest that enter into the problem of just insurance rates, and as a characteristic application of the actuarial formulas and concepts, the subject has mathematical bearings.

9. Professor Ford discussed the modifications required in the usual theory of constant annuities when these annuities are variable. Some rather surprising paradoxes resulting from simple conditions upon the variable annuity were suggested and explained on the basis of the equations involved.

10. Professor Fleck outlined the topics required in the mathematical preparation of students of business methods, as suggested by the nature of the work to be undertaken. He dwelt also upon the added advantages accruing to the individual who has profited by the training in accuracy, abstraction, and system, incident to mathematical discipline.

11. Professor Evans discussed the mathematical aspects of exchange contained in the article appearing in this MONTHLY, *1922*, 371–380.

Reported by A. A. BENNETT (in absence of the Secretary).

THE DEVELOPMENT OF "PARTITIO NUMERORUM," WITH PARTICULAR REFERENCE TO THE WORK OF MESSRS. HARDY, LITTLEWOOD AND RAMANUJAN.

By AUBREY J. KEMPNER, University of Illinois.

Part II.

SYNOPSIS OF HARDY AND RAMANUJAN'S PAPER: ASYMPTOTIC FORMULÆ IN COMBINATORY ANALYSIS (13).[1]

12. To illustrate by a special example the nature of the new method of dealing with the problems of partition, we proceed to give a synopsis of one of the earliest

[1] See (**13**) of the bibliography printed at the beginning of Part I, pp. 355, 356.

Article (**22 b.**), mentioned in footnote 1, 362, came to my notice recently. The main result of this important paper follows: Assuming the hypothesis mentioned above concerning the ζ-function, *almost all* even numbers are the sum of two primes. The expression "almost all" means roughly that, while there may be an infinite number or even numbers which require more than two primes, the percentage of such numbers decreases towards zero as more and more numbers are taken into account. AUTHOR.

papers in which most of the typical features of the theory are employed or foreshadowed: Hardy and Ramanujan, *Asymptotic Formulæ in Combinatory Analysis* (**13**).

It is possible that Hardy and Littlewood's work on Waring's problem is of greater interest for the development of the additive theory of numbers, but in the paper under discussion some of the difficulties which occur in Waring's problem do not make their appearance.

The problem treated is the following:

To determine the order of magnitude of $p(n)$, the number of partitions of n into summands 1, 2, $\cdots$ (Section 6, II). Before this paper was written, nothing was known concerning the order of magnitude of $p(n)$, as a function of n, for large n. Didactically, this paper is of particular interest, because it discusses not one, but several, lines of attack, starting from elementary methods, passing on to methods employing infinite series, then to methods employing Cauchy's integral and finally to methods making use of the most powerful tools to be found in the workshop of the theory of analytic functions of a complex variable.

The more elementary methods are avowedly not carried out so far as might be possible, so that the full power of the elementary methods, as compared with the advanced methods, still remains to be determined. It may be surmised that the untimely death of Ramanujan will have adversely affected this particular line of investigation.

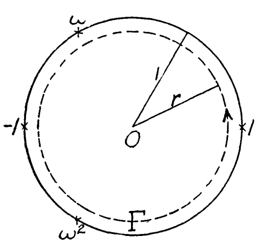

FIG. 1.

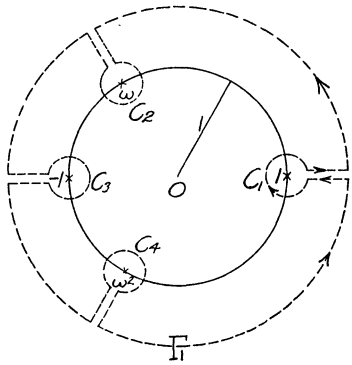

FIG. 2.

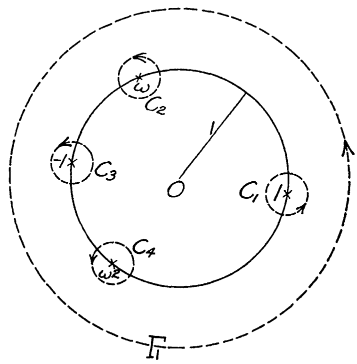

FIG. 3.

13. The results obtained by the various methods may be listed as follows:

I. By elementary methods:

There exist two positive constants, A and B, such that for sufficiently large n[1]

$$e^{A\cdot\sqrt{n}} < p(n) < e^{B\cdot\sqrt{n}},$$

or, *i.e.*,

$$A\cdot\sqrt{n} < \log p(n) < B\cdot\sqrt{n}.$$

II. By less elementary methods, admitting the use of some delicate theorems

[1] And therefore, by choosing if necessary a larger value of B and a smaller value of A, for all positive integers n.

from the modern theory of infinite series, I is refined to: There exists a positive constant C, such that $\log p(n) \sim C \cdot \sqrt{n}$; read: $\log p(n)$ "equivalent to" $C \cdot \sqrt{n}$.

The mathematical contents of this relation are expressed by

$$\limsup_{n=\infty} \frac{\log p(n)}{\sqrt{n}} = \liminf_{n=\infty} \frac{\log p(n)}{\sqrt{n}} = C.$$

It is also possible by these methods to determine the value of C, $C = 2\pi/\sqrt{6}$, and thus to obtain the remarkable equation

$$p(n) = e^{\frac{2\pi}{\sqrt{6}} \cdot \sqrt{n}(1+\epsilon)}, \qquad \lim_{n=\infty} \epsilon = 0.$$

III. By making use of Cauchy's integral theorem, this relation is still further refined:

$$p(n) \sim \frac{1}{4\sqrt{3} \cdot n} \cdot e^{\frac{2\pi}{\sqrt{6}} \cdot \sqrt{n}},$$

or, *i.e.*,

$$\limsup_{n=\infty} \frac{p(n) \cdot 4n\sqrt{3}}{e^{\frac{2\pi}{\sqrt{6}} \cdot \sqrt{n}}} = \liminf_{n=\infty} \frac{p(n) \cdot 4n\sqrt{3}}{e^{\frac{2\pi}{\sqrt{6}} \cdot \sqrt{n}}} = 1.$$

IV. Finally, using the wonderfully powerful new function theoretic methods, the following results are obtained:

$$p(n) = \frac{1}{2\pi\sqrt{2}} \cdot \frac{d}{dn} \left\{ \frac{e^{\pi\sqrt{2/3} \cdot \sqrt{n-1/24}}}{\sqrt{n - 1/24}} \right\} + O(e^{D \cdot \sqrt{n}}),$$

where D is any positive quantity $> \frac{\pi}{2}\sqrt{\frac{2}{3}}$, and where, as usual, $f(n) = \varphi(n) + O(\psi(n))$ means that, for large n, $|f(n) - \varphi(n)|$ is at most of the order of magnitude of the (positive) function $\psi(n)$, that is,

$$\limsup_{n=\infty} \{ |f(n) - \varphi(n)| / \psi(n) \} \leqq k, \text{ where } k \geqq 0 \text{ is a finite quantity.}$$

While the last equation for $p(n)$ gives explicitly only the largest term of a certain expansion of $p(n)$ together with an estimate of the order of magnitude of the error committed by breaking off after the first term, the authors show by some numerical examples that this error is surprisingly small. Thus, the first term contributes to $p(100)$ about 99.9998 per cent. of its value, and to $p(200)$ about 99.999999 per cent. of its value (compare tables below).

The authors show that the further application of their method permits the determination of as many terms as are desired, and permits the determination of the order of magnitude of the error committed by breaking off after a certain, sufficiently large, number of terms. The first terms of this expansion are:

$$p(n) = \frac{1}{2\pi\sqrt{2}} \cdot \frac{d}{dn} \left\{ \frac{e^{\pi\sqrt{2/3} \cdot \sqrt{n-1/24}}}{\sqrt{n - 1/24}} \right\} + \frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\frac{2n\pi}{3} - \frac{\pi}{18}\right) \cdot \frac{d}{dn} \left\{ \frac{e^{\frac{1}{3}\pi\sqrt{2/3} \cdot \sqrt{n-1/24}}}{\sqrt{n - 1/24}} \right\}$$
$$+ \frac{(-1)^n}{2\pi} \cdot \frac{d}{dn} \left\{ \frac{e^{\frac{1}{2}\pi\sqrt{2/3} \cdot \sqrt{n-1/24}}}{\sqrt{n - 1/24}} \right\} + \frac{\sqrt{2}}{\pi} \cdot \cos\left(\frac{n\pi}{2} - \frac{\pi}{8}\right) \cdot \frac{d}{dn} \left\{ \frac{e^{\frac{1}{4}\pi\sqrt{2/3} \cdot \sqrt{n-1/24}}}{\sqrt{n - 1/24}} \right\} + \cdots.$$

Hardy and Ramanujan give the following numerical illustrations. For $n = 100$ and $n = 200$, resp., the first six, resp. eight, terms yield:

190 568 944.783	3 972 998 993 185.896
+ 348.872	+ 36 282.978
− 2.598	− 87.555
+ .685	+ 5.147
+ .318	+ 1.424
− .064	+ .071
	+ .000
190 569 291.996	+ .043
	3 972 999 029 388.004

The true values are 190 569 292 and 3 972 999 029 388, resp., as was verified in independent computations, by means of combinatory analysis methods, by P. A. MacMahon, after two months' effort.[1]

We notice that our expression for $p(n)$ is an expansion of the type

$$p(n) = \varphi_1(n) + \varphi_2(n) + \varphi_3(n) + \cdots,$$

where each φ is of lower order of magnitude than the preceding terms. Now it would be entirely compatible with such an expansion to yield, for any given number of terms on the right side, an expression which differs from $p(n)$ by a value $R(n)$ which approaches ∞ for $\lim n = \infty$. It would already be almost more than might be expected if this difference were to be large, but under a finite value. A most remarkable aspect of the formula is that, by continuing the expansion to a certain number of terms,[2] the error can be made much smaller than unity, so that *the expansion gives for* $p(n)$ (which is necessarily an integer) *its exact value.* This fact by itself makes sufficiently clear to anyone familiar with additive number theory the revolutionizing character of the new work.

14. *What are the methods by which this marvelous accuracy is attained?*

Without stopping to indicate the proof of any of the theorems derived by elementary methods (see section 13), we turn to the (transcendental) method of determining the value of $p(n)$ (see section 13, IV).

We know already (section **6,** II) that, for the generating function

$$\begin{aligned} F(x) &= [(1 - x^1)(1 - x^2)(1 - x^3) \cdots \mathit{in\ inf.}]^{-1} \\ &= c_0 + c_1x^1 + c_2x^2 + \cdots + c_nx^n + \cdots, \\ p(n) &= c_n, \end{aligned}$$

[1] The following little table which is given by Hardy and Ramanujan in connection with their numerical computations may be of some independent interest as a warning against the use of incomplete induction in the theory of numbers. A certain function a_n of n is introduced —its exact definition is given by $\log_{10} p(n) = \frac{10}{9}\{\sqrt{n+10} - a_n\}$—for which $a_1 = 3.317$, $a_3 = 3.176$, $a_{10} = 3.011$, $a_{100} = 3.036$, $a_{1000} = 3.537$, $a_{10000} = 4.148$, $a_{100000} = 4.448$, with intermediate values of a_n fitting in in a natural manner. Yet, as $n \to (+\infty)$, $a_n \to (-\infty)$; a behavior which one would hardly expect.

[2] The number of terms required depends on the value of the number n.

and that

$$p(n) = \frac{1}{2\pi i} \cdot \int_{\Gamma} \frac{F(x)}{x^{n+1}} dx,$$

where Γ is the circumference of a circle concentric with the unit circle, of radius $R < 1$ (see section 9). We are also aware of the difficulties to be overcome in the evaluation of this integral.

The principles of the method to be explained apply to other fundamental papers of Hardy and Littlewood, notably to the papers on Waring's problem; however, as was noted above, in these papers additional serious difficulties are encountered.

Since we are interested in the underlying principles only, not in the technical details of the work, we shall attempt to exhibit the bare skeleton of the method.

One of the roots of the proof is the old inequality

$$\left| \int_C g(x)dx \right| \leqq \text{Max. } (|g(x)|) \times \text{Length of path.}$$

Our problem is greatly simplified by the fact that the function $F(x)$ is closely connected with a well-known modular function $h(\tau)$. In Tannery and Molk's[1] notation, let $x = q^2 = e^{2\pi i\tau}$, then

$$h(\tau) = q^{1/12} \cdot \prod_{n=1}^{\infty} (1 - q^{2n}) = \frac{x^{1/24}}{F(x)} \cdot$$

(The exponent 1/24 introduces into the work troublesome 24th roots of unity.) The function $h(\tau)$ has been carefully studied by many authors. Starting from the known theory, Hardy and Ramanujan are enabled to study the behavior of $F(x)$ at the singular points of the function, that is, at the points $e^{2\pi i\theta}$, θ real. It turns out that, as $x \to 1$ $(= e^{2\pi i.0})$ along the axis of reals, $F(x) \to \infty$ in the same general manner as does e raised to the power $\pi^2/6(1 - x)$; and for $x = r \cdot e^{2i\pi p/q}$, $(p, q) = 1$,[2] $r \to 1$, we have $F(x) \to \infty$ as e raised to the power $\pi^2/6q^2(1 - r)$, that is, while $F(x)$ approaches ∞ when we approach any "rational" point $(e^{2\pi ip/q})$ along a radius, the order of magnitude of $F(x)$ *for this method of approach* is the weaker, the larger the denominator, and it may already from this be expected that, if we approach an "irrational" point $x = r \cdot e^{2\pi i\theta}$, θ irrational, $r \to 1$, the order of magnitude of $F(x)$ will be smaller than for the rational points. As a matter of fact, for θ irrational, $F(x) \to 0$ as $r \to 1$, *in general.* (For no irrational θ does $F(x)$ approach infinity as fast as it does for any one rational $\theta = p/q$.) It is due to this behavior of $F(x)$ for θ irrational that Hardy and Ramanujan obtain, not only the order of magnitude of $p(n)$, but even its accurate value. It turns out that to determine $\int_\Gamma$ it is only necessary to take into consideration the "heaviest singularities" of $F(x)$, *i.e.*,

[1] Tannery and Molk, *Fonctions elliptiques*, vol. 2 (1896).

[2] It is assumed throughout, without future mention, that p and q are relatively prime, $(p, q) = 1$.

those corresponding to the points on the unit circle represented by $e^{2\pi i p/q}$, $q = 1, 2, 3, \cdots$, up to a certain value of q. As a first guess, it might perhaps be expected that all such points have to be considered for which $q \leqq n$. In reality, it is only necessary to consider the singularities at $e^{2\pi i p/q}$, $q = 1, 2, \cdots, [\sqrt{n}]$,[1] approximately, and it is even shown that something like the range $q = 1, 2, \cdots, [\sqrt{n}/\log n]$ would be sufficient, since for larger q the order of magnitude of the new terms involved in the integral becomes sufficiently small.

15. The second root of the method lies in a very interesting application of the so-called *Farey* series. We assume a fixed n which is not changed throughout our work. Assume for example $n = 30$, to fix the ideas. Then the singular points of $F(x)$ to be considered are $e^{2\pi i p/q}$, $q = 1, 2, 3, 4, 5 = [\sqrt{30}]$, $p \leqq q$. We obtain thus the following singular points:

$$e^{2\pi i p/q},\ p/q = 0,\ 1,\ 1/2,\ 1/3,\ 2/3,\ 1/4,\ 3/4,\ 1/5,\ 2/5,\ 3/5,\ 4/5,$$

that is, all points $e^{2\pi i p/q}$ where p/q is any positive rational number $\leqq 1$ and with denominator $\leqq 5$. This suggests the application of a Farey series, and, as a matter of fact, the clever application of Farey series is one of the outstanding features of the transcendental work of Hardy, Ramanujan and Littlewood on partitions.

The Farey series (of order k) is constructed in the following manner. Starting with $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$ we insert the intermediate number $\frac{0+1}{1+1} = \frac{1}{2}$; between $\frac{0}{1}$ and $\frac{1}{2}$ we insert $\frac{0+1}{1+2} = \frac{1}{3}$; between $\frac{1}{2}$ and $\frac{1}{1}$, $\frac{1+1}{1+2} = \frac{2}{3}$, etc. In general, between any two consecutive fractions a/b, a'/b' already obtained by this process, insert $(a+b)/(a'+b')$, not allowing, however, any denominator to exceed k. Thus, for $k = 5$, the Farey series is given below.

We try to place each one of the elements of the Farey series into a separate interval by dividing up the space between any two consecutive numbers. This is done as follows: Let three successive fractions of the Farey series of order k be p''/q'', p/q, p'/q'. Then we construct around p/q the interval with the end-points $p/q - 1/\{q(q+q'')\}$, $p/q + 1/\{q(q+q')\}$. It is easily seen that successive intervals adjoin. For $k = 5$ the following schedule gives in the first line the elements of the Farey series, in the second line the starting points and end-points of the intervals (the last entry in the second line is obtained by closing the series cyclically, recalling that $e^{2\pi i p/q}$ has for $p/q = 0/1$ the same value as for $p/q = 1/1$), while in the third line the lengths of the intervals are noted.

Farey series:	0/1	1/5	1/4	1/3	2/5	1/2	3/5	2/3	3/4	4/5	1/1	
Intervals:		1/6	2/9	2/7	3/8	3/7	4/7	5/8	5/7	7/9	5/6	1/6
Length of Interval:		1/18	4/63	5/56	3/56	1/7	3/56	5/56	4/63	1/18	1/3.	

We have thus, for example, the fraction 1/5 isolated in an interval of length 1/18, the fraction 1/4 isolated in an interval of length 4/63, the fraction 1/1 (which is counted as identical with 0/1) is enclosed in an interval of length 1/3. After

[1] $[a] = a$ for a integer; $[a]$ = largest integer contained in a, otherwise.

fixing in this fashion the *length* of the intervals which are constructed around the individual fractions of the Farey series, we consider the position of each fraction within its proper interval; for example, the position of the fraction 1/3 within its interval $2/7 \cdots 3/8$, of length 5/56. The fraction 1/3 divides this interval into a left-hand part α and a right-hand part β. It is at once verified that $\alpha > 1/(3\cdot 5)$, where the 3 of the denominator is the denominator of the element 1/3 while the 5 is the order $k = 5$ of the Farey series. On the other hand, α is larger than 1/2 of this quantity, $\alpha > 1/(2\cdot 3\cdot 5)$. Indeed, $\alpha = 1/3 - 2/7 = 1/21$, and $1/30 < 1/21 < 1/15$. For the other part, β, the same inequality holds, $\beta = 3/8 - 1/3 = 1/24$, $1/30 < 1/24 < 1/15$. These inequalities are typical of all intervals: *thus, if p''/q'', p/q, p'/q' are three successive elements of a Farey series of order k, and we insert between p''/q'' and p/q the number $p/q - 1/\{q(q+q'')\}$ and between p/q and p'/q' the number $p/q + 1/\{q(q+q')\}$, we shall have enclosed p/q in the interval $p/q - 1/\{q(q+q'')\} \cdots p/q + 1/\{q(q+q')\}$, of length $\alpha+\beta$, $\alpha = 1/\{q(q+q'')\}$, $\beta = 1/\{q(q+q')\}$, and where*

$$\frac{1}{2\cdot kp} < \alpha < \frac{1}{kp}, \qquad \frac{1}{2\cdot kp} < \beta < \frac{1}{kp}\cdot$$

We have already noted that our whole interval $0 \cdots 1$ is exactly covered by these intervals, each of which is broken up by an element of the Farey series into its α and β.

We transfer this division of the interval $0 \cdots 1$ in an obvious manner to the circumference of the unit circle. All we have to do is to associate with the point $e^{2\pi i p/q}$ the fractional value p/q while with the one point $e^{2\pi i\cdot 0} = e^{2\pi i\cdot 1} = 1$ are associated the two values 0/1 and 1/1, thus cyclically closing the interval. For $n = 30$, for example, the Farey series of order $k = [\sqrt{30}] = 5$ leads to the following Figure (4) of the unit circle, in which the singular points $e^{2\pi i p/q}$ are indicated by their corresponding fractions and where, outside of the circle, each point has the length of its corresponding interval indicated.

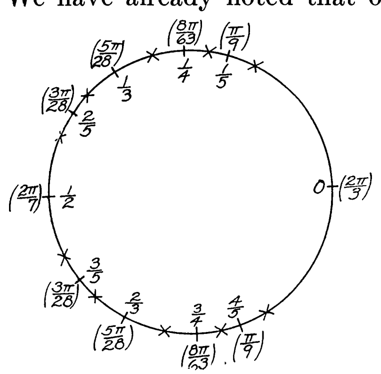

FIG. 4.

Such a separation is called by the authors a Farey dissection, and we have insisted at some length on its structure an account of its fundamental importance, already emphasized, for the new theory.

16. *What has been accomplished by the Farey dissection?* We have already remarked that the "worst" singularities of our function occur at the points $e^{2\pi i p/q}$ with the smallest q and that, the larger q, the "milder" the singularity; and also that from, say, $q > [\sqrt{n}]$ on, the singularities are so mild that they need not be taken into serious account.

It is thus seen that we have caged the dangerous singularities, each one in a separate interval, and the lengths of these intervals, as determined by the Farey

dissection, turn out to be such that, by means of an extremely delicate analysis, the Cauchy integral around the contour Γ (of a circle of radius < 1, about the origin) can be evaluated; in fact, the intervals are sufficiently *large* to ensure that *outside* of the interval the influence of the singularity may be ignored along the unit circle, but sufficiently *small* to ensure that, in estimating the integral by $|\int| \leqq$ *length of interval* $\times$ *Max.* (|*function*|), the length of the interval will not make the right side too large.

The path is now prepared for an attack along the following lines: Since we know the type of singularity which $F(x)$ possesses at each of the points which count, we shall attempt to form a function, as simple as possible, which shall have at the point $e^{2\pi i}$, that is, at the heaviest singularity of $F(x)$, a singularity of the same type as $F(x)$ has at this point,[1] and regular everywhere else on, in, and in some neighborhood around, the unit circle. Let $F_{1,1}(x) = F_{11}$ be such a function. Then $F(x) - F_{1,1}(x)$ is regular, or nearly regular, at $x = 1$. Similarly, a function F_{12} is determined, regular in the domain just described except at the point $e^{i\pi} = -1$, *i.e.*, $e^{2\pi i p/q}$, $p = 1$, $q = 2$, and whose singularity at this point is, as closely as possible, of the type of the singularity of F at -1. Then $F - F_{11} - F_{12}$ is regular, or nearly regular, at the two "worst" singular points of F, at $+1$, -1. We proceed similarly for each point of the Farey dissection. Thus we shall have constructed for each point $e^{2\pi i p/q}$, $0 \leqq p < q$, $q \leqq [\sqrt{n}]$, a function F_{pq} such that $F(x) - \sum_{p,q} F_{p,q}(x)$, $0 \leqq p < q$, $q \leqq [\sqrt{n}]$ is essentially regular everywhere on the unit circle. In this last statement, the expression "essentially regular" serves to cover up a multitude of sins. Of course, $F - \sum_{p,q} F_{pq}$ still is not regular at any point of the circumference of the unit circle, except at the finite number of points $e^{2\pi i p/q}$, $0 \leqq p < q$, $q \leqq [\sqrt{n}]$. But we have stated above that at all points except these the singularities are of so mild a type that they, for certain modes of approach at least, may in our work be ignored. These facts have to be carefully taken into account in determining the path of integration. We only state that this path of integration is essentially a circle, concentric to the unit circle and of radius $R = 1 - (\beta/n)$ (β a certain constant), but which is modified in a certain manner which we do not describe in this sketch. For F_{pq} the following complicated function is chosen:

$$F_{pq} = F_{p,q}(x) = \omega_{p,q} \cdot \frac{\sqrt{q}}{\pi\sqrt{2}} \cdot F_{\left(\frac{\pi\sqrt{2}}{q\sqrt{3}}\right)}(x \cdot e^{-2\pi i p/q}),$$

where

$$F_\alpha(x) = \sum_{n=1}^{\infty} \psi_\alpha(n) \cdot x^n, \qquad \psi_\alpha(n) = \frac{d}{dn} \frac{e^{\alpha \cdot \lambda_n} + e^{-\alpha \cdot \lambda_n} - 1}{\lambda_n}, \qquad \lambda_n = \sqrt{n - 1/24},$$

$\omega_{p,q}$ a certain 24th root of unity. We may fancy each $F_{p,q}(x)$ expanded into a power series in x. Let

$$F_{pq} = F_{p,q}(x) = c_{pq0} + c_{pq1}x^1 + c_{pq2}x^2 + \cdots.$$

[1] At least for certain methods of approach to the singular point.

Next, let

$$\Phi(x) = F(x) - \sum_{p,q} F_{p,\,q}(x), \qquad 0 \leqq p < q \leqq [\sqrt{n}].$$

Then, remembering that $F(x) = 1 + p_1x^1 + p_2x^2 + \cdots + p_nx^n + \cdots$, we obtain, by comparing coefficients in our last eqùation:

$$p(n) - \sum_{p,q} c_{pqn} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx,$$

where the path of integration is the circumference of the circle of radius $R=1-\beta/n$ about the origin as center, and β a certain positive constant not depending on x or n. Since the c_{pqn} are known quantities as coefficients of the power-series expansions of the F_{pq}, defined above, all hinges on the question whether we can integrate $\Phi(x)/x^{n+1}$ around Γ. Onto this circle Γ we project, by means of radii from the origin, the singular points $e^{2\pi ip/q}$ $(0 \leqq p < q \leqq [\sqrt{n}])$, as well as the whole Farey dissection. From now on, we refer the Farey dissection with all of its segments to this smaller circle Γ. Let ξ_{pq} on this circle be the interval of the Farey dissection surrounding the point $R\cdot e^{2\pi ip/q}$ $(0 \leqq p < q \leqq [\sqrt{n}])$—corresponding to the singular point $e^{2\pi ip/q}$ on the unit circle—and let η_{pq} be the arc complementary to ξ_{pq}. Then:

$$\begin{aligned} p(n) - \sum_{p,q} c_{pqn} &= \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx = \frac{1}{2\pi i}\int_\Gamma \frac{F(x)}{x^{n+1}}\,dx - \frac{1}{2\pi i}\int_\Gamma \sum_{p,q} \frac{F_{pq}(x)}{x^{n+1}}\,dx \\ &= \frac{1}{2\pi i}\left\{\sum_{p,q}\int_{\xi_{pq}} \frac{F(x)}{x^{n+1}}\,dx - \sum_{p,q}\int_{\xi_{pq}+\eta_{pq}=\Gamma} \frac{F_{pq}(x)}{x^{n+1}}\,dx\right\} \\ &= \frac{1}{2\pi i}\cdot\sum_{p,q}\int_{\xi_{pq}} \frac{F-F_{pq}}{x^{n+1}}\,dx - \frac{1}{2\pi i}\cdot\sum_{p,q}\int_{\eta_{pq}} \frac{F_{pq}}{x^{n+1}}\,dx \\ &= \sum_{p,q} J_{pq} - \sum_{p,q} j_{pq}. \end{aligned}$$

The rest of the very delicate work consists in approximating the values of the J_{pq} and the j_{pq}. We make no attempt to reproduce this highly technical examination. The final result is that

$$\sum_{p,q} J_{pq} = O(n^{-1/4}), \qquad \sum_{p,q} j_{pq} = O(n^{-1/4}),$$

that is, for all integral positive n

$$\Big|\sum_{p,q} J_{pq}\Big| : n^{-1/4} < K_1, \qquad \Big|\sum_{p,q} j_{pq}\Big| : n^{-1/4} < K_2,$$

where K_1, K_2 are two finite positive numbers, independent of n. Therefore,

$$p(n) = \sum_{p,q} c_{pqn} + O(n^{-1/4}).$$

17. *What does this formula prove?* For a given n the number of partitions is given by $p(n)$. After choosing our n, we take $q \leqq [\sqrt{n}]$[1] and construct the power-series for each of F_{01}, F_{12}, F_{13}, F_{23}, F_{14}, F_{34}, $\cdots$, $F_{q-1,\,q}$. For any p, q let

[1] Compare top of page 421.

$F_{pq} = c_{pq0} + c_{pq1}x^1 + \cdots + c_{pqn}x^n + \cdots$, then, for a given n the c_{pqn}, if written in the order $(c_{01n}) + (c_{12n}) + (c_{13n} + c_{23n}) + (c_{14n} + c_{34n}) + (c_{15n} + c_{25n} + c_{35n} + c_{45n}) + \cdots$, will be so arranged that the c_{ijn} in any pair of parentheses correspond to weaker singularities of $F(x)$ than the c_{ijn} in any preceding parentheses; and for $q > [\sqrt{n}]$ the singularities of $F(x)$ are so weak (for certain modes of approach) that they may be ignored if we are satisfied with an accuracy for $p(n)$ of order of magnitude $n^{-1/4}$. But with increasing n, $n^{-1/4} \to 0$; therefore the value of $p(n)$ will be, for sufficiently large n, the integer closest to $\sum_{p,q} c_{pqn}$. *We shall thus have obtained, as was stated before, not only the order of magnitude of $p(n)$, but its actual value.* We do not quote in full the (very complicated) wording of the theorem. Its character will be clear from the formula for $p(n)$ given in section 13, IV, in which each term takes care of one of the singular points, and in such manner that the heaviest singularity is accounted for by the first term, the next heaviest by the second term, etc.

To avoid incorrect conclusions, it must be said that the theorem does not mean that the series for $p(n)$, if we continue it to infinity by omitting the restriction $q \leqq [\sqrt{n}]$, is necessarily a convergent series, still less that it must converge to the value $p(n)$. But it does mean that, if we break off our series at $q = [\sqrt{n}]$[1], this finite series will, for sufficiently large values of n, give us $p(n)$ with an error of less than, say, $\frac{1}{2}$, and since it is by its nature a positive integer, we obtain thus the actual value of $p(n)$.

THE GEOMETRY OF RIEMANN AND EINSTEIN.[2]

By JAMES PIERPONT, Yale University.

PART I.

Introduction. In Einstein's theory our space is not Euclidean; aside from local disturbances its geometry is of the kind first studied by Riemann, Klein and Newcomb. The widespread interest in Einstein's theory has led the writer to believe that an elementary presentation of this kind of non-euclidean geometry would prove acceptable to a not inconsiderable class of readers.

The usual method of developing non-Euclidean geometry is to begin at the very foundations and build up the theory in a highly abstract and logical manner from a set of definitions relative to three classes of things which we do not see but which are called points, lines and planes. To follow such reasoning requires

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[1] See the last sentence of section **14**.

[2] Read at the summer meeting of the Association at Vassar College, Poughkeepsie, New York, September 6, 1923.

An alternative title for the paper is "Elliptic Geometry." Riemann showed that the metric of a space of constant curvature is defined by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{[1 + (x_1^2 + \cdots + x_n^2)/4R^2]^2}.$$

The resulting geometries are called elliptic, parabolic or hyperbolic according as R^2 is positive, ∞ or negative. There are two kinds of elliptic geometry which have received various names, as polar and antipodal, spherical and elliptic. If the last terminology were adopted, we would have to give up the threefold classification of space first mentioned.

a great effort on the part of the reader and this no doubt explains why relatively few persons have devoted much attention to this subject. There is another way quite as rigorous and, as the author believes, more attractive to the ordinary mathematician and physicist. We will admit that the propositions of Euclidean geometry have been established with entire rigor and proceed by its aid to construct another geometry which for short we call Riemannian or R-geometry. Our method is analogous to that used in general arithmetic: granting that the arithmetic of ordinary real numbers has been established, we may take pairs of them as (a, b) to define a new class of numbers ordinarily denoted by $a + ib$. Or we may take sets of four (a, b, c, d) to form a class of new numbers usually denoted by $a + bi + cj + dk$ and called quaternions. No one thinks it necessary to go back to first principles and develop an abstract theory of magnitude in order to develop the arithmetic of quaternions.

A great merit of the method adopted in this paper, so it seems to the writer, lies in its intuitiveness; the reader has the things dealt with constantly under his eyes. As will be seen, the geometry of R-space in the immediate vicinity of the observer is sensibly the same as in ordinary geometry, it is only when large portions of space are considered that the great difference between the two geometries, Euclidean and Riemannian, becomes apparent.

1. **The Metric of Riemann.** We begin by recalling a few facts of E-geometry.[1] We take a rectangular coördinate system whose origin we denote by O; the coördinates of a point x we denote by x_1, x_2, x_3, etc. If $x + dx$ is a point near by, its distance $d\sigma$ from x is given by

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{1}$$

Let

$$x_1 = \varphi_1(t) \qquad x_2 = \varphi_2(t), \qquad x_3 = \varphi_3(t) \tag{2}$$

be the equations of a curve; if to $t = a, t = b$ correspond the points A, B on (2), the length of the arc AB is

$$\sigma = \int_a^b \frac{d\sigma}{dt} dt.$$

The curve (2) is a *straight* when σ is less than for any adjacent curve If x, $x + dx$, $x + \delta x$ are the coördinates of three nearby points, they determine a little triangle the length of whose sides may be denoted by $d\sigma$, $\delta\sigma$, $\Delta\sigma$. If θ is the angle between $d\sigma$, $\delta\sigma$, that is, the angle whose vertex is the point x, we have

$$\Delta\sigma^2 = d\sigma^2 + \delta\sigma^2 - 2d\sigma \cdot \delta\sigma \cos\theta, \tag{3}$$

which gives at once, on using (1),

$$\cos\theta = \frac{dx_1}{d\sigma}\frac{\delta x_1}{\delta\sigma} + \frac{dx_2}{d\sigma}\frac{\delta x_2}{\delta\sigma} + \frac{dx_3}{d\sigma}\frac{\delta x_3}{\delta\sigma}. \tag{4}$$

This may be taken as the definition of the angle θ between two curves meeting at x.

[1] The letters E and R before a word are to be read Euclidean, Riemannian.

We propose now to introduce *another definition of distance or length.* The coördinates of a point remaining the same as before, *i.e.*, ordinary rectangular coördinates, let R be an arbitrary constant > 0. We now say the distance ds between the points x and $x + dx$ is given by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{[1 + (x_1^2 + x_2^2 + x_3^2)/4R^2]^2} \tag{5}$$

and the length s of the arc AB considered above shall be defined by

$$s = \int_a^b \frac{ds}{dt} dt. \tag{6}$$

Let us compare (5) with (1). We see that the numerator of (5) is $d\sigma^2$ while $r^2 = x_1^2 + x_2^2 + x_3^2$ is the square of the distance from the origin O to the point x in E-measure. We may set thus

$$ds = \frac{d\sigma}{1 + r^2/4R^2} = \mu d\sigma, \tag{7}$$

where μ is the factor multiplying $d\sigma$. For points x such that r is small compared with the space constant, R, μ is nearly 1 and hence $ds = d\sigma$ sensibly. As x recedes from O, μ decreases and $ds \,/\, d\sigma \doteq 0$.

To illustrate this new definition of length, let us find the length s in R-measure of the segment OP of an E-straight through the origin, whose direction cosines are l_1, l_2, l_3. Its equations are

$$x_1 = l_1 t, \qquad x_2 = l_2 t, \qquad x_3 = l_3 t; \qquad l_1^2 + l_2^2 + l_3^2 = 1.$$

Here $r^2 = x_1^2 + x_2^2 + x_3^2 = t^2 \quad \therefore \quad r = t$; also $d\sigma = dt$. If the length of OP in E-measure is σ, we have by (7)

$$s = \int_0^\sigma \frac{dt}{1 + t^2/4R^2} = 2R \operatorname{arctg} \frac{\sigma}{2R}$$

as its length in R-measure. If σ is small compared with R,

$$\operatorname{arctg} \frac{\sigma}{2R} = \frac{\sigma}{2R}$$

nearly, and $s = \sigma$ sensibly. On the other hand if $\sigma \doteq \infty$, $s \doteq \pi R$. Thus the length of any E-straight through the origin [1] is double this or $2\pi R$ in R-measure.

In E-geometry we often find it convenient to use polar coördinates r, θ, φ, where

$$x_1 = r \sin\theta \cos\varphi, \qquad x_2 = r \sin\theta \sin\varphi, \qquad x_3 = r\cos\theta.$$

In R-measure the length of the radius vector r is, as just seen,

$$\rho = 2R \operatorname{arctg} \frac{r}{2R}. \tag{8}$$

[1] As we shall see these lines are also straight lines in R-geometry.

We call ρ, θ, φ, *R-polar coördinates.* The only difference between these two kinds of polar coördinates is that in one case we express the length of the radius vector in E-measure, and in the other case in R-measure.

From (8) we have

$$r = 2R \tan \frac{\rho}{2R}, \qquad dr = \sec^2 \frac{\rho}{2R} \cdot d\rho. \tag{9}$$

In E-polar coördinates,

$$d\sigma^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2.$$

Replacing r by ρ in (7) we find [1]

$$ds^2 = d\rho^2 + R^2 \sin^2 \frac{\rho}{R} (d\theta^2 + \sin^2 \theta d\varphi^2) \tag{10}$$

in R-polar coördinates.

As an application of R-polar coördinates, let us find the length in R-measure of the curve

$$x_1 = r \cos \varphi, \qquad x_2 = r \sin \varphi, \qquad x_3 = 0.$$

In E-geometry this is a circle of radius r in the x_1x_2-plane.[2] Here $\theta = \pi/2$, $d\theta = 0$, $dr = 0$ $\therefore$ $d\rho = 0$; thus (10) gives

$$ds = R \sin \frac{\rho}{R} \cdot d\varphi.$$

Therefore,

$$s = R \sin \frac{\rho}{R} \int_0^{2\pi} d\varphi = 2\pi R \sin \frac{\rho}{R}$$

is the length of this curve in R-measure. If r, and hence ρ, is small compared with R, $\sin \rho/R = \rho/R$ nearly, and $s = 2\pi\rho = 2\pi r$, sensibly, as in E-geometry.

Let the two curves C, C' meet at the point x; on C take the point $x + dx$, and on C' the point $x + \delta x$. Let ds, δs, Δs be the distances in R-measure between x and $x + dx$, x and $x + \delta x$, $x + dx$ and $x + \delta x$. Analogous to (3) we define the angle φ between C, C' by

$$\cos \varphi = \frac{ds^2 + \delta s^2 - \Delta s^2}{2ds \cdot \delta s}.$$

By (7) $ds = \mu d\sigma$, $\delta s = \mu \delta\sigma$, $\Delta s = \mu \Delta\sigma$; hence

$$\cos \varphi = \frac{d\sigma^2 + \delta\sigma^2 - \Delta\sigma^2}{2d\sigma \cdot \delta\sigma} = \cos \theta. \qquad \therefore \quad \varphi = \theta \cdot$$

This gives the important theorem: *The angle between two curves has the same value in R- as in E-measure.*

[1] This is the expression that Einstein employs in his paper. *Sitzungsber. d. Preuss. Akad. d. Wiss.*, 1917, p. 142.

[2] As will be seen later, this is also a circle in R-geometry.

2. **Plane *R*-geometry.** Before considering the geometry of 3 dimensions, we will take up the geometry of the plane $x_3 = 0$. In this plane our metric is defined by

$$ds^2 = \frac{dx_1^2 + dx_2^2}{[1 + (x_1^2 + x_2^2)/4R^2]^2} \tag{11}$$

or by

$$ds = \frac{d\sigma}{1 + r^2/4R^2}, \tag{12}$$

where $d\sigma^2 = dx_1^2 + dx_2^2$ and $r^2 = x_1^2 + x_2^2$.

Let us change the variables to

$$z_1 = \frac{4R^2x_1}{\lambda}, \qquad z_2 = \frac{4R^2x_2}{\lambda}, \qquad z_3 = \frac{R}{\lambda}(4R^2 - r^2), \tag{13}$$

where

$$\lambda = r^2 + 4R^2.$$

We find at once that

$$z_1^2 + z_2^2 + z_3^2 = R^2, \tag{14}$$

while (11) gives

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2. \tag{15}$$

Thus while x ranges over the x_1x_2-plane, the point z ranges over the E-sphere (14), which we shall call the S-sphere, whose center is O and radius is R. Moreover when x describes an element of arc ds, the relation (15) shows that z describes an element of arc on this sphere of equal length. The relation between the points x and z is uniform. For, solving (13), we get

$$x_1 = \frac{2Rz_1}{R + z_3}, \qquad x_2 = \frac{2Rz_2}{R + z_3}. \tag{16}$$

There is one exception in this correspondence, *viz.*: when $z_3 = -R$, x_1, x_2 are infinite.

Analogous to E-geometry we say an R-straight in the x_2x_3-plane is a curve of minimum length in R-measure. Now the element of arc on the sphere S is given by (15) which is precisely that of E-geometry. Hence the length of any arc in the R-plane x_1x_2 is the length of the corresponding arc on the sphere S in E-measure. Now on a sphere the curves of shortest length are great circles, hence R-straights in the x_1x_2-plane are the images of great circles of the sphere S. To find the equations of these straights, let the great circle lie in the plane

$$A_1z_1 + A_2z_2 + A_3z_3 = 0. \tag{17}$$

Substituting from (13) into (17) we get

$$A_3(x_1^2 + x_2^2) - 4R(A_1x_1 + A_2x_2) = 4A_3R^2, \tag{18}$$

the equation of an E-circle. This circle cuts the circle

$$(F), \qquad x_1^2 + x_2^2 = 4R^2, \tag{19}$$

in the two points lying on the E-straight

$$A_1x_1 + A_2x_2 = 0$$

as is seen by setting (19) in (18). As this line goes through the origin, we see that all E-circles in the family of circles (18) cut the circle (19) in diametral points. We call (19) the *fundamental circle, F,* and the circles (18) *diametral circles.* Hence the result: *Diametral circles in E-geometry are straight lines in R-geometry.*

The fundamental circle, being the image of the great circle $z_3 = 0$, is an R-straight. To the great circle lying in the plane $A_1z_1 + A_2z_2 = 0$ corresponds in the x_1x_2-plane the E-straight $A_1x_1 + A_2x_2 = 0$. Hence: *All E-straights through the origin are also R-straights.*

We have seen there is no point in the x_1x_2-plane corresponding to the point $z_3 = -R$ on the sphere S. To avoid this exception we adjoin an *ideal point* to the R-plane but not to the E-plane. We may now say that R-straights are closed curves which all have the same length $2\pi R$. Any two of them cut in 2 points. Any two points at a distance $< \pi R$ apart determine uniquely a point. Since the angle between two R-straights is the same as that between the corresponding great circles on S, and since moreover the length of a segment on an R-straight is the same as the length in E-measure of the corresponding arc on S, we have at once the theorem: *The relations between the angles A, B, C and lengths of the opposite sides a, b, c of an R-triangle are those of triangles in E-geometry on a sphere of radius R.* Thus, in particular,

$$\begin{aligned} \sin A : \sin B : \sin C &= \sin\frac{a}{R} : \sin\frac{b}{R} : \sin\frac{c}{R}, \\ \cos\frac{a}{R} &= \cos\frac{b}{R}\cos\frac{c}{R} + \sin\frac{b}{R}\sin\frac{c}{R}\cos A. \end{aligned} \tag{20}$$

Since similar triangles do not exist in E-spherical geometry, there are no similar triangles in R-geometry The sum of the angles of an R-triangle is > 2 right angles. All R-straights perpendicular to a given R-straight meet at a point at a distance $\pi R/2$ or as we may say at a *quadrant's distance.*

There remains only one more notion to discuss, that of *motion* or *displacement.* In E-geometry a figure may be moved about freely without altering the distance between any two of its points, or the angle between any two of its straights. Is this possible in the R-plane, distance being of course expressed in R-measure? The answer is "Yes"; for to any figure in the x_1x_2-plane corresponds its image on the sphere S. As this image may be moved freely about on S without altering any of its dimensions, and as to each position of the image corresponds a position of the original figure, we have the result stated above.

The reader may wish to know why we have adopted the metric defined by (11). Since the days of the Greeks it has been known that the geometry on a sphere is non-euclidean; its straights (great circles) have a finite length, there

are no parallels. To obtain a similar geometry on the plane we may project stereographically the points z of the S sphere (14) on the x_1x_2-plane tangent to S at the point $z_1 = z_2 = 0$, $z_3 = R$, the center of projection being the diametrically opposite point. The x_1, x_2-axes we take parallel to the z_1, z_2-axes respectively. The relation between the point z and its projection x is precisely that defined by (13). If ds' is the element of arc PQ on the sphere, the length of the corresponding arc $P'Q'$ in the x_1x_2-plane is

$$ds = \frac{\sqrt{dx_1^2 + dx_2^2}}{1 + (x_1^2 + x_2^2)/4R^2}, \tag{21}$$

where x_1, x_2 are the coördinates of P'. If we want $ds' = ds$, that is, if we want arcs in the x_1x_2-plane to have the same length as the corresponding arcs on the sphere, we must change our definition of length in accordance with (21), that is, we must define our metric by (11). Since stereographic projection leaves angles unchanged, we do not need to adopt a new definition of measure for angles.

3. Two Kinds of R-geometry. Since diametral circles cut in two points, one inside, the other outside the fundamental circle F, two R-straights cut in two points, and not in one as in E-geometry. Also two points do not always determine an R-straight, *viz.:* two diametrically opposite points on the fundamental circle.

On this account we may define another geometry in which these exceptions do not occur, as follows:

(1) Diametrically opposite points on the fundamental circle are regarded as one and the same point,[1]

(2) All points without the fundamental circle are regarded as non-existent, or imaginary.

Let a diametral circle cut the fundamental circle F in the points A, A'. If B is a point of this diametral circle lying within F, the arc ABA' is the R-straight in this geometry. A is identical with A', this line is closed and its length is obviously only πR.

This new geometry may be called the *restricted R-geometry,* and may be denoted by R^*.

THREE-DIMENSIONAL R-GEOMETRY.

4. The Coördinates $z_1, \cdots, z_4$. We now return to general considerations not restricted to a plane. Our first task is to determine the nature of R-straights in 3-way space. Here we do not have the geometry on an E-sphere to aid us and we must rely on analysis, but an analysis guided by the results and methods employed in the plane.

To avoid analytical difficulties it will be convenient to introduce the z-variables:

$$z_i = \frac{4R^2x_i}{\lambda}, \qquad i = 1, 2, 3; \qquad z_4 = \frac{R}{\lambda}(4R^2 - r^2), \tag{22}$$

[1] Readers familiar with projective geometry will find nothing strange in this, where the line at infinity plays a rôle somewhat analogous to the fundamental circle.

where

$$\lambda = r^2 + 4R^2, \qquad r^2 = x_1^2 + x_2^2 + x_3^2. \tag{23}$$

We see these are in definition entirely analogous to the z-variables in (13). We find at once that

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2. \tag{24}$$

If we solve (22) for the x's we find

$$x_i = \frac{2Rz_i}{R + z_4}, \qquad i = 1, 2, 3. \tag{25}$$

Thus to each set[1] of values: $z_1, \cdots, z_4$, satisfying (24) corresponds a single point (x_1, x_2, x_3) and conversely. We may regard $z_1, \cdots, z_4$, therefore, as the coördinates of a point whose x-coördinates are given by (25).

It may aid the reader to have a geometric interpretation of the z-coördinates. Let x_1, x_2, x_3 be the x-coördinates of a point P, the length of the segment OP in E-measure is r; in R-measure its length is, by (8),

$$\rho = 2R \operatorname{arctg} \frac{r}{2R}\cdot \tag{26}$$

If $\overline{OP}$ makes the angles α_i with the x_i-axis, $x_i = r \cos \alpha_i$, Therefore,

$$z_i = \frac{4R^2 r \cos \alpha_i}{\lambda}\cdot \tag{27}$$

To eliminate the x-coördinates which enter λ let us show that

$$\frac{4Rr}{\lambda} = \sin \frac{\rho}{R}\cdot \tag{28}$$

In fact by (26)

$$\sin \frac{\rho}{R} = \sin\left(2 \operatorname{arctg} \frac{r}{2R}\right) = 2 \sin\left(\operatorname{arctg} \frac{r}{2R}\right) \cos\left(\operatorname{arctg} \frac{r}{2R}\right)$$

Now

$$\operatorname{arctg} \frac{r}{2R} = \operatorname{arc\,sin} \left(\frac{\dfrac{r}{2R}}{\sqrt{1 + \dfrac{r^2}{4R^2}}} \right) = \operatorname{arc\,sin} \frac{r}{\sqrt{\lambda}}\cdot$$

Also

$$\operatorname{arctg} \frac{r}{2R} = \operatorname{arc\,cos} \frac{1}{\sqrt{1 + \dfrac{r^2}{4R^2}}} = \operatorname{arc\,cos} \frac{2R}{\sqrt{\lambda}}\cdot$$

[1] The exceptional case $z_1 = z_2 = z_3 = 0$, $z_4 = -R$, may be treated as in § 2.

Therefore,

$$\sin\frac{\rho}{R} = 2\sin\left(\text{arc sin}\frac{r}{\sqrt{\lambda}}\right)\cos\left(\text{arc cos}\frac{2R}{\sqrt{\lambda}}\right)$$
$$= 2\cdot\frac{r}{\sqrt{\lambda}}\cdot\frac{2R}{\sqrt{\lambda}} = \frac{4Rr}{\lambda},$$

which establishes (28).

We show further that

$$\frac{4R^2 - r^2}{\lambda} = \cos\frac{\rho}{R}\cdot \tag{29}$$

For

$$\cos\frac{\rho}{R} = \cos\left(2\,\text{arctg}\frac{r}{2R}\right) = \left\{\cos\left(\text{arctg}\frac{r}{2R}\right)\right\}^2 - \left\{\sin\left(\text{arctg}\frac{r}{2R}\right)\right\}^2$$
$$= \left\{\cos\left(\text{arc cos}\frac{2R}{\sqrt{\lambda}}\right)\right\}^2 - \left\{\sin\left(\text{arc sin}\frac{r}{\sqrt{\lambda}}\right)\right\}^2$$
$$= \frac{4R^2}{\lambda} - \frac{r^2}{\lambda} = \frac{4R^2 - r^2}{\lambda},$$

which establishes (29). Setting (28), (29) in (22) gives

$$z_i = R\sin\frac{\rho}{R}\cos\alpha_i, \qquad i = 1, 2, 3; \qquad z_4 = R\cos\frac{\rho}{R}\cdot \tag{30}$$

Thus the z-coördinates are simple functions of the distance in R-measure of the point P from O and the direction cosines of the line OP.

A linear relation [1] between the z's as

$$A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = 0 \tag{31}$$

defines a locus which we shall call an *R-plane*. The justification of such a name will follow presently; we introduce it now merely to have a name for such a relation inasmuch as it enters into our analysis in a vital manner.

If we replace the z's in terms of the x's as defined in (22), the relation (31) becomes

$$A_4(x_1^2 + x_2^2 + x_3^2) - 4R(A_1x_1 + A_2x_2 + A_3x_3) = 4A_4R^2. \tag{32}$$

This in E-geometry defines a sphere; any such sphere we will call a *diametral sphere*, on account of its relation to the sphere,

$$x_1^2 + x_2^2 + x_3^2 = 4R^2, \tag{33}$$

which we call the *fundamental* or F sphere. In fact, all the spheres (32) cut (33) in the diametral plane

$$A_1x_1 + A_2x_2 + A_3x_3 = 0.$$

Thus *diametral spheres cut the fundamental sphere, F, along a great circle.*

Another property of the z-coördinates is the simple form they give to ds^2.

[1] It is always supposed that the z-coördinates satisfy the fundamental relation (24).

In fact, by direct calculation we find that (5) becomes

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2. \tag{34}$$

The radius of F is $2R$ in E-measure; its length in R-measure is $\pi R/2$ by (8). Hence by (30) the z-coördinates of a point, A, on F are:

$$A; \qquad z_i = R \cos \alpha_i, \qquad i = 1, 2, 3; \qquad z_4 = 0.$$

The coördinates of the diametrically opposite point, A', are:

$$A'; \qquad z_i' = -z_i, \qquad i = 1, 2, 3; \qquad z_4 = 0.$$

5. Straight Lines in 3-way *R*-geometry. Analogous to E-geometry, we say an R-straight is a curve of minimum length in R-measure. To find these curves we may proceed as follows. Let C be the curve defined by

$$x_1 = \varphi_1(t), \qquad x_2 = \varphi_2(t), \qquad x_3 = \varphi_3(t).$$

Let A, B be two points on C corresponding to $t = a$, $t = b$; then the length of the arc AB in R-measure is

$$s = \int_a^b \frac{ds}{dt} \cdot dt = \int_a^b F dt, \tag{35}$$

where $F = ds/dt$ is defined by (5), or in terms of the z variables by (34). As the latter variables are easier to manage, we shall employ them; thus

$$F^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dz_1}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2 + \left(\frac{dz_3}{dt}\right)^2 + \left(\frac{dz_4}{dt}\right)^2 \tag{36}$$
$$= \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 = \Sigma \zeta_i^2,$$

setting

$$\zeta_i = \frac{dz_i}{dt}, \qquad i = 1, 2, 3, 4.$$

Let us now deform the curve C slightly keeping the end points A, B fixed so that the point (x_1, x_2, x_3) has on the new curve $\bar{C}$ the coördinates:

$$\bar{x}_1 = x_1 + \delta x_1, \qquad \bar{x}_2 = x_2 + \delta x_2, \qquad \bar{x}_3 = x_3 + \delta x_3,$$

the δx being small quantities. At the same time each z_i becomes $\bar{z}_i = z_i + \delta z_i$, while ζ_i goes over into $\bar{\zeta}_i = \zeta_i + \delta \zeta_i$ and F becomes $\bar{F} = F + \delta F$. The length of the new curve $\bar{C}$ between A, B is now

$$\bar{s} = s + \delta s = \int_a^b \bar{F} dt;$$

that is, in passing from C to $\bar{C}$ the length of the arc AB has been increased by δs. For C to have minimum length it is necessary that this increment should vanish, neglecting small quantities of higher order than the first, *i.e.*, when C is replaced by the adjacent curve $\bar{C}$, the integral (35) receives no increment in small quantities of the first order. In other words the curve C renders the integral (35) sta-

tionary. In accordance with modern usage, we shall now define more specifically: *A straight line in R-geometry is a curve which renders s in* (35) *stationary, i.e., the curve for which*

$$\delta s = \delta \int_a^b \frac{ds}{dt} dt = 0, \tag{37}$$

neglecting small quantities of higher order.

We shall establish in the next article the Fundamental Theorem: *Straight lines in R-geometry are the intersection of two diametral spheres, i.e.,* of two R-planes, and conversely.

From this follows that *if two points of an R-straight lie on an R-plane, all its points lie on this plane.*

Now these two properties are the fundamental properties of E-planes, and herein lies the justification of regarding the diametral spheres in E-geometry as planes in R-geometry.

The following demonstration is the only difficult piece of analysis in this paper; we have therefore placed it in a separate section so that the reader may omit it and pass to the next section, § 7. He should, however, note the equations (49) which are the parametric equations of our R-straights.

6. Demonstration of the Fundamental Theorem. We start with

$$\delta s = \bar{s} - s = \int_a^b \bar{F} dt - \int_a^b F dt = \int_a^b (\bar{F} - F) dt = \int_a^b \delta F \cdot dt. \tag{38}$$

Now

$$\bar{F}^2 = (F + \delta F)^2 = F^2 + 2F \cdot \delta F + (\delta F)^2,$$

or by (36) $\qquad = \sum(\zeta_i + \delta\zeta_i)^2 = \sum\zeta_i^2 + 2\sum\zeta_i\delta\zeta_i + \sum(\delta\zeta_i)^2, \qquad i = 1, \cdots, 4.$

We shall consider only curves $\bar{C}$ such that $(\delta\zeta_i)^2$ is small compared with the increments $\delta\zeta_i$. Thus, neglecting $(\delta F)^2$ and the $(\delta\zeta_i)^2$, the above gives

$$F\delta F = \sum\zeta_i\delta\zeta_i.$$

Thus the condition (37) in (38) gives

$$\int_a^b \frac{\sum\zeta_i\delta\zeta_i}{F} \cdot dt = 0. \tag{39}$$

We can get rid of F in the denominator if we take s instead of t as the independent variable. In this case $F = 1$, by (36), since $ds/ds = 1$. Thus (39) becomes

$$\int_\alpha^\beta ds \sum\zeta_i\delta\zeta_i = \int_\alpha^\beta ds \sum \frac{dz_i}{ds} \delta\left(\frac{dz_i}{ds}\right) = 0, \qquad i = 1, \cdots, 4, \tag{40}$$

where $s = \alpha$ when $t = a$, and $s = \beta$ when $t = b$.

To transform (40), we note that[1]

$$\frac{d}{ds}(\delta z) = \frac{d}{ds}(\bar{z} - z) = \frac{d\bar{z}}{ds} - \frac{dz}{ds} = \delta\left(\frac{dz}{ds}\right) \cdot \tag{41}$$

[1] For brevity we have dropped the subscript i for the moment.

Also,

$$\frac{d}{ds}\left(\frac{dz}{ds}\delta z\right) = \delta z\cdot\frac{d^2z}{ds^2} + \frac{dz}{ds}\cdot\frac{d}{ds}(\delta z)$$

or

$$\frac{d}{ds}\left(\frac{dz}{ds}\delta z\right) = \delta z\cdot\frac{d^2z}{ds^2} + \frac{dz}{ds}\cdot\delta\left(\frac{dz}{ds}\right), \tag{42}$$

on using (41). Integrating (42) between the limits α, β gives

$$\left[\frac{dz}{ds}\delta z\right]_\alpha^\beta = \int_\alpha^\beta \delta z\cdot\frac{d^2z}{ds^2}\cdot ds + \int_\alpha^\beta \frac{dz}{ds}\delta\left(\frac{dz}{ds}\right)\cdot ds. \tag{43}$$

Now in passing from C to $\overline{C}$ the end points A, B remained fixed; hence the δz vanish at the limits. Thus the left side of (43) vanishes and this gives

$$\int_\alpha^\beta \frac{dz_i}{ds}\cdot\delta\left(\frac{dz_i}{ds}\right)ds = -\int_\alpha^\beta \delta z_i\cdot\frac{d^2z_i}{ds^2}\cdot ds.$$

This in (40) gives

$$\int_\alpha^\beta ds\sum\delta z_i\frac{d^2z_i}{ds^2} = 0. \tag{44}$$

Here the δz_i are not independent, for from (24) we have

$$z_1\delta z_1 + \cdots + z_4\delta z_4 = 0 \qquad \text{or} \qquad \sum z_i\delta z_i = 0,$$

which shows that only three of the four δz_i are independent. Let H be an arbitrary function of the z_i, then the last equation gives

$$H\cdot\sum z_i\delta z_i = 0 \qquad \text{or} \qquad \sum Hz_i\delta z_i = 0.$$

If with Lagrange we introduce this sum in (44) we do not destroy its validity, since it $= 0$, hence

$$\int_\alpha^\beta ds\sum\left\{\frac{d^2z_i}{ds^2} + Hz_i\right\}\delta z_i = 0. \tag{45}$$

We now determine H so that the coefficient of δz_4 or

$$\frac{d^2z_4}{ds^2} + Hz_4 = 0. \tag{46}$$

Then the term in δz_4 drops out of (45) and we may regard δz_1, δz_2, δz_3 as arbitrary. Hence for (45) to subsist it is now necessary that the coefficients of these three quantities vanish. These three equations and (46) give us

$$\frac{d^2z_i}{ds^2} + Hz_i = 0, \qquad i = 1, 2, 3, 4. \tag{47}$$

We may eliminate H as follows: we multiply (47) by z_i and add, getting

$$\sum z_i\frac{d^2z_i}{ds^2} + H\sum z_i^2 = 0.$$

But by (24), $\sum z_i^2 = R^2$. Therefore,

$$\sum z_i \frac{d^2 z_i}{ds^2} + R^2 H = 0. \tag{47'}$$

On the other hand, (24) gives on differentiating

$$\sum z_i \frac{dz_i}{ds} = 0.$$

Differentiating this gives

$$\sum z_i \frac{d^2 z_i}{ds^2} + \sum \left(\frac{dz_i}{ds}\right)^2 = 0.$$

Now we have already seen, using (36), that the second term on the left = 1.
Therefore,

$$\sum z_i \frac{d^2 z_i}{ds^2} + 1 = 0.$$

This in (47′) gives

$$-1 + R^2 H = 0 \qquad \text{or} \qquad H = \frac{1}{R^2};$$

which in (47) gives the four equations

$$\frac{d^2 z_i}{ds^2} + \frac{z_i}{R^2} = 0, \qquad i = 1, \cdots, 4, \tag{48}$$

which are therefore *the differential equations defining an R-straight.* These equations having constant coefficients belong to a well-known type; their integrals have the form

$$z_i = a_i \cos\frac{s}{R} + b_i \sin\frac{s}{R}, \qquad i = 1, 2, 3, 4. \tag{49}$$

Here the constants of integration, a_i, b_i, are subject to the condition that the z_i satisfy the relation (24). For $s = 0$ this gives $z_i = a_i$ while, for $s = \pi R/2$, we get $z_i = b_i$. Thus the a_i and the b_i are coördinates of two points a, b and by (24)

$$a_1^2 + \cdots + a_4^2 = R^2; \qquad b_1^2 + \cdots + b_4^2 = R^2. \tag{50}$$

These 8 constants satisfy another relation, for squaring (49) and adding gives

$$\sum z_i^2 = \cos^2\frac{s}{R}\sum a_i^2 + \sin^2\frac{s}{R}\sum b_i^2 + 2\sin\frac{s}{R}\cos\frac{s}{R}\cdot\sum a_i b_i.$$

Therefore,

$$R^2 = \left(\cos^2\frac{s}{R} + \sin^2\frac{s}{R}\right)R^2 + 2\sin\frac{s}{R}\cos\frac{s}{R}\cdot\sum a_i b_i$$

on using (24) and (50). Thus

$$0 = 2\sin\frac{s}{R}\cos\frac{s}{R}\cdot\sum a_i b_i,$$

Therefore,

$$\sum a_i b_i = 0. \tag{51}$$

Thus the 8 constants of integration a_i, b_i in (49) satisfy the three relations (50), (51). *The equations* (49) *together with* (50), (51) *are the equations of an R-straight.* We show now that we may determine the a_i, b_i so that the z_i satisfy

$$A_1 z_1 + \cdots + A_4 z_4 = 0, \qquad B_1 z_1 + \cdots + B_4 z_4 = 0, \tag{52}$$

where the A's and B's are arbitrary. Geometrically, these two equations define two R-planes or, from the standpoint of E-geometry, two diametral spheres. But if the coördinates z_i of (49) satisfy these equations (52), this means that the points of an R-straight lie on the intersection of two R-planes (52), and this is our fundamental theorem. To prove it we set (49) in (52); we get

$$\cos\frac{s}{R}\cdot\sum a_i A_i + \sin\frac{s}{R}\cdot\sum b_i A_i = 0$$

and a similar equation in the B_i. This requires that

$$\begin{array}{l} a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 = 0, \\ b_1 A_1 + \quad \cdots \quad + b_4 A_4 = 0, \\ a_1 B_1 + \quad \cdots \quad + a_4 B_4 = 0, \\ b_1 B_1 + \quad \cdots \quad + b_4 B_4 = 0. \end{array} \tag{53}$$

If the two R-planes are distinct, that is, if the B's are not proportional to the A's, we may for example express a_1, a_2 in terms of a_3, a_4 and similarly b_1, b_2 in terms of b_3, b_4. If we put these values of a_1, a_2, b_1, b_2 in terms of a_3, a_4, b_3, b_4 in the three equations (50) and (51), we see that there is only one degree of freedom left. That means, for example, that we may take the point a, from which we measure s, at any point on the intersection of the two R-planes (52).

Conversely, the a's and b's being chosen so as to satisfy the relations (50), (51), the equations (52) determine the coefficients A, B of the two R-planes (52), aside of course from constant factors. Thus the fundamental theorem is established.

Let us multiply the first equation of (52) by B_4, the second equation by A_4 and subtract; we get

$$C_1 z_1 + C_2 z_2 + C_3 z_3 = 0,$$

where $C_i = A_i B_4 - B_i A_4$, $i = 1, 2, 3$. Replacing the z's by their values in the x's by (22), we get

$$C_1 x_1 + C_2 x_2 + C_3 x_3 = 0,$$

a plane through the origin. Hence R-straights are also the intersection of a diametral sphere with a plane through the origin. We may also say from the standpoint of E-geometry: an R-straight is a circle cutting the fundamental sphere in diametral points.

(*To be concluded in the next issue.*)

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTION.

50. Professor R. C. Archibald of Brown University, Providence, R. I., will be glad to learn if any library in the United States contains the following volumes: (*a*) *Mathematisch-naturwissenschaftliche Blätter*, 1. Jahrgang (1904) and 15. Jahrgang (1918); (*b*) *Nieuwe Wiskundig Voorstellen met derzelver Outbindingen*, vol. 2 (1846) *with* title page and index.

DISCUSSIONS.

I. APPROXIMATION TO LOG $(1\cdot 2\cdot 3\cdots x)$.

By E. B. ESCOTT, Walton School of Commerce.

By the Euler-Maclaurin Summation Formula, we have the well-known expansion

$$\log x! = \log \sqrt{2\pi} + (x + \tfrac{1}{2}) \log x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \cdots. \tag{1}$$

We may also get the following development in powers of $x + \frac{1}{2}$ by the method given in Boole's *Finite Differences* (2d ed., p. 98):

$$\begin{aligned}\log x! = \log \sqrt{2\pi} &+ (x + \tfrac{1}{2}) \log (x + \tfrac{1}{2}) - (x + \tfrac{1}{2}) - \frac{1}{24(x + \frac{1}{2})} \\ &+ \frac{7}{2880(x + \frac{1}{2})^3} - \frac{31}{40320(x + \frac{1}{2})^5} + \cdots \\ &+ (-1)^{n+1} \frac{(2^{2n+1} - 1)B_{2n+1}}{(n + 1)(2n + 1)2^{2n+2}(x + \frac{1}{2})^{2n+1}} + \cdots,\end{aligned} \tag{2}$$

where the B's are Bernoulli's numbers.

To find an approximate expression for the remainder

$$-\frac{1}{24(x + \frac{1}{2})} + \frac{7}{2880(x + \frac{1}{2})^3} - \cdots,$$

let

$$\frac{1}{k} \log \left(1 + \frac{1}{a(x + \frac{1}{2})^2}\right) = -\frac{1}{24(x + \frac{1}{2})^2} + \frac{7}{2880(x + \frac{1}{2})^4} - \cdots.$$

Expanding the left member,

$$\frac{1}{k} \left(\frac{1}{a(x + \frac{1}{2})^2} - \frac{1}{2a^2(x + \frac{1}{2})^4} + \cdots\right) = -\frac{1}{24(x + \frac{1}{2})^2} + \frac{7}{2880(x + \frac{1}{2})^4} - \cdots.$$

Then

$$\frac{1}{ka} = -\frac{1}{24},$$

and

$$-\frac{1}{2ka^2} = \frac{7}{2880};$$

whence

$$a = \frac{60}{7},$$

$$k = -\frac{14}{5}.$$

Substituting these values in (2), we have

$$\log x! = \log \sqrt{2\pi} + (x + \tfrac{1}{2})\left[\log (x + \tfrac{1}{2}) - 1 - \frac{5}{14}\log\left(1 + \frac{7}{60(x+\frac{1}{2})^2}\right)\right]$$

approximately.

Or, finally

$$\log x! = \log \sqrt{2\pi} + (x + \tfrac{1}{2})\left[\frac{12}{7}\log (x + \tfrac{1}{2}) - \frac{5}{14}\log\left(x^2 + x + \frac{11}{30}\right) - 1\right] \qquad (3)$$

which is the approximate formula sought.

The above logarithms are natural logarithms. In terms of common logarithms, 1 would be replaced by $\log_{10} e$, *i.e.*,

$$\log_{10} x! = \log_{10} \sqrt{2\pi} + (x+\tfrac{1}{2})\left[\frac{12}{7}\log_{10} (x+\tfrac{1}{2}) - \frac{5}{14}\log_{10}\left(x^2+x+\frac{11}{30}\right) - \log_{10} e\right]. \qquad (4)$$

Prof. A. R. Forsyth in the *Report of the British Association for the Advancement of Science*, vol. 53 (1883), pp. 407–8, gives the following approximate formula:

$$\log_{10} x! = \log_{10} \sqrt{2\pi} + (x + \tfrac{1}{2})[\tfrac{1}{2} \log_{10} (x^2 + x + \tfrac{1}{6}) - \log_{10} e]. \qquad (5)$$

To estimate the degree of the approximation in (4) and (5), let us expand them in powers of $(x + \frac{1}{2})$ and compare with (2).

The expansion of (4) gives

$$\log x! = \log \sqrt{2\pi} + (x + \tfrac{1}{2}) \log (x + \tfrac{1}{2}) - (x + \tfrac{1}{2}) - \frac{1}{24(x+\frac{1}{2})} + \frac{7}{2880(x+\frac{1}{2})^3} - \frac{49}{259200(x+\frac{1}{2})^5} + \cdots. \qquad (6)$$

The expansion of (5) gives

$$\log x! = \log \sqrt{2\pi} + (x + \tfrac{1}{2}) \log (x + \tfrac{1}{2}) - (x + \tfrac{1}{2}) - \frac{1}{24(x+\frac{1}{2})} - \frac{1}{576(x+\frac{1}{2})^3} - \frac{1}{10368(x+\frac{1}{2})^5} + \cdots. \qquad (7)$$

This shows that (4) is a better approximation than Forsyth's approximation (5).

A well-known approximation is that due to Stirling,

$$\log_{10} x! = \log_{10} \sqrt{2\pi} + (x + \tfrac{1}{2}) \log_{10} x - x \log_{10} e. \tag{8}$$

To compare the accuracy of the three formulæ compare the value of $\log_{10} 50!$ as given by each.

By (1) or (2)	$\log_{10} 50! =$ 64.48307 48724 72035	(correct value),
By (4)	$\log_{10} 50! =$ 64.48307 48724 72802	(true to 11 decimals),
By (5) (Forsyth)	$\log_{10} 50! =$ 64.48307 48584 22193	(true to 7 decimals),
By (8) (Stirling)	$\log_{10} 50! =$ 64.48235 10579 85416	(true to 2 decimals).

II. Note on Total Representations as Sums of Squares.

By E. T. Bell, University of Washington.

Let $E(n)$, $O(n)$ denote respectively the total number of representations of the positive integer n as a sum of an even, an odd number of squares of integers with roots $\gtreqless 0$, the order of the squares in each representation being essential. Then

$$E(2n) > O(2n), \qquad O(2n + 1) > E(2n + 1).$$

This curious fact was observed while constructing a short table for numerical verifications of certain results in the theory of numbers. For example,

$n =$	1	2	3	4	5	6	7,
$E(n) =$	0	4	0	16	8	64	64,
$O(n) =$	2	0	8	2	32	24	128.

The algebraic proof is immediate. It is well known that the series $\sum_t (-x)^{t^2}$, where t takes the values $\pm 1, \pm 2, \pm 3, \cdots$, is absolutely convergent for some $|x| > 0$. Within the region of convergence let $s(x)$ represent the sum of this series. The series $s(-x)$ also is absolutely convergent for some $|x| > 0$, and for $|x|$ sufficiently small[1] the following is an absolutely convergent power series in x,

$$\frac{1}{1 + s(x)} = 1 + \sum_{n=1}^{\infty} (-1)^n \{s(x)\}^n = 1 + \sum_{n=1}^{\infty} (-1)^n \{E(n) - O(n)\} x^n,$$

the last following at once from the obvious remark that the coefficient of x^n in $\dfrac{1}{1 + s(-x)}$ is $E(n) - O(n)$.

Evidently our theorem will be proved when we show, as next, that the coefficient $(-1)^n\{E(n) - O(n)\}$ of x^n in this expansion is greater than zero for all integers $n > 0$.

[1] From the simplest considerations on the partition of numbers (as in the *Annals of Mathematics*, vol. 23 (1921), p. 61), it can be shown that an interval of convergence is $0 < |x| < \frac{1}{4}$. The upper bound $\frac{1}{4}$ can be increased to 1 by more complicated methods, but this is immaterial for the present discussion.

From the well-known identity[1]

$$1 + s(x) = \Pi(1 - x^{2n}) \times \Pi(1 - x^m)^2,$$

in which the products extend to $n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots$, it is obvious that the coefficient of each power of x occurring in the power series expansion of $\dfrac{1}{1 + s(x)}$ is an integer > 0. But in this development it is evident from the form of the products that each exponent of x is of the form $al_1 + bm_1$ in which $a, b \geqq 0$ with the exception of $a = b = 0$, $l_1 > 0$ is an even integer, $m_1 > 0$ is an odd integer. Now every integer > 0 is of this form, and in the development l_1, m_1 run through all even integers > 0, all odd integers > 0. Hence every integer $n > 0$ occurs as an exponent of x, and therefore $(-1)^n\{E(n) - O(n)\} > 0$ for every integer $n > 0$.

III. A Graphical Method of Solving Simultaneous Linear Equations.

By J. P. Ballantine, Columbia University.

Suppose it is desired to find a solution of the following simultaneous linear equations:

$$3x - 2y = 4,$$
$$2x + y = 5.$$

It is desired to find a number x which multiplied by the vector (3, 2) and a number y which multiplied by the vector $(-2, 1)$ will be so chosen that the sum of the two resulting vectors is the vector (4, 5). This gives rise to the following geometrical representation of the problem. The vectors (3, 2) and $(-2, 1)$ are laid off, and it is geometrically apparent that, in order to obtain (4, 5), it is necessary to take the first vector twice and the second vector once. Hence the solution is obtained by taking $x = 2$ and $y = 1$. It is also clear that if the solution were to come out fractional, as good estimates of these fractions could be obtained by the above geometrical interpretation as from the usual method of plotting the two loci, and estimating the coördinates of the point of intersection.

Each of the two geometrical methods has its own advantages. For instance, if the problem were varied by altering one of the two equations, the method which plots the loci of the two equations would be preferable. On the other hand, if the problem were varied by replacing the right-hand members, 4 and 5, by different pairs of numbers, then the method suggested in this paper would be preferable. In practical work this situation often arises. Suppose, for instance, one is using the so-called method of diminishing the constant terms. In this method one assumes an approximate solution X and Y, and finds that the differences $x - X$ and $y - Y$ satisfy simultaneous equations differing from the original ones only in the right-hand members. The same figure which suggested that X and Y were approximations can be used with the new right-hand members to suggest second approximations. Thus a process similar to Horner's method may be carried out.

[1] See any text on elliptic functions; *e.g.*, Hancock, *Theory of Elliptic Functions*, p. 397.

No novelty is claimed for the above method. It is one in common use in general analysis. It is laid before the readers of the MONTHLY as a method of familiarizing the freshman with the notion of a vector as a number pair.

RECENT PUBLICATIONS.

REVIEWS.

Statistical Method. By T. L. KELLEY. New York, The Macmillan Co., 1923. xi + 390 pages. Price $4.00.

"This book has been written with a view to serving two needs; that of biologists, economists, educators and psychologists, who know little of higher mathematics, possibly care less, and who use statistical methods merely as a device to portray the facts of their group investigations; and that of those in the same fields who resort to mathematics to aid in the discovery of new truths."

The treatment is inductive; usually a set of data is taken, the best method of analysis for the particular set is discussed and exemplified, general formulas and statistical constants are derived, and their meanings and significance are clearly interpreted with reference to the concrete problem considered. In this way the student is taught to examine data carefully and intelligently, and to endeavor to invent new methods of analysis if necessary, instead of using known formulas mechanically, regardless of whether they are applicable to the observed situation.

The material dealt with is practical data from widely diversified fields, rather than such nearly ideal distributions as are obtained by card drawing, dice throwing, and the like.

The methods and notations employed are, in the main, those of the English school of statisticians.

The book could hardly be read without a considerable knowledge of statistical method, although the beginner could undoubtedly profit by reading the first few chapters. It does not, however, presuppose an extensive mathematical training on the part of the student.

Some of the general topics considered are tabulation and plotting of series, graphic methods, measures of central tendency and dispersion, the normal probability distribution, methods of fitting curves to distributions. An extensive discussion is given of methods of measuring relationship. In connection with multiple correlation the author presents a method of successive approximations that he has developed for finding the values of the regression coefficients and the multiple correlation coefficient. In a chapter on sundry special problems—and elsewhere throughout the book—Professor Kelley has incorporated certain of his original investigations. The concluding chapter is devoted to index numbers.

In an appendix are to be found a list of more than one hundred important symbols, a bibliography of nearly three hundred titles, and the Kelley-Wood table of the normal probability integral. This table gives the values of the

abscissa x and the ordinate z of the normal probability curve $z = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, corresponding to values of the area $I = \int_0^x zdx$, for every thousandth from 0.000 to 0.499. It also gives values of q $(= 0.5 - I)$, the area of the smaller portion cut off by a given ordinate, of p $(= 0.5 + I)$, the area of the larger portion, and of z/q, z/p, and pq. The table should prove a valuable supplement to such tables as those of Sheppard.

At the end of the book is an alignment chart, devised by the author, which appears to be very useful in computations, especially in those connected with multiple correlation work.

The reviewer detected incidentally a number of typographical errors—perhaps not an undue number for a book containing so many formulas and symbols—most of which are not serious, consisting principally of carelessness in the use of exponents and subscripts (the use of one in place of the other, etc.), and the use of wrong letters (*e.g.*, u for μ, n for η). Attention ought to be called, however, to the important formula [22], p. 79, for the standard deviation, which should read

$$\sigma = \sqrt{\frac{\Sigma \xi^2}{N} - \delta^2};$$

and also to the formula for σ_{μ_n}, p. 84, which should read

$$\sigma_{\mu_n} = \sqrt{\frac{\Sigma \left[\frac{S(x^n - \mu_n)}{N} \right]^2}{M}}.$$

On the whole, Professor Kelley's book is a decided contribution to the literature on statistical method. It is thorough in its treatment, and the material is perhaps as systematically organized as is possible in a book with such a comprehensive scope. It will undoubtedly prove extremely valuable to the student of statistics, who will find in it an abundance of interesting material, no matter in what direction his inclinations may lie.

P. R. RIDER.

Introduction to the Calculus. By W. F. OSGOOD. The Macmillan Co., New York, 1922. 449 pp. and 133 figs. Price \$2.90.

Professor Osgood's first text-book on calculus which appeared in 1907 under the title of *A First Course in the Differential and Integral Calculus* gained a well-deserved popularity and passed through four reprintings. In January, 1921, the first part on differential calculus was revised and appeared under the title *Elementary Calculus.* The present book contains the eight chapters contained in the latter together with seven additional chapters.

The chief changes in the order of presentation and content may be briefly summarized as follows: (1) the chapter on *Infinitesimals and Differentials* is

placed before the treatment of the derivatives of the transcendental functions, (2) the chapter on *Differentiation of Transcendental Functions* has been divided into three separate chapters and appears under the titles *Trigonometric Functions, Logarithms and Exponentials,* and *Inverse Trigonometric Functions,* (3) some of the applications given in the last chapter of the original text appear in the earlier parts of the new text, notably Newton's Method for solving an equation approximately appears in Chapter VII under *Applications* and Simpson's Rule for the approximate evaluation of a definite integral in Chapter VII on *Definite Integrals,* (4) the latter part of Chapter XIV as well as Chapters XV–XX are omitted (with the exception of the topics noted under (3) dealing with *Applications* of partial derivatives to the *Geometry of Space, Taylor's Theorem for Functions of Several Variables, Envelopes, Double Integrals, Triple Integrals, and Approximate Computations*).

The point of view of the revised text is essentially that of the original, viz., to afford the student an opportunity to develop skill in the use of the methods of the calculus both by solving formal problems and by putting these methods to work in geometrical and mechanical settings. For the latter purpose Professor Osgood draws freely on the wealth of problems to be had in these two fields. The number of these excellent problems has been increased. The very interesting problems on small errors have been omitted though they form a most valuable application of differentials. Despite this fact, the student working through a moderate number of the problems of this text cannot help but emerge with a definite picture of the calculus at work in a manner which illustrates its true value.

Professor Osgood's style is very clear and this virtue is emphasized in the revised text by the relegation to the background of proofs and lengthy definitions until the student by actually rubbing shoulders with his new acquaintances learns to call them by their first names. In some cases, proofs are omitted entirely with a reference to the original text, *e.g.*, the proof for $\lim\limits_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$. As a student, the reviewer recalls Professor Osgood's statement that rigor in mathematics is nothing other than clearness. Without sacrificing rigor, Professor Osgood has undoubtedly gained in simplicity by stating certain fundamental theorems without proof.

The subject of definite integrals is undoubtedly the hardest topic for treatment in a beginning calculus course. A first course must necessarily rely at this point on the intuition of the student, with an occasional glimpse at the facts. However, unnecessary restrictions can only introduce complications without materially simplifying the subject. For instance, to restrict continuous curves to mean only those with a finite number of extremes without making use of this restriction is needless and beclouds the issue. If complete proofs cannot be given in the less restricted case, would it not be wiser to state the facts in the more general case deferring the proof until a later course? Professor Osgood seems to depart at this point from the procedure adopted in the earlier chapters.

The reviewer has called attention in a recent number of the MONTHLY (*1922,*

239–250) to Professor Osgood's original text-book treatment of Duhamel's Theorem, and others have done the same before him. On p. 302, the statement of the theorem is modified to meet this criticism, by the addition of the phrase "*i.e.*, $\lim_{n\to\infty} \epsilon_k = 0$, *k varying in any manner whatever as n increases.*" In applying the test in the form $\lim_{n\to\infty} \frac{\alpha_k}{\beta_k} = 1$, we are obliged to make the two unnecessary restrictions that x and y shall each be positive. This would have been obviated by changing the test to that of $\lim_{n\to\infty} (\alpha_k - \beta_k) = 0$. The range of application of this theorem in the new edition is not as wide as that in the old, nor is the ease of application any greater.

On the whole the excellent qualities of the original text are to be found in the present, together with improvements in simplicity and new problems. The only typographical error noticed is in the seventh line of p. 302, where "high" should be replaced by "higher."

H. J. ETTLINGER.

NOTES ON RECENT PUBLICATIONS.

Wissenschaftliche Vorträge gehalten auf dem 5. Kongress der skandinavischen Mathematiker in Helsingfors vom 4. bis 7. Juli 1922 (Helsingfors, Akademische Buchhandlung, 1923, 4 + 315 pp.) is by far the largest of the reports of the five congresses held; the others were in 1909, 1911, 1913, and 1916. The next is to be at Copenhagen in 1924. The report of the fourth congress, which was printed at Upsala in 1920, is excessively rare owing to some misunderstanding regarding the number of copies to be published. Possibly there is not more than one copy in America.

An extended paper (52 pages) by Professor W. E. MILNE on "Damped vibrations" appears as vol. 2, No. 2, of the University of Oregon Publications. This consists mainly of the study of the equation

$$A\frac{dv}{dt} + R(v) + Cy = 0, \qquad \frac{dy}{dt} = v,$$

the resistance $R(v)$ being an arbitrary function of the velocity. Fifteen pages of tables furnish numerical values for the solution of the important cases.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 45, January, 1923: "On the number of solutions in positive integers of the equation $yz + zx + xy = n$" by L. J. Mordell, 1–4; "A closed set of normal orthogonal functions" by J. L. Walsh, 5–24; "Congruences determined by a given surface" by Claribel Kendall, 25–41; "Linear partial differential equations with a continuous infinitude of variables" by I. A. Barnett, 42–53; "On the ordering of the terms of polars and transvectants of binary forms" by L. Isserlis, 54–71.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, May, 1923: "Sur une classe d'équations fonctionnelles" by G. Julia (continuation), 129–150; "La théorie des marées et les équations intégrales" by G. Bertrand, 151–160 (continued).—June, 1923: "La théorie des marées et les équations intégrales" by G. Bertrand (continuation), 161–192.

Annals of Mathematics, second series, volume 24, September, 1922: "Periodicities in the theory of partitions" by E. T. Bell, 1–22; "Functionals of summable functions" by W. L. Hart, 23–38; "Periodically closed chains of reduced fractions" by A. Arwin, 39–68; "On certain linear differential equations of the second order" by F. H. Murray, 69–88.

Bulletin des Sciences Mathématiques, second series, volume 47, June, 1923: Review by E. Cartan of Assier de Pompignan, *Note sur le calcul tensoriel* (Paris, 1923), 193; "Une lettre inédite de Descartes au Père Mersenne" by H. Omont, 194–195; "Démonstration du théorème de Riesz-Fischer et du théorème de Weyl sur les suites convergentes en moyenne" by M. Plancherel, 195–204; "Mouvement d'un solide pesant fixé par un point voisin de son centre de gravité" by H. Vergne, 204–224 (continued).—July, 1923: Review by A. Lebeuf of H. Andoyer, *Cours d'Astronomie. 1re partie* (3d ed., Paris, 1923), 225–229; "Sur les modes de continuité de certaines fonctionnelles" by G. Bouligand, 229–244; "Mouvement d'un solide pesant fixe par un point voisin de son centre de gravité" by H. Vergne (continuation), 244–256 (continued).

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'Ueber S. Lies Geometrie der Kreise und Kugeln' " by H. Mohrmann, 315–319; Nachwort von E. Study, 319–320.

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PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3044. Proposed by EUGENE M. BERRY, West Lafayette, Indiana.

1. Express $\prod\limits_{k=1}^{k=n} \cos a_k$ as a summation of cosines, each term to be of the form $C \cos (\pm a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_n)$.

2. Express $\prod\limits_{k=1}^{k=n} \sin a_k$ as a summation of sines or cosines according as n is odd or even.

3. Express $\prod\limits_{k=1}^{k=n} \sin a_k \cdot \prod\limits_{j=1}^{j=m} \cos b_j$ as a summation of sines or cosines according as n is odd or even.

3045. Proposed by S. A. COREY, Des Moines, Iowa.

If in the equation

$$s!\left[\frac{1}{(s+1)!}+\frac{c_3}{4!(s-1)!}+\frac{c_5}{6!(s-3)!}+\frac{c_7}{8!(s-5)!}+\cdots+\frac{c_s}{(s+1)!2!}\right]=0 \qquad (1)$$

$c_3, c_5, c_7, \cdots c_s$ be given and retain such constant values that (1) is satisfied for all positive odd integral values of s, $(s > 1)$, prove that if s be decreased by unity (so that $s = 2n$), then the left member will become equal to $\pm B_n$, according as n is odd or even, B_n being Bernoulli's nth number. Also show how any one of the constants c may be found without first finding all the preceding constants.

3046. Proposed by A. L. WECHSLER, New York City.

What is the probability that there will be at least r consecutive heads out of n tosses of a coin?

3047. Proposed by ARNOLD DRESDEN, University of Wisconsin.

Prove that for any positive integer n,

$$\Sigma \frac{1}{\Pi_{i=1}^{t} p_i!k_i^{p_i}} = 1,$$

where k_i and p_i are positive integers, k_i being the distinct elements of any t-partite partition of n $(t = 1 \cdots n)$ and p_i the number of parts of the partition which are equal to k_i.

3048. Proposed by H. C. Bradley, Massachusetts Institute of Technology.

Cut two equilateral triangles of any relative proportions into not more than five pieces which can be assembled to form a single equilateral triangle.

3049. Proposed by H. Grossman, (Student) College of the City of New York.

Prove that every factor of $2^{2^n} + 1$ is congruent to 1 mod 2^{n+1}, and that no two different numbers of the form $2^{2^n} + 1$ have a common factor.

SOLUTIONS.

2980 [1922, 271]. Proposed by J. Rosenbaum, Milford, Connecticut.

Locate a point such that the sum of its distances from n given points shall be a minimum.

Solution by H. M. Lufkin, Dunkirk, N. Y.

Let the rectangular coördinates of the n given points P_1, P_2, P_3, $\cdots$, P_n taken counter clockwise be (x_1, y_1), (x_2, y_2), (x_3, y_3), $\cdots$, (x_n, y_n), respectively, and let the coördinates of the required point, P, be (x, y).

Then the sum of the distances from the required point, P, to the n given points is

$$f(x, y) = \sum_{i=1}^{i=n} \sqrt{(x - x_i)^2 + (y - y_i)^2} = \sum_{i=1}^{i=n} \rho_i,$$

say. The necessary condition that $f(x, y)$ shall be a minimum is that

$$\frac{\partial f}{\partial x} = \frac{x - x_1}{\rho_1} + \frac{x - x_2}{\rho_2} + \cdots + \frac{x - x_n}{\rho_n} = 0$$

and

$$\frac{\partial f}{\partial y} = \frac{y - y_1}{\rho_1} + \frac{y - y_2}{\rho_2} + \cdots + \frac{y - y_n}{\rho_n} = 0.$$

Denote by θ_i the angle between the positive sense of the x axis and the line PP_i $(i = 1, 2, \cdots, n)$; then

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{i=n} \cos \theta_i, \quad (1) \qquad \frac{\partial f}{\partial y} = \sum_{i=1}^{i=n} \sin \theta_i. \quad (2)$$

A necessary condition that $f(x, y)$ shall be a relative minimum is that these shall be equal to zero. The point (x, y) can then be determined by solving (1) and (2).

The function $f(x, y)$ is continuous at all points of the plane and the first derivatives exist at every point except P_1, P_2, $\cdots$, P_n. If we enclose the polygon in a large boundary, the value of the function $f(x, y)$ is less within the boundary than on it. Hence the function $f(x, y)$ takes on its absolute minimum either at a point where the first derivatives vanish or at P_1, P_2, etc.

In the case of the triangle, none of whose angles equal or are greater than 120° this point is one of the isogonal points, and where one angle equals or is greater than 120° it occurs at the vertex of the obtuse angle. In the case of the quadrilateral the point is the intersection of the diagonals.

Also solved by T. M. Blakslee and F. L. Wilmer.

Note by Otto Dunkel, Washington University.—If a convex polygon be drawn containing all the points in its interior or on its boundary and having for its vertices points of the given set, then the minimum must occur either within or on the boundary. If we suppose that each point P_i exerts a unit attractive force upon a particle P in any position of the plane, then the vanishing of the derivatives in (1) means that equilibrium occurs. This gives at once the position of P in the case of a triangle having no angle as great as 120° and in that of a *convex* quadrilateral. In the case of four points such that one point lies within or on a side of the triangle formed by the other three, it is easily

shown by elementary geometry that the minimum occurs at this point. The case of the convex quadrilateral is also easy by elementary geometry. In the case of a triangle having no angle as great as 120° there is a known simple geometrical proof as follows: Let P be the point within the triangle ABC at which the sides subtend angles of 120°. Perpendiculars to PA, PB, PC at A, B, C, respectively, form an equilateral triangle. If Q is any point within this equilateral triangle, the sum of its distances from the sides is constant and equal to the altitude of the triangle. But this sum is less than $QA + QB + QC$ unless Q is at P. Hence the minimum is at P. If an angle is equal to or greater than 120° it may be shown by elementary geometry that the minimum occurs at the vertex of the angle. See *1920*, 38 for four solutions in the case of the triangle and references on pages 40, 41.

2983 [1922, 313]. Proposed by R. S. UNDERWOOD, Alabama Polytechnic Institute.

Prove that the following terminating series have the sums indicated, where by 0! is meant 1!

$$\frac{1}{n!1!} + \frac{1}{(n-2)!3!} + \frac{1}{(n-4)!5!} + \cdots = \frac{2^n}{(n+1)!}; \tag{1}$$

$$\frac{1}{(n+1)!0!} + \frac{1}{(n-1)!2!} + \frac{1}{(n-3)!4!} + \cdots = \frac{2^n}{(n+1)!}. \tag{2}$$

(These are coincident for n even but are distinct for n odd.)

$$\frac{1}{n!1!} - \frac{1}{(n-2)!3!} + \frac{1}{(n-4)!5!} - \cdots = \frac{\alpha(-4)^{[n/4]}}{(n+1)!}, \tag{3}$$

where $[n/4]$ denotes the greatest integer not exceeding $n/4$, and where $\alpha = 1, 2, 2, 0$ according as n is congruent to 0, 1, 2, 3 (mod 4).

Obtain a similar form for the series

$$\frac{1}{(n+1)!0!} - \frac{1}{(n-1)!2!} + \frac{1}{(n-3)!4!} - \cdots. \tag{4}$$

Obtain also the sums of the following terminating series and of the four series obtained from them by changing the signs of the alternate terms:

$$\frac{1}{n!1!} + \frac{1}{(n-4)!5!} + \frac{1}{(n-8)!9!} + \cdots; \tag{5}$$

$$\frac{1}{(n+1)!0!} + \frac{1}{(n-3)!4!} + \frac{1}{(n-7)!8!} + \cdots; \tag{6}$$

$$\frac{1}{(n-2)!3!} + \frac{1}{(n-6)!7!} + \frac{1}{(n-10)!11!} + \cdots; \tag{7}$$

$$\frac{1}{(n-1)!2!} + \frac{1}{(n-5)!6!} + \frac{1}{(n-9)!10!} + \cdots. \tag{8}$$

SOLUTION BY J. F. REILLY, University of Iowa.

From the binomial theorem

$$2^{n+1} = (1+1)^{n+1} = 1 + \frac{(n+1)!}{n!1!} + \frac{(n+1)!}{(n-1)!2!} + \cdots, \tag{A}$$

$$0 = (1-1)^{n+1} = 1 - \frac{(n+1)!}{n!1!} + \frac{(n+1)!}{(n-1)!2!} - \cdots. \tag{B}$$

Adding (A) and (B) and dividing by $2(n+1)!$, we have equation (2). Subtracting (B) from (A) and dividing by $2(n+1)!$, we have equation (1).

Again from the binomial theorem, together with De Moivre's theorem

$$(1+i)^{n+1} = 2^{(n+1)/2}\left(\cos\frac{(n+1)\pi}{4} + i\sin\frac{(n+1)\pi}{4}\right)$$
$$= \left\{1 - \frac{(n+1)!}{(n-1)!2!} + \frac{(n+1)!}{(n-3)!4!} - \cdots\right\}$$
$$+ i\left\{\frac{(n+1)!}{n!1!} - \frac{(n+1)!}{(n-2)!3!} + \frac{(n+1)!}{(n-4)!5!} - \cdots\right\}.$$

Now $\sin\frac{(n+1)\pi}{4} = \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0$ according as n is congruent to 0, 1, 2, 3, 4, 5, 6, 7 (mod 8). Hence, equating coefficients of i and dividing by $(n+1)!$, we obtain equation (3). Equating the real terms in the above equation and dividing by $(n+1)!$ and noting that $\cos\frac{(n+1)\pi}{4} = \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1$ according as n is congruent to 0, 1, 2, 3, 4, 5, 6, 7 (mod 8), we have

$$\text{(4)}\qquad \frac{1}{(n+1)!0!} - \frac{1}{(n-1)!2!} + \frac{1}{(n-3)!4!} - \cdots = \frac{\beta(-4)^{[n/4]}}{(n+1)!},$$

where $\beta = 1, 0, -2, -4$ according as n is congruent to 0, 1, 2, 3 (mod 4).

Adding equations (1) and (3) and dividing by 2, we obtain

$$\text{(5)}\qquad \frac{1}{n!1!} + \frac{1}{(n-4)!5!} + \frac{1}{(n-8)!9!} + \cdots = \frac{2^n + \alpha(-4)^{[n/4]}}{2(n+1)!}.$$

Similarly from equations (2) and (4), we obtain

$$\text{(6)}\qquad \frac{1}{(n+1)!0!} + \frac{1}{(n-3)!4!} + \frac{1}{(n-7)!8!} + \cdots = \frac{2^n + \beta(-4)^{[n/4]}}{2(n+1)!}.$$

By subtracting (3) from (1) and dividing by 2, we obtain

$$\text{(7)}\qquad \frac{1}{(n-2)!3!} + \frac{1}{(n-6)!7!} + \cdots = \frac{2^n - \alpha(-4)^{[n/4]}}{2(n+1)!},$$

and by subtracting (4) from (2) and dividing by 2, we obtain

$$\text{(8)}\qquad \frac{1}{(n-1)!2!} + \frac{1}{(n-5)!6!} + \cdots = \frac{2^n - \beta(-4)^{[n/4]}}{2(n+1)!}.$$

In order to sum the series formed from (5), (6), (7), and (8) by changing the signs of alternate terms, we expand

$$\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{n+1} \quad\text{and}\quad \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{n+1},$$

and in each case equate reals to reals and imaginaries to imaginaries.

The following formulas result:

$$(2+\sqrt{2})^{(n+1)/2}\cos\frac{(n+1)\pi}{8} = 1 + \frac{1}{\sqrt{2}}(n+1) - \frac{1}{\sqrt{2}}\frac{(n+1)n(n-1)}{3!}$$
$$- \frac{(n+1)n(n-1)(n-2)}{4!} - \frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-3)}{5!} + \frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-5)}{7!} \qquad (C)$$
$$+ \frac{(n+1)n\cdots(n-6)}{8!} + \cdots,$$

$$(2+\sqrt{2})^{(n+1)/2}\sin\frac{(n+1)\pi}{8} = \frac{1}{\sqrt{2}}(n+1) + \frac{(n+1)n}{2!} + \frac{1}{\sqrt{2}}\frac{(n+1)n(n-1)}{3!}$$
$$- \frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-3)}{5!} - \frac{(n+1)n\cdots(n-4)}{6!} - \frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-5)}{7!} + \cdots, \qquad (D)$$

$$(2-\sqrt{2})^{(n+1)/2}\cos\frac{(n+1)3\pi}{8}=1-\frac{1}{\sqrt{2}}(n+1)+\frac{1}{\sqrt{2}}\frac{(n+1)n(n-1)}{3!}$$
$$-\frac{(n+1)n(n-1)(n-2)}{4!}+\frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-3)}{5!}-\frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-5)}{7!}$$
$$+\frac{(n+1)n\cdots(n-6)}{8!}-\cdots, \tag{E}$$

$$(2-\sqrt{2})^{(n+1)/2}\sin\frac{(n+1)3\pi}{8}=\frac{1}{\sqrt{2}}(n+1)-\frac{(n+1)n}{2!}+\frac{1}{\sqrt{2}}\frac{(n+1)n(n-1)}{3!}$$
$$-\frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-3)}{5!}+\frac{(n+1)n\cdots(n-4)}{6!}-\frac{1}{\sqrt{2}}\frac{(n+1)n\cdots(n-5)}{7!}+\cdots. \tag{F}$$

Numbering the series with alternate signs (9), (10), (11), and (12), we have

$$(9)\quad \frac{1}{(n+1)!}\left[(n+1)-\frac{(n+1)n\cdots(n-3)}{5!}+\frac{(n+1)n\cdots(n-7)}{9!}-\cdots\right]$$
$$=\frac{\sqrt{2}}{4(n+1)!}[(C)+(D)+(E)+(F)]$$
$$=\frac{\sqrt{2}}{4(n+1)!}\left[(2+\sqrt{2})^{(n+1)/2}\left\{\cos\frac{(n+1)\pi}{8}+\sin\frac{(n+1)\pi}{8}\right\}\right.$$
$$\left.-(2-\sqrt{2})^{(n+1)/2}\left\{\cos\frac{(n+1)3\pi}{8}-\sin\frac{(n+1)3\pi}{8}\right\}\right]$$
$$=\frac{1}{2(n+1)!}\left[(2+\sqrt{2})^{(n+1)/2}\cos\frac{(n-1)\pi}{8}+(2-\sqrt{2})^{(n+1)/2}\sin\frac{(3n+1)\pi}{8}\right],$$

$$(10)\quad \frac{1}{(n+1)!}\left[1-\frac{(n+1)n(n-1)(n-2)}{4!}+\frac{(n+1)n\cdots(n-6)}{8!}-\cdots\right]$$
$$=\frac{1}{2(n+1)!}[(C)+(E)]$$
$$=\frac{1}{2(n+1)!}\left[(2+\sqrt{2})^{(n+1)/2}\cos\frac{(n+1)\pi}{8}+(2-\sqrt{2})^{(n+1)/2}\cos\frac{(n+1)3\pi}{8}\right],$$

$$(11)\quad \frac{1}{(n+1)!}\left[\frac{(n+1)n(n-1)}{3!}-\frac{(n+1)n\cdots(n-5)}{7!}+\frac{(n+1)n\cdots(n-9)}{11!}-\cdots\right]$$
$$=\frac{\sqrt{2}}{4(n+1)!}[-(C)+(D)-(E)+(F)]$$
$$=\frac{\sqrt{2}}{4(n+1)!}\left[-(2+\sqrt{2})^{(n+1)/2}\left\{\cos\frac{(n+1)\pi}{8}-\sin\frac{(n+1)\pi}{8}\right\}\right.$$
$$\left.+(2-\sqrt{2})^{(n+1)/2}\left\{\cos\frac{(n+1)3\pi}{8}+\sin\frac{(n+1)3\pi}{8}\right\}\right]$$
$$=\frac{1}{2(n+1)!}\left[(2+\sqrt{2})^{(n+1)/2}\sin\frac{(n-1)\pi}{8}+(2-\sqrt{2})^{(n+1)/2}\cos\frac{(3n+1)\pi}{8}\right],$$

$$(12)\quad \frac{1}{(n+1)!}\left[\frac{(n+1)n}{2!}-\frac{(n+1)n\cdots(n-4)}{6!}+\frac{(n+1)n\cdots(n-8)}{10!}-\cdots\right]$$
$$=\frac{1}{2(n+1)!}[(D)-(F)]$$
$$=\frac{1}{2(n+1)!}\left[(2+\sqrt{2})^{(n+1)/2}\sin\frac{(n+1)\pi}{8}-(2-\sqrt{2})^{(n+1)/2}\sin\frac{(n+1)3\pi}{8}\right].$$

Taking half the sum and half the difference of series (5) and (9); of (6) and (10); of (7) and (11); and of (8) and (12), we have eight other series of positive terms whose sums will thus be known.

If now we form the eight corresponding series with alternate signs, we can sum them by employing eighth roots of -1 in a manner similar to that in which we have employed square roots and fourth roots. Then sixteen new series of positive terms can be formed, and the sixteenth roots of -1 employed as before to sum the corresponding series with alternate signs. Evidently this may be carried on indefinitely.

Also solved by A. PELLETIER and M. POUTSKY.

2987 [1922, 356]. Proposed by PHILIP FITCH, North Denver High School, Colorado.

A flexible chain of length l and uniform weight is fastened at one end to the ridge of a roof with pitch p and slant height L. If the eaves of the roof are at a height h from the ground and the coefficient of friction between the chain and the roof is μ, how long will it take, after releasing the chain, for the highest end to reach the ground?

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The whole motion is supposed to be through the distances L and h, with no crumpling of the chain excepting in the last stage; also, $l < L$ and $l < h$.

(*i*) For the time through $L - l$.

Let w = the distance passed over by the whole chain at the end of time t from the beginning of motion; then the moving mass being constant, m being the mass of a unit of length of the chain, resolving forces along the plane, the equation of motion is

$$ml\frac{d^2w}{dt^2} = mgl \sin p - \mu mgl \cos p$$

or, putting $\mu = \tan \alpha$,

$$\ddot{w} = g \sec \alpha \sin (p - \alpha). \tag{1}$$

Multiplying by $2\dot{w}$ and integrating, taking $\dot{w} = 0$, when $w = 0$,

$$\dot{w}^2 = 2g \sec \alpha \sin (p - \alpha)w = 2k^2w,$$

say, or,

$$dw/2\sqrt{w} = k/\sqrt{2}\, dt. \tag{2}$$

Integrating

$$t_1 = \frac{\sqrt{2}}{k}\sqrt{w}\Big]_0^{L-l} = \frac{\sqrt{2}}{k}\sqrt{L - l}.$$

Also, when $w = L - l$, $\dot{w} = v_1 = k\sqrt{2(L - l)}$.

(*ii*) For the time from the moment t_1 till the whole chain first becomes vertical.

Let x = the distance the upper end of the chain has moved down the plane after time t in the second stage of motion; then $l - x$ and x are the parts of the chain on the plane and vertical; also let T = the tension acting on the two parts.

The equations of motion then are

$$m(l - x)\frac{d^2}{dt^2}(l - x) = mg(l - x) \sin p - \mu mg \cos p(l - x) - T$$

or

$$m(l - x)\frac{d^2x}{dt^2} = - mg(l - x) \sec \alpha \sin (p - \alpha) + T$$

and

$$mx\frac{d^2x}{dt^2} = mgx - T.$$

Eliminating T and arranging,

$$\frac{d^2x}{dt^2} = \frac{g}{l}[\{1 + \sec \alpha \sin (p - \alpha)\}x - l \sec \alpha \sin (p - \alpha)]. \tag{3}$$

Multiplying by $2\frac{dx}{dt}$ and integrating,

$$\dot{x}^2 = \frac{g}{l}[\{1 + \sec \alpha \sin (p - \alpha)\}x^2 - 2l \sec \alpha \sin (p - \alpha)x] + C_1.$$

When $x = 0$, $\dot{x}^2 = v_1^2 = 2k^2(L - l) = C_1$, and

$$t = t_2 = \int_0^l \frac{dx}{\sqrt{g/l[\{1 + \sec \alpha \sin (p - \alpha)\}x^2 - 2l \sec \alpha \sin (p - \alpha)x] + 2k^2(L - l)}}, \tag{4}$$

giving a transcendental form for the second period of time.

Putting C_1 in the equation for x and then $x = l$, we have $\dot{x} = v_2 = \sqrt{l(g - 3k^2) + 2Lk^2}$, the velocity of the chain when starting in its motion wholly vertical.

(*iii*) For the time in falling freely under gravity through the vertical distance $h - l$.

We have

$$v_3 = \{2g(h - l) + v_2^2\}^{1/2},$$

and then

$$t_3 = (v_3 - v_2)/g.$$

(*iv*) For the time required for the chain to become wholly heaped up at the foot of h, the initial velocity being v_3.

Let z = the length of chain in motion at any time after the moment t_3; then the equation of motion is

$$\frac{d}{dt}(mz\dot{z}) = mgz.$$

Multiplying by $2z\dot{z}$ and integrating,

$$z^2\dot{z}^2 = \frac{2g}{3}z^3 + C_2. \tag{5}$$

When $z = l$, $\dot{z} = v_3$, and $C_2 = l^2v_3^2 - \frac{2g}{3}l^3$, and (5) is

$$z^2\dot{z}^2 = \frac{2g}{3}z^3 + l^2\left(v_3^2 - \frac{2g}{3}l\right),$$

or

$$t_4 = -\int_0^l \frac{zdz}{\left\{\frac{2g}{3}z^3 + l^2\left(v_3^2 - \frac{2g}{3}l\right)\right\}^{1/2}}.$$

The required time is $t = t_1 + t_2 + t_3 + t_4$.

The last integration is made by the usual methods under elliptic functions or those of Weierstrass.

Also solved by J. E. Redden and J. B. Reynolds.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to R. W. Burgess, Brown University, Providence, R. I.

At the meeting of the American Philosophical Society on April 10, the James Scott medal was awarded to Sir Joseph Thomson, for his work on the physics of the electron.

Drury College, on the occasion of its fiftieth anniversary, conferred an honorary doctorate on Professor B. F. Finkel, of the department of mathematics, founder of this Monthly.

The Tables for interior ballistics, which were issued some time ago in blue print form, have now been published in printed form, "For Official Use Only," as "Ordnance Document, No. 2369." These tables were computed in the Ballistic Section of the Ordnance Department, under direction of Professor A. A. Bennett. They are the first tables of the sort available for the U. S. Army, and are made possible by the use of certain mathematical transformations discovered by Professor Bennett. They render obsolete the method of individual trajectory computations for interior ballistics, which has been hitherto the only theoretically accurate procedure.

Professor P. J. Daniell, of Rice Institute, has been appointed to the Town Trust chair of mathematics at the University of Sheffield, effective January, 1924.

Professor W. B. FITE, of Columbia University, has been granted leave of absence for the first half of the academic year 1923–24.

Professor B. H. CAMP, of Wesleyan University, during his leave of absence for the academic year 1923–24, plans to study at the University of Paris and with Professor Karl Pearson at the Biometric Laboratory of University College, London.

Professor E. A. KHOLODOVSKY, assistant professor of mathematics at the Polytechnic Institute of Petrograd, has been appointed assistant in the Lick Observatory.

Dr. V. D. GOKHALE, of the University of Chicago, has been appointed associate professor of mathematics at the University of the Philippines.

Assistant Professor E. W. PEHRSON, of the University of Utah, has been promoted to an associate professorship of mathematics.

The following appointments to instructorships of mathematics are announced: Mr. J. P. BALLANTINE, of the University of Chicago, at Columbia University; Dr. I. A. BARNETT, of the University of Saskatchewan, at the University of Cincinnati; Dr. H. W. CHANDLER, of the University of Minnesota, at the University of Florida; Mr. B. F. DOSTAL, of Bradley Polytechnic Institute, at the University of Michigan; Dr. M. M. FELDSTEIN, of the University of Chicago, at the University of West Virginia; Miss MINNIE HOLMAN, of the University of Oregon, at the University of Wyoming; Mr. B. F. KIMBALL, of Harvard University, at Cornell University; Mr. T. H. MILNE, of the University of Alberta, at the University of Buffalo; Dr. JESSE OSBORN, of Cornell University, at the University of Iowa; Mr. S. T. SPARKMAN, at the University of South Carolina; Mr. C. E. STOUT, of Case School, at Heidelberg University. Dr. I. MAIZLISH, of the University of Minnesota, has been appointed instructor in physics at Lehigh University.

Captain D. M. GARRISON, of the corps of Professors of Mathematics, U. S. Navy, is retiring from the Navy, and has accepted the professorship of mathematics at St. John's College, Annapolis, Md. Captain Garrison had been a member of the department of mathematics at the U. S. Naval Academy for twenty years, and for the past five years, during the difficult period of readjustment following the world war, has been head of the department. He will be succeeded as head of the department by Commander A. J. CHANTRY, Jr., of the Corps of Naval Constructors.

At the University of Iowa, Assistant Professor W. H. WILSON is on leave of absence for the academic year 1923–24. Mr. R. E. KENNON, instructor of mathematics, has resigned to accept a position as Examiner in the Iowa State Insurance Department.

At the University of Texas, Associate Professor E. L. DODD has been promoted to a full professorship and made chairman of the department. Associate Professor R. L. MOORE has been promoted to a full professorship, Adjunct Professor H. J. ETTLINGER to an associate professorship, and Dr. P. M. BATCHELDER and Miss MARY DECHERD to adjunct professorships.

Professor W. H. METZLER, of Syracuse University, has been appointed dean of the New York State College for Teachers at Albany.

Dr. B. M. TURNER, of the University of Illinois, has been appointed assistant professor of mathematics at the University of West Virginia.

Dr. F. W. REED, of Cornell University, has been appointed assistant professor of mathematics at Ohio State University.

Dr. GEORGE RUTLEDGE, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Assistant Professor J. E. DAVIS, of the University of Arkansas, has been appointed assistant professor of mathematics at Drexel Institute.

Mr. H. T. DAVIS, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Indiana.

Mr. M. L. MACQUEEN, of the University of Wisconsin, has been appointed a member of the department of mathematics at the Southwestern Presbyterian College.

Professor W. H. KIRCHNER, of the department of drawing and descriptive geometry at the University of Minnesota, has sabbatical leave of absence for the year 1923–24, and expects to spend much of his time studying at the University of Palermo.

Miss LENA R. COLE has been made head of the department of mathematics at Central Normal College, Danville, Ind.

Dr. NINA M. ALDERTON has been promoted to be head of the department of mathematics at Mills College.

Professor B. R. ALLEN, of Westmoorland College, has been appointed associate professor of mathematics in the Kansas State Teachers College at Emporia.

Professor A. A. MCSWEENY, of the College of Agriculture and Mechanic Arts of the University of Montana, has been appointed head of the department of mathematics at John Tarleton Agricultural College, Stephenville, Texas.

Professor W. W. WEBER, dean and professor of mathematics at Southern College, Lakeland, Florida, has been appointed to the chair of mathematics in Lander College, Greenwood, S. C.

Professor W. A. HAMILTON, who resigned his position at Beloit College because of the manner of dismissal of one of his colleagues, has been appointed lecturer in mathematics at the University of Wisconsin for the coming year.

Professor J. E. PIERCE, of James Millikin University, has been made head of the department of mathematics at Heidelberg University.

Professor J. A. CRAGWALL, head of the department of mathematics at Wabash College, has been granted a year's leave of absence because of poor health. His duties are being temporarily taken over by Professor G. E. CARSCALLEN.

Professor B. F. JOHNSON, of the Missouri State Normal School, has been granted leave of absence for the current academic year, and expects to spend the year, in part at least, at Oxford University.

Mr. W. E. ARMENTROUT, who had been instructor of mathematics at the University of Wisconsin, and was under appointment as instructor at Cornell

University for the year 1923–24, was struck by lightning June 18, 1923, and instantly killed.

Professor H. B. LEONARD of the University of Arizona represented the Mathematical Association at the inauguration of Dr. C. H. Marvin as president of the university. The American Mathematical Society was represented by Professor G. H. CRESSE.

The International Congress of Mathematicians in 1924 is to meet at the University of Toronto, Toronto, Canada, August 4–9. The British Association for the Advancement of Science will hold its meetings at the same place, August 6–13.

The MONTHLY is in receipt of a letter from Dr. Georg Wolff of Hannover, Germany, stating that, owing to the low purchasing power of the mark, it will no longer be possible to publish the periodical entitled *Unterrichtsblätter für Mathematik und Naturwissenschaften* unless aid is secured from abroad. To this end American subscriptions are solicited, the price being but 60 cents a year. Such subscriptions should be addressed to Studienrat E. Zieprecht, Hannover, Germany, Am Schatzkampe 11.

With a view to stimulating on the part of the public some appreciation of various phases of mathematical development, the American Mathematical Society has established an honorary lectureship. Under the auspices of the Society, a scientist of the highest distinction will be invited to deliver a popular address on a topic in mathematics or its applications. This lectureship is to be called the Josiah Willard Gibbs Lectureship in honor of the greatest mathematical physicist which America has produced. It is proposed to have lectures at intervals of a year or more and it is expected that the first will be given in New York during the winter of 1923–24.

J. M. WILLARD, Professor of Mathematics at Pennsylvania State College, died at his home December 10, 1923. He was born at Orford, New Hampshire, February 1, 1865, and graduated from Dartmouth with honors in Mathematics in 1887. He taught at Pinkerton Academy at Derry, N. H., for three years and entered Johns Hopkins University for graduate study in 1890. In 1893 he came to the Pennsylvania State College as Professor of Mathematics, in which position he remained until his death.